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Perturbation based stochastic finite element method for homogenization of two-phase elastic composites

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Abstract

The main idea of the paper is to apply the second order perturbation and stochastic second central moment technique to solve the homogenization problem. In order to determine the effective elasticity tensor, the prevailing computational methodology discussed in the literature so far was the Monte-Carlo simulation providing appropriate expected values and higher order probabilistic moments of the effective tensor components. The technique applied in this paper aims at significantly reducing the computational cost of the simulation without sacrificing the solution accuracy. The numerical example substantiates this claim in the case of a periodic fiber-reinforced plane strain composite with random fiber and matrix Young's moduli. © 2000 Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

It is widely known that the role of composite materials in modern engineering has been increasing in the last decades at an enormously fast pace. The most important problem while designing structures made of such materials, fiber-reinforced composites in particular, is the description of overall material properties on the basis of material parameters of the constituents. Such approach enable one to simplify significantly the discretization process of a composite [13,15,33,35] and, at the same time, to speed up computations even for deterministic models. To this purpose the effective parameters definition is introduced (i.e., effective elasticity tensor) which is derived for the given geometry of the constituents directly or by the use of the upper and lower bounds for the volume fractions of the constituents [11,20,24,32]. The first of these methods needs an appropriate variational formulation together with the finite element method (FEM) implementation and application of the special probabilistic tools analyzing the random fields of displacements and stresses resulted. In the case of upper and lower bounds with significantly shortened computational algorithms (no need to use the FEM-based numerical procedures), the whole space of the effective parameters is generated. From the engineering point of view, it seems to be very interesting to compare both of these methods considering the effectiveness in the context of effective properties computed and the time costs. The most complete and actual reviews of homogenization methods elaborated are done in Refs. [3,9,13,24].

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In general, the stochastic concepts which appeared in the homogenization theory can be divided into two groups. The first one assumes that the randomness occurs in the macro-geometry (unperiodic fibers array) [10] or microgeometry (periodic array with stochastic interfaces) [25–27] of the composite structure considered. The existing numerical algorithms are based generally on the geometrical tesselations approaches which are known as the Delaunay networks approach [31], the Voronoi cell finite element method (VCFEM) [17] or using the Monte-Carlo simulation (MCS) technique presented in Refs. [23,25,34]. The main reason for this approach is that composite geometry is obtained using digital image analysis of its structure [38]. On the other hand, we consider composite materials assuming deterministically defined internal (both macro- and micro-) geometry, but having randomized elastic (or generally material) characteristics. Mechanical problems for the second group of composites are solved by the use of one of the following computational methodologies: the Stochastic Finite Element Methods (SFEM) [28], stochastic spectral techniques [16] or the MCS approach [12,21,24]. Furthermore, there are numerous purely mathematical concepts dealing with these problems; however, they do not enable to provide the corresponding numerical implementations nor probabilistic sensitivity studies [1,2,6,14,29,30].

The main idea of this article is to introduce the stochastic second order and second moment perturbation analysis for homogenization of the two-phase periodic composite structure. The starting point for the stochastically perturbed solution is the effective modules method derived deterministically in Ref. [36] and probabilistically with MCS computational realization. It should be underlined that the probabilistic approach is very important considering the fact that all elastic constants of composite constituents are statistically estimated by the respective mean values and standard deviations. However, observing the time costs of MCS simulations, it was necessary to implement decisively faster methods taking into account detailed, future stochastic sensitivity studies with elastic or thermoelastic properties as design variables of the problem.

Considering this fact and expected complexity of homogenization equations, the stochastic second order and second moment analysis has been proposed to get the mathematical description and computer code to analyze the first two probabilistic moments of the effective elasticity tensor components. The values of these moments are compared with those obtained by using MCS technique that enable to verify the SFEA solution accuracy. Using the SFEM approach it should be remembered that the approach has its limitations consisting in upper bounds on the coefficient of variation of input random variables which, due to numerous computational studies [28], should be generally smaller than 0.15. However, neglecting stochasticity of the interface geometry, input random parameters (especially elastic) are approximately in the range of 0.1 [26] of this coefficient and can be properly analyzed by the use of the stochastic finite elements. Finally, it should be mentioned that this study can be further extended on homogenization of stochastic dynamics of composites related to reliability studies being for now one of the most developed research fields [18].

2. Mathematical model of the problem

2.1. Periodic fiber-reinforced two-phase composite

The main object of the considerations is the random periodic fiber-reinforced composite structure in the plane strain. Let us denote the representative volume element (RVE) of Y as Ω; $Y \subset \Re^2$ denotes here the section of this composite with $x_3 = 0$ plane and is constant along the x_3 axis being parallel to the fibers direction (see Fig. 1).

Let us assume that the region Ω contains two perfectly bonded, coherent and disjoint subsets Ω_1 (fiber) and Ω_2 (matrix) and let the scale between corresponding geometrical diameters of Ω and Y be described by the small parameter $\varepsilon > 0$. The parameter ε is indexing further all the tensors written for the geometrical scale of Ω and let $\partial\Omega$ denote the external boundary of the Ω while $\partial\Omega_{12}$ the interface boundary between Ω_1 and Ω_2 regions.

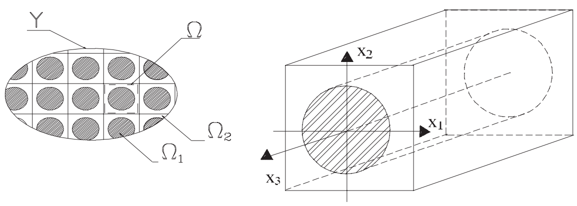

Fig. 1. Periodic fiber-reinforced composite.

Further, it is assumed that the composite is periodic in a random sense if for an additional ω belonging to a suitable probability space there exists such a homothety that transforms Ω onto the whole composite Y. Next, let us introduce two different coordinate systems $\mathbf{y} = (y_1, y_2, y_3)$ at the microscale of the composite and $\mathbf{x} = (x_1, x_2, x_3)$ at the macroscale and let us consider any periodic state function F defined on the region Y; this function can be expressed as

$$F^{\varepsilon}(\mathbf{x}) = F\left(\frac{\mathbf{x}}{\varepsilon}\right) = F(\mathbf{y}). \tag{1}$$

This expression makes it possible to describe the macrofunctions (connected with the macroscale of a composite) in terms of microfunctions and vice versa. The elasticity coefficients can be defined, for instance, as

$$C^{\varepsilon}_{ijkl}(\mathbf{x}) = C_{ijkl}(\mathbf{y}). \tag{2}$$

To characterize elastic properties of the composite constituents let us assume that Ω_1 and Ω_2 contain linear elastic and transversely isotropic materials, where the Young's moduli are Gaussian random variables bounded to the nonnegative values only. Practically, probabilistic distributions considered are to have such probabilistic moments that probability of negative value occurrence (for Young's moduli) is infinitesimal (the so called cut-off Gaussian distribution). The expected values and the variances of these variables are given as follows:

$$0 < e(\mathbf{x}; \omega) < \infty, \tag{3}$$

$$E[e(\mathbf{x}; \omega)] = \begin{cases} e_1, & \mathbf{x} \in \Omega_1, \\ e_2, & \mathbf{x} \in \Omega_2, \end{cases} \tag{4}$$

$$\mathrm{cov}\big(e_i(\mathbf{x}; \omega); e_j(\mathbf{x}; \omega)\big) = \begin{bmatrix} \mathrm{var}\, e_1 & 0 \\ 0 & \mathrm{var}\, e_2 \end{bmatrix}, \tag{5}$$

where correlations equal to 0 are assumed due to the lack of any appropriate experimental data. The Poisson's ratios are assumed to be given deterministically so that

$$-1 < \nu(\mathbf{x}) < \tfrac{1}{2}, \tag{6}$$

$$\nu(\mathbf{x}) = \begin{cases} \nu_1, & \mathbf{x} \in \Omega_1, \\ \nu_2, & \mathbf{x} \in \Omega_2. \end{cases} \tag{7}$$

The elasticity tensor components for both matrix and fiber fulfill the following conditions:

$$C_{ijkl} \in L^{\infty}(\Re^3), \quad \text{for } i, j, k, l = 1, 2, 3, \tag{8}$$

$$C_{ijkl} = C_{klij} = C_{jikl}, \tag{9}$$

$$\exists\, C_0 > 0;\ C_{ijkl}\xi_{ij}\xi_{kl} \geqslant C_0\xi_{ij}\xi_{ij} \quad \forall_{i,j}\ \xi_{ij} = \xi_{ji}. \tag{10}$$

Moreover, for any of the composite constituents this tensor is defined as

$$C_{ijkl}(e(\mathbf{x}; \omega); \mathbf{x}) = e(\mathbf{x}; \omega)\left\{\delta_{ij}\delta_{kl}\frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{2(1+\nu(\mathbf{x}))}\right\}. \tag{11}$$

By observing that the elasticity tensor coefficients depend linearly on the Young's modulus, the first two probabilistic moments of the tensor can be derived explicitly as

$$E[C_{ijkl}(e(\mathbf{x}; \omega); \mathbf{x})] = A_{ijkl}(\mathbf{x}) \cdot E[e(\mathbf{x}; \omega)], \tag{12}$$

and

$$\begin{gathered} \mathrm{cov}\big(C_{ijkl}(e_r(\mathbf{x}; \omega); \mathbf{x}); C_{mnpq}(e_s(\mathbf{x}; \omega); \mathbf{x})\big) = A_{ijkl}(\mathbf{x})A_{mnpq}(\mathbf{x})\mathrm{cov}(e_r(\mathbf{x}; \omega); e_s(\mathbf{x}; \omega)) \\ (\text{no sum on } i, j, k, l, m, n, p, q), \end{gathered} \tag{13}$$

where

$$A_{ijkl}(\mathbf{x}) = \delta_{ij}\delta_{kl}\frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{2(1+\nu(\mathbf{x}))}. \tag{14}$$

Finally, the effective moduli tensor $C^{(\mathrm{eff})}_{ijkl}$ is introduced as such a tensor replacing C^{ε}_{ijkl} with $C^{(\mathrm{eff})}_{ijkl}$ in the following equilibrium equations:

$$C^{\varepsilon}_{ijkl}\,\varepsilon_{kl}(\mathbf{u}^{\varepsilon}) + f_i = 0; \quad \mathbf{x}\in\Omega, \tag{15}$$

$$\varepsilon_{ij}(\mathbf{u}^{\varepsilon}) = \tfrac{1}{2}\left(u^{\varepsilon}_{i,j} + u^{\varepsilon}_{j,i}\right); \quad \mathbf{x}\in\Omega, \tag{16}$$

$$C^{\varepsilon}_{ijkl} = \psi^{(a)}(\mathbf{x})\, C^{\varepsilon(a)}_{ijkl}, \tag{17}$$

where $\mathbf{u}^0$ is obtained as a solution being a weak limit of $\mathbf{u}^{\varepsilon}$ with $\varepsilon \to 0$. The characteristic function in Eq. (17) is defined as

$$\psi^{(a)}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x}\in\Omega_a, \\ 0, & \text{elsewhere} \end{cases} \quad a = 1,2, \tag{18}$$

while the boundary conditions

$$\mathbf{u}^{\varepsilon} = 0, \quad \mathbf{x}\in\partial\Omega. \tag{19}$$

Detailed mathematical considerations and especially the proof of existence and uniqueness of this system solution has been provided in Refs. [22,24,36].

2.2. Variational formulation of the homogenization procedure

The homogenization problem is to find the limit of solution $\mathbf{u}^{\varepsilon}$ with ε tending to 0. To this purpose, let us consider a bilinear form $a^{\varepsilon}(\mathbf{u},\mathbf{v})$ defined as follows:

$$a^{\varepsilon}(\mathbf{u},\mathbf{v}) = \int_{\Omega} C_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right)\varepsilon_{ij}(\mathbf{u})\,\varepsilon_{kl}(\mathbf{v})\,\mathrm{d}\Omega, \tag{20}$$

and a following linear form

$$L(\mathbf{v}) = \int_{\Omega} f_i v_i\,\mathrm{d}\Omega + \int_{\partial\Omega_{\sigma}} p_i v_i\,\mathrm{d}(\partial\Omega). \tag{21}$$

The variational statement equivalent to the equilibrium problem (15–19) is to find $\mathbf{u}^{\varepsilon}$ fulfilling the following equation:

$$a^{\varepsilon}(\mathbf{u}^{\varepsilon},\mathbf{v}) = L(\mathbf{v}) \tag{22}$$

for any kinematic admissible displacement $\mathbf{v}$. To this purpose, let us define a space of periodic functions $P(\Omega)$ so that the trace of $\mathbf{v}$ is equal on the opposite sides of Ω. Let us denote for any $\mathbf{u},\mathbf{v}\in P(\Omega)$

$$a_y(\mathbf{u},\mathbf{v}) = \int_{\Omega} C_{ijkl}(\mathbf{y})\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega, \tag{23}$$

and introduce a homogenization function $\chi_{(ij)k}\in P(\Omega)$ as a solution for the local problem on a periodicity cell:

$$a_y\left(\left(\chi_{(ij)k} + y_j\delta_{ki}\right)\mathbf{n}_k,\mathbf{w}\right) = 0 \tag{24}$$

for any $\mathbf{w}\in P(\Omega)$, here δ_{ki} denotes the Kronecker delta while $\mathbf{n}_k$ is the unit coordinate vector. Now, we are looking for the solution $\mathbf{u}^{\varepsilon}$ that converges weakly

$$\mathbf{u}^{\varepsilon} \to \mathbf{u} \tag{25}$$

if the tensor $C^{\varepsilon}_{ijkl}(\mathbf{y})$ is Ω-periodic and its components fulfill conditions (8–10). Solution $\mathbf{u}$ is the unique one for the boundary value problem

$$\mathbf{u}\in V : D(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \tag{26}$$

Table 1
The components of homogenization boundary forces $F_{(pq)i}$

	$\chi_{(11)}$	$\chi_{(12)}$	$\chi_{(22)}$
$F_{(pq)1}$	$C^{(2)}_{1111} - C^{(1)}_{1111}$	$C^{(2)}_{1212} - C^{(1)}_{1212}$	$C^{(2)}_{1122} - C^{(1)}_{1122}$
$F_{(pq)2}$	$C^{(2)}_{2211} - C^{(1)}_{2211}$	$C^{(2)}_{1212} - C^{(1)}_{1212}$	$C^{(2)}_{2222} - C^{(1)}_{2222}$

for any admissible displacement $\mathbf{v}$ and

$$D(\mathbf{u},\mathbf{v}) = \int_Y D_{ijkl}\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}Y, \tag{27}$$

where

$$D_{ijkl} = \frac{1}{|\Omega|}\,a_y\Big(\Big(\chi_{(ij)p} + y_i\delta_{pj}\Big)\mathbf{n}_p, \Big(\chi_{(kl)q} + y_l\delta_{qk}\Big)\mathbf{n}_q\Big). \tag{28}$$

As a result, a homogeneous orthotropic elastic material is obtained as the tensor

$$C^{(\mathrm{eff})}_{ijkl} = \frac{1}{|\Omega|}\int_\Omega \Big(C_{ijkl}(\mathbf{y}) + C_{ijmn}(\mathbf{y})\varepsilon_{mn}\Big(\chi_{(kl)}(\mathbf{y})\Big)\Big)\,\mathrm{d}\Omega. \tag{29}$$

Homogenization functions $\chi_{(pq)i}$ for i, p, $q=1$, 2 are computed as some specific elastostatic plane strain problems displacement solutions. For this purpose, the RVE is considered under the displacement symmetry conditions imposed on its external boundaries (zeroing of the displacements perpendicular to $\partial\Omega$ and rotational degrees of freedom). The stress boundary conditions are applied along the interface (if only fiber and matrix are perfectly bonded) in the following form (cf. Eq. (A.4), Appendix A):

$$\sigma_{ij}\Big(\chi_{(pq)}\Big)n_j = \big[C_{ijpq}\big]n_j = F_{(pq)i}; \quad \mathbf{x} \in \partial\Omega_{12}, \tag{30}$$

where n_j is the component of the unit vector normal to the fiber–matrix boundary and directed to the fiber interior, while $[f]$ denotes the difference of the function f values

$$[f] = f^{(2)} - f^{(1)}. \tag{31}$$

The stress boundary conditions corresponding to different homogenization problems are specified in Table 1.

It should be underlined that taking into account the interface phenomena in engineering composites, the fiber and matrix boundaries may be partially different contours (the lack of contact between the components) which may be the result of composite processing thermal stresses.

Finally, neglecting the body forces vector and taking into account all equations posed above, the variational statement for the homogenization problem can be formulated as follows:

$$\int_\Omega \delta v_{i,j}C_{ijkl}\chi_{(pq)k,l}\,\mathrm{d}\Omega = -\int_{\partial\Omega_{12}} \delta v_i F_{(pq)i}\,\mathrm{d}(\partial\Omega). \tag{32}$$

2.3. Stochastic second order perturbation of the homogenization equations

The homogenization model presented is combined now with the stochastic second order perturbation second central probabilistic moment method. Let us denote the random variable vector of the problem as $\{b^{\mathrm{r}}(\mathbf{x};\omega)\}$, and the probability density functions of random variable by $g(b^{\mathrm{r}}(\mathbf{x};\omega))$ and the joint probability functions of random variables pair $b^{\mathrm{r}}(\mathbf{x};\omega)$ and $b^{\mathrm{s}}(\mathbf{x};\omega)$ by $g(b^{\mathrm{r}}(\mathbf{x};\omega), b^{\mathrm{s}}(\mathbf{x};\omega))$, respectively. Indices r, s running over 1 to R, where R denotes the total number of input random vector components. Thus, the expected value of the vector $\{b^{\mathrm{r}}(\mathbf{x};\omega)\}$ can be expressed as [28,39]

$$E[b^{\mathrm{r}}] = \int_{-\infty}^{+\infty} b^{\mathrm{r}} g(b^{\mathrm{r}})\,\mathrm{d}b^{\mathrm{r}} \tag{33}$$

while the covariance in the form

$$\mathrm{cov}(b^{\mathrm{r}}, b^{\mathrm{s}}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (b^{\mathrm{r}} - E[b^{\mathrm{r}}])(b^{\mathrm{s}} - E[b^{\mathrm{s}}])g(b^{\mathrm{r}}, b^{\mathrm{s}})\,\mathrm{d}b^{\mathrm{r}}\,\mathrm{d}b^{\mathrm{s}}. \tag{34}$$

The coefficient of variation of the input random vector components is defined as

$$\alpha(b(\mathbf{x};\omega)) = \sqrt{\frac{\mathrm{var}(b(\mathbf{x};\omega))}{E^2[b(\mathbf{x};\omega)]}}. \tag{35}$$

The variational principle (32) can be extended to the stochastic one by the use of the second order second moment perturbation method based on the Taylor series expansion. According to the method described in Ref. [29] all functions in the last equation are expressed in the form similar to the following extension of function G:

$$G(\mathbf{x}) = G^0(\mathbf{x}) + \theta G^{,\mathrm{r}}(\mathbf{x})\,\Delta b^{\mathrm{r}} + \tfrac{1}{2}\theta^2 G^{,\mathrm{rs}}(\mathbf{x})\,\Delta b^{\mathrm{r}}\,\Delta b^{\mathrm{s}}, \tag{36}$$

where θ is given small perturbation, $\theta\Delta b^{\mathrm{r}}$ denotes the first order variation of b^{r} about its expected value. Moreover, symbols $(.)^0$, $(.)^{,\mathrm{r}}$ and $(.)^{,\mathrm{rs}}$ represent the expected value, the first and the second partial derivatives with respect to the random variables evaluated at the expected values of input random parameters. To rewrite the stochastic formulation of the variational formulation (32), the interface forces following the stress interface conditions should be stochastically perturbed first. It is known from the classical theory of homogenization [9,24,36] that in the case of ideal bonding between fiber and matrix, the interface load components are obtained, as it was mentioned above, in the form of the following difference:

$$F_{(pq)i} = F^{(2)}_{(pq)i} - F^{(1)}_{(pq)i}. \tag{37}$$

Taking into account the Taylor series expansion given by Eq. (36) it is obtained that

$$F_{(pq)i} = \left(F_{(pq)i}\right)^0 + \theta\left(F_{(pq)i}\right)^{,\mathrm{r}}\,\Delta b^{\mathrm{r}} + \tfrac{1}{2}\theta^2\left(F_{(pq)i}\right)^{,\mathrm{rs}}\,\Delta b^{\mathrm{r}}\,\Delta b^{\mathrm{s}}. \tag{38}$$

Rewriting the forces $F^{(t)}_{(pq)i}$ for $t = 0,1,2$, comparing the respective terms of the zeroth, first and second order and, at last, dividing the last two equations by $\theta\Delta b^{\mathrm{r}}$ and $\frac{1}{2}\theta^2\,\Delta b^{\mathrm{r}}\,\Delta b^{\mathrm{s}}$, respectively, we obtain

$$\left(F_{(pq)i}\right)^0 = \left(F^{(2)}_{(pq)i}\right)^0 - \left(F^{(1)}_{(pq)i}\right)^0, \tag{39}$$

$$\left(F_{(pq)i}\right)^{,\mathrm{r}} = \left(F^{(2)}_{(pq)i}\right)^{,\mathrm{r}} - \left(F^{(1)}_{(pq)i}\right)^{,\mathrm{r}}, \tag{40}$$

$$\left(F_{(pq)i}\right)^{,\mathrm{rs}} = \left(F^{(2)}_{(pq)i}\right)^{,\mathrm{rs}} - \left(F^{(1)}_{(pq)i}\right)^{,\mathrm{rs}}. \tag{41}$$

Thus, the stochastic version of the minimum potential energy principle for the homogenization problem has the following form:

- one zeroth order equation:

$$\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^0_{ijkl}\left(\chi_{(pq)k,l}\right)^0 \mathrm{d}\Omega = -\int_{\partial\Omega_{12}}\delta v_i\left(F_{(pq)i}\right)^0 \mathrm{d}(\partial\Omega), \tag{42}$$

- R first order equations:

$$\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^0_{ijkl}\left(\chi_{(pq)k,l}\right)^{,\mathrm{r}} \mathrm{d}\Omega = -\int_{\partial\Omega_{12}}\delta v_i\left(F_{(pq)i}\right)^{,\mathrm{r}} \mathrm{d}(\partial\Omega) - \sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^{,\mathrm{r}}_{ijkl}\left(\chi_{(pq)k,l}\right)^0 \mathrm{d}\Omega, \tag{43}$$

- one second order equation:

$$\begin{aligned}\left(\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^0_{ijkl}\left(\chi_{(pq)k,l}\right)^{,\mathrm{rs}} \mathrm{d}\Omega\right)\cdot \mathrm{cov}(b^{\mathrm{r}},b^{\mathrm{s}}) &= -\left(\int_{\partial\Omega_{12}}\delta v_i\left(F_{(pq)i}\right)^{,\mathrm{rs}} \mathrm{d}(\partial\Omega)\right)\cdot \mathrm{cov}(b^{\mathrm{r}},b^{\mathrm{s}}) \\ &\quad - \left(2\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^{,\mathrm{r}}_{ijkl}\left(\chi_{(pq)k,l}\right)^{,\mathrm{s}} \mathrm{d}\Omega \right. \\ &\quad \left. + \sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j} C^{,\mathrm{rs}}_{ijkl}\left(\chi_{(pq)k,l}\right)^0 \mathrm{d}\Omega\right)\cdot \mathrm{cov}(b^{\mathrm{r}},b^{\mathrm{s}}).\end{aligned} \tag{44}$$

Considering the nature of the problem (Young's moduli of the fiber and matrix are the components of the input random variable vector) there holds

$$\frac{\partial\big(C_{ijkl}(e_a(\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_a} = \psi^{(a)}A^{(a)}_{ijkl}(\mathbf{x}), \quad a=1,2, \tag{45}$$

where $A^{(a)}_{ijkl}$ is the tensor given by Eq. (14) and calculated for the elastic characteristics of respective material indexed by 'a' while $\psi^{(a)}$ is the characteristic function given by Eq. (18). Thus, the first order derivatives of the elasticity tensor with respect to the input random variable vector are obtained as

$$\frac{\partial\big(C_{ijkl}(e_a(\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_a} = \left\{\frac{\partial\big(C_{ijkl}(e_a(\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_1}, \frac{\partial\big(C_{ijkl}(e_a(\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_2}\right\}. \tag{46}$$

Hence, the second order derivatives have the form

$$\frac{\partial^2\big(C_{ijkl}((\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_a^2} = \psi^{(a)}\,\frac{\partial A^{(a)}_{ijkl}(\mathbf{x})}{\partial e_a} = 0, \quad \text{for } a=1,2, \tag{47}$$

while mixed second order derivatives can be written as

$$\frac{\partial^2\big(C_{ijkl}((\mathbf{x};\omega);\mathbf{x})\big)}{\partial e_1\,\partial e_2} = \psi^{(1)}\frac{\partial A^{(1)}_{ijkl}(\mathbf{x})}{\partial e_2} = \psi^{(2)}\frac{\partial A^{(2)}_{ijkl}(\mathbf{x})}{\partial e_1} = 0. \tag{48}$$

Therefore, all components of the second order derivatives of the stiffness matrices $K^{(pq)}_{\alpha\beta}$ of the problem are equal to 0. Moreover, since the assumption on uncorrelation of input random variables there holds

$$\mathrm{cov}(e_1;e_2) = \begin{bmatrix} \mathrm{var}\,e_1 & 0 \\ 0 & \mathrm{var}\,e_2 \end{bmatrix} \tag{49}$$

and thus, the first and the second partial derivatives of the vectors $F^{(a)}_{(pq)i}$ with respect to the random variables vector are calculated as

$$\frac{\partial F^{(a)}_{(pq)i}}{\partial e_a} = \frac{\partial C^{(a)}_{ijpq}}{\partial e_a}n_j = A^{(a)}_{ijpq}\,n_j,\ \ \mathbf{x}\in\partial\Omega_a, \quad a=1,2, \tag{50}$$

and

$$\frac{\partial^2 F^{(a)}_{(pq)i}}{\partial e_a^2} = \frac{\partial^2 C^{(a)}_{ijpq}}{\partial e_a^2}n_j = \frac{\partial A^{(a)}_{ijpq}}{\partial e_a}\,n_j = 0,\ \ \mathbf{x}\in\partial\Omega_a, \quad a=1,2. \tag{51}$$

Considering all these simplifications, the set of Eqs. (42)–(44) can be written in the following form:

- one zeroth order equation:

$$\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j}C^0_{ijkl}\,(\chi_{(pq)k,l})^0\,\mathrm{d}\Omega = -\int_{\partial\Omega_{12}}\delta v_i\big(F_{(pq)i}\big)^0\,\mathrm{d}(\partial\Omega), \tag{52}$$

- R first order equations:

$$\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j}C^0_{ijkl}\,(\chi_{(pq)k,l})^{,r}\,\mathrm{d}\Omega = -\int_{\partial\Omega_{12}}\delta v_i\big[A_{pqij}\big]n_j\,\mathrm{d}(\partial\Omega) - \sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j}A^{(a)}_{ijkl}\,(\chi_{(pq)k,l})^0\,\mathrm{d}\Omega \tag{53}$$

- one second order equation:

$$\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j}C^0_{ijkl}\,(\chi_{(pq)k,l})^{(2)}\,\mathrm{d}\Omega = -\sum_{a=1,2}\int_{\Omega_a}\delta v_{i,j}A^{(a)}_{ijkl}\,(\chi_{(pq)k,l})^{,s}\,\mathrm{d}\Omega\cdot\mathrm{cov}(b^r,b^s), \tag{54}$$

where

$$(\chi_{(pq)k,l})^{(2)} = -\tfrac{1}{2}(\chi_{(pq)k,l})^{,rs}\cdot\mathrm{cov}(b^r,b^s). \tag{55}$$

It should be noted that Eqs. (52)–(55) give the set of fundamental variational equations of the homogenization problem in the second order stochastic perturbation language. Next, these equations will be discretized by the use of classical finite element technique and, as a result, the zeroth, first and second order algebraic equations will be obtained.

3. Computational implementation

3.1. Stochastic finite element discretization of the homogenization procedure

Let us introduce the following discretization of the displacement function and its derivatives with respect to the random variables using the shape functions $\varphi_{i\alpha}(\mathbf{x})$:

$$\left(\chi_{(pv)i}(\mathbf{x})\right)^0 = \varphi_{i\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^0, \quad \mathbf{x} \in \Omega,\ p,v = 1,2, \tag{56}$$

$$\left(\chi_{(pv)i}(\mathbf{x})\right)^{,r} = \varphi_{i\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^{,r}, \quad \mathbf{x} \in \Omega,\ p,v = 1,2, \tag{57}$$

$$\left(\chi_{(pv)i}(\mathbf{x})\right)^{,rs} = \varphi_{i\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^{,rs}, \quad \mathbf{x} \in \Omega,\ p,v = 1,2, \tag{58}$$

where $i = 1,2$; $r,s = 1,\ldots,R$; $\alpha = 1,\ldots,N$ (N is the total number of degrees of freedom in the region Ω). By the analogous way, the approximation of the strain tensor components is introduced as

$$\varepsilon_{ij}^0\left(\chi_{(pv)}(\mathbf{x})\right) = B_{ij\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^0, \quad \mathbf{x} \in \Omega, \tag{59}$$

$$\varepsilon_{ij}^{,r}\left(\chi_{(pv)}(\mathbf{x})\right) = B_{ij\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^{,r}, \quad \mathbf{x} \in \Omega, \tag{60}$$

$$\varepsilon_{ij}^{,rs}\left(\chi_{(pv)}(\mathbf{x})\right) = B_{ij\alpha}(\mathbf{x}) \cdot \left(q_{(pv)\alpha}\right)^{,rs}, \quad \mathbf{x} \in \Omega, \tag{61}$$

where $B_{ij\alpha}(\mathbf{x})$ is the typical FEM strain–displacement matrix [4]

$$B_{ij\alpha}(\mathbf{x}) = \tfrac{1}{2}\left[\varphi_{i\alpha,j}(\mathbf{x}) + \varphi_{j\alpha,i}(\mathbf{x})\right], \quad \mathbf{x} \in \Omega. \tag{62}$$

Introducing the equations stated above to the zeroth, first and second order statements of the homogenization problem represented by Eqs. (52)–(55), we arrive at the stochastic formulation of the problem which can be posed in the form of the following algebraic linear equations:

$$\mathbf{K}^0\left(\mathbf{q}_{(pv)}\right)^0 = \left(\mathbf{Q}_{(pv)}\right)^0, \tag{63}$$

$$\mathbf{K}^0\left(\mathbf{q}_{(pv)}\right)^{,r} = \left(\mathbf{Q}_{(pv)}\right)^{,r} - \mathbf{K}^{,r}\left(\mathbf{q}_{(pv)}\right)^0, \tag{64}$$

$$\mathbf{K}^0\left(\mathbf{q}_{(pv)}\right)^{(2)} = -\mathbf{K}^{,r}\left(\mathbf{q}_{(pv)}\right)^{,s}, \tag{65}$$

where

$$\left(\mathbf{q}^{(pv)}\right)^{(2)} = \tfrac{1}{2}\left(\mathbf{q}^{(pv)}\right)^{,rs} \operatorname{cov}(b^r, b^s) \tag{66}$$

and $\mathbf{K}$, $\mathbf{q}_{(pv)}$, $\mathbf{Q}_{(pv)}$ denote the global stiffness matrix, discretized homogenization functions and external load vectors, respectively. Considering the plane strain formulation of the problem, the global stiffness matrix and its partial derivatives with respect to the random variable of the problem can be rewritten as follows:

$$K_{\alpha\beta}^0 = \sum_{e=1}^{E} \int_{\Omega_e} C_{ijkl}^0 B_{ij\alpha} B_{kl\beta}\, \mathrm{d}\Omega = \sum_{e=1}^{E} \frac{e(1-v)}{(1+v)(1-2v)} \int_{\Omega_e} \begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ & 1 & 0 \\ \text{symm.} & & \frac{1-2v}{2(1-v)} \end{bmatrix} B_{ij\alpha} B_{kl\beta}\, \mathrm{d}\Omega, \tag{67}$$

$$K_{\alpha\beta}^{,\mathrm{r}} = \sum_{e=1}^{E} \int_{\Omega_e} C_{ijkl}^{,\mathrm{r}} B_{ij\alpha} B_{kl\beta}\, \mathrm{d}\Omega = \sum_{e=1}^{E} \frac{(1-v)}{(1+v)(1-2v)} \int_{\Omega_e} \begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ & 1 & 0 \\ \text{symm.} & & \frac{1-2v}{2(1-v)} \end{bmatrix} B_{ij\alpha} B_{kl\beta}\, \mathrm{d}\Omega, \tag{68}$$

$$K_{\alpha\beta}^{,\mathrm{rs}} = \sum_{e=1}^{E} \int_{\Omega_e} C_{ijkl}^{,\mathrm{rs}} B_{ij\alpha} B_{kl\beta}\, \mathrm{d}\Omega = 0. \tag{69}$$

Computing from the above equations successively the zeroth order displacement vector $(\mathbf{q}_{(pv)})^0$ from Eq. (63), the first order displacement vector $(\mathbf{q}_{(pv)})^{,\mathrm{r}}$ from Eq. (64) and the second order displacement vector $(\mathbf{q}_{(pv)})^{(2)}$ from the last two Eqs. (65)–(66), the expected values of the homogenization function can be derived as

$$E[\mathbf{q}_{(pv)}] = (\mathbf{q}_{(pv)})^0 + \tfrac{1}{2}(\mathbf{q}_{(pv)})^{,\mathrm{rs}} \operatorname{cov}(b^\mathrm{r}, b^\mathrm{s}), \tag{70}$$

and their covariance matrix in the form:

$$\operatorname{cov}\left(q_{(pv)\alpha}, q_{(pv)\beta}\right) = \left(q_{(pv)\alpha}\right)^{,\mathrm{r}} \left(q_{(pv)\beta}\right)^{,\mathrm{s}} \operatorname{cov}(b^\mathrm{r}, b^\mathrm{s}), \tag{71}$$

where α, β are running over all the degrees of freedom of the system. Moreover, the expected values of the stress tensor can be computed as

$$E\left[\sigma_{ij}^{(f)}(\mathbf{q}_{(\mathrm{pv})})\right] = C_{ijkl}^{0(f)} B_{kl\alpha}^{(f)} \left(\mathbf{q}_{(pv)}^{(f)}\right)^0 + \tfrac{1}{2}\left[2C_{ijkl}^{,\mathrm{r}(f)}\left(\mathbf{q}_{(pv)}^{(f)}\right)^{,\mathrm{s}} + C_{ijkl}^{0(f)}\left(\mathbf{q}_{(pv)}^{(f)}\right)^{,\mathrm{rs}}\right] \mathbf{B}_{kl}^{(f)} \operatorname{cov}(b^\mathrm{r}, b^\mathrm{s}), \tag{72}$$

while its covariances from the following equation:

$$\begin{aligned} \operatorname{cov}\left(\sigma_{ij}^{(d)}, \sigma_{kl}^{(f)}\right) = {} & \Big[C_{ijmn}^{0(d)} C_{klgh}^{0(f)} \left(\mathbf{q}_{(pv)}^{(d)}\right)^{,\mathrm{r}} \left(\mathbf{q}_{(pv)}^{(f)}\right)^{,\mathrm{s}} + C_{ijmn}^{,\mathrm{r}(d)} C_{klgh}^{,\mathrm{s}(f)} \left(\mathbf{q}_{(pv)}^{(d)}\right)^0 \left(\mathbf{q}_{(pv)}^{(f)}\right)^0 + C_{ijmn}^{,\mathrm{r}(d)} C_{klgh}^{0(f)} \left(\mathbf{q}_{(pv)}^{(d)}\right)^{,\mathrm{s}} \left(\mathbf{q}_{(pv)}^{(f)}\right)^0 \\ & + C_{ijmn}^{0(d)} C_{klgh}^{,\mathrm{s}(f)} \left(\mathbf{q}_{(pv)}^{(d)}\right)^{,\mathrm{r}} \left(\mathbf{q}_{(pv)}^{(f)}\right)^0\Big] \mathbf{B}_{mn}^{(d)} \mathbf{B}_{gh}^{(f)} \operatorname{cov}(b^\mathrm{r}, b^\mathrm{s}), \end{aligned} \tag{73}$$

for $i, j, k, l, g, h, p, v = 1, 2$ and $1 \leqslant d, f \leqslant E$ indexing the numbers of the finite elements in the discretized system.

3.2. Stochasticity of effective elasticity tensor

The objective of a homogenization procedure is to determine the effective elasticity tensor components – Eq. (29) is used to this purpose in deterministic problems. In accordance with the methodology adopted in this paper, the first two probabilistic moments (expected values and covariances) of the elasticity tensor components are to be found in the corresponding stochastic problem. There holds

$$E\left[C_{ijpq}^{(\mathrm{eff})}\right] = \frac{1}{|\Omega|} \int_\Omega \left(E\left[C_{ijpq}\right] + E\left[C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\right]\right) \mathrm{d}\Omega. \tag{74}$$

The second term in the above integral can be extended as follows [28]:

$$\begin{aligned} E\left[C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\right] = {} & \int_{-\infty}^{+\infty} \left(C_{ijkl}^0 + \Delta b^\mathrm{r} C_{ijkl}^{,\mathrm{r}} + \tfrac{1}{2}\Delta b^\mathrm{r}\, \Delta b^\mathrm{s} C_{ijkl}^{,\mathrm{rs}}\right) p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} \int_{-\infty}^{+\infty} \Big((\chi_{(pq)k,l})^0 + \Delta b^u (\chi_{(pq)k,l})^{,u} \\ & + \tfrac{1}{2}\Delta b^u \Delta b^v (\chi_{(pq)k,l})^{,uv}\Big) p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b}. \end{aligned} \tag{75}$$

By observing that

$$\int_{-\infty}^{+\infty} p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} = 1, \tag{76}$$

$$\int_{-\infty}^{+\infty} \Delta b_r p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} = 0, \tag{77}$$

$$\int_{-\infty}^{+\infty} \Delta b_r\, \Delta b_s p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} = \operatorname{cov}(b^\mathrm{r}, b^\mathrm{s}) \tag{78}$$

there holds

$$E\Big[C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\Big] = \int_{-\infty}^{+\infty} C^0_{ijkl}(\chi_{(pq)k,l})^0 p_R(\mathbf{b}(\mathbf{x}))\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \Delta b^{\mathrm{r}} C^{,\mathrm{r}}_{ijkl}\, \Delta b^u (\chi_{(pq)k,l})^{,\mathrm{u}} p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b}$$
$$+\frac{1}{2}\int_{-\infty}^{+\infty} C^0_{ijkl}\,\Delta b^u\,\Delta b^{\mathrm{r}}(\chi_{(pq)k,l})^{,uv} p_R(\mathbf{b}(x))\,\mathrm{d}\mathbf{b}$$
$$= C^0_{ijkl}\Big(\chi_{(pq)k,l}\Big)^0 + \Big\{C^{,\mathrm{r}}_{ijkl}(\chi_{(pq)k,l})^{,\mathrm{s}} + \tfrac{1}{2}C^0_{ijkl}(\chi_{(pq)k,l})^{,\mathrm{rs}}\Big\}\cdot \mathrm{cov}(b^{\mathrm{r}}, b^{\mathrm{s}}). \quad (79)$$

Averaging both sides of this equation over the region Ω and including it in statement (74) together with spatially averaged expected values of elasticity tensor, the expected values of the homogenized elasticity tensor are obtained. Next, the covariances of the effective elasticity tensor components can be derived similarly as

$$\mathrm{cov}\Big(C^{(\mathrm{eff})}_{ijkl}, C^{(\mathrm{eff})}_{mnpq}\Big) = \mathrm{cov}\big(C_{ijkl}, C_{mnpq}\big) + \mathrm{cov}(C_{ijkl}, C_{mnuv}\chi_{(pq)u,v}) + \mathrm{cov}(C_{ijrs}\chi_{(kl)r,s}, C_{mnpq})$$
$$+\,\mathrm{cov}(C_{ijrs}\chi_{(kl)r,s}, C_{mnuv}\chi_{(pq)u,v}). \quad (80)$$

Taking into account all mathematical transformations provided in Appendix B, the final form of this covariance is obtained as

$$\mathrm{cov}\Big(C^{(\mathrm{eff})}_{ijkl}; C^{(\mathrm{eff})}_{mnpq}\Big) = \Big\{C^{,\mathrm{r}}_{ijkl}C^{,\mathrm{s}}_{mnpq} + C^{,\mathrm{r}}_{ijtw}(\chi_{(kl)t,w})^0 C^{,\mathrm{s}}_{mnpq} + C^{,\mathrm{r}}_{ijtw}(\chi_{(kl)t,w})^{,\mathrm{s}}C^0_{mnpq} + C^{,\mathrm{r}}_{ijkl}C^{,\mathrm{s}}_{mnuv}(\chi_{(pq)u,v})^0$$
$$+ C^{,\mathrm{r}}_{ijkl}C^0_{mnuv}(\chi_{(pq)u,v})^{,\mathrm{s}} + C^{,\mathrm{r}}_{ijtw}C^{,\mathrm{s}}_{mnuv}(\chi_{(kl)t,w})^0(\chi_{(pq)u,v})^0 + C^{,\mathrm{r}}_{ijtw}C^0_{mnuv}(\chi_{(kl)t,w})^0(\chi_{(pq)u,v})^{,\mathrm{s}}$$
$$+ C^0_{ijtw}C^{,\mathrm{r}}_{mnuv}(\chi_{(kl)t,w})^{,\mathrm{s}}(\chi_{(pq)u,v})^0 + C^0_{ijtw}C^0_{mnuv}(\chi_{(kl)t,w})^{,\mathrm{r}}(\chi_{(pq)u,v})^{,\mathrm{s}}\Big\}\mathrm{cov}(b^{\mathrm{r}}, b^{\mathrm{s}}) \quad (81)$$

and it should be underlined that the above equations give complete description of the effective elasticity tensor components in the stochastic second moment and second order perturbation approach. Finally, let us note that many simplifications resulted here thanks to the assumption that the input random variables of the homogenization problem are the Young's moduli in the fiber and matrix only. If the Poisson's ratios were treated as random, the second order derivatives of the constitutive tensor would generally be different from zero and the stochastic finite element formulation of homogenization procedure would be much more complicated.

4. Numerical illustration

The procedure has been implemented in the homogenization-oriented computer program MCCEFF [24–26] used previously for computations of the effective elasticity tensor components probabilistic moments by the use of the MCS technique [2,3,25]. For the periodicity cell and its discretization shown in Fig. 2, the elastic properties of the glass fiber and resin matrix are adopted as follows: the Young's moduli expected values $E[e_1] = 84$ GPa, $E[e_2] = 4.0$ GPa while the deterministic Poisson's ratios are taken as equal $v_1 = 0.22$ for fiber and $v_2 = 0.34$ for matrix.

Four different sets of the Young's moduli coefficients of variation have been analyzed as it is specified in Table 2. Different combinations of the values 0.10 and 0.05 have been tested to analyze the influence of the component data

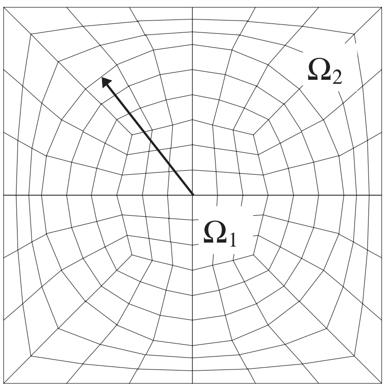

Fig. 2. Periodicity cell tested.

Table 2
The coefficient of variation of the input random variables

Test number	$\alpha(e_1)$	$\alpha(e_2)$
1	0.10	0.10
2	0.10	0.05
3	0.05	0.10
4	0.05	0.05

Table 3
Expected values and coefficients of variation for the effective elasticity tensor components

Test no	$E\left[C^{(\mathrm{eff})}_{1111}(\omega)\right]$		$\alpha\left(C^{(\mathrm{eff})}_{1111}(\omega)\right)$		$E\left[C^{(\mathrm{eff})}_{1122}(\omega)\right]$		$\alpha\left(C^{(\mathrm{eff})}_{1122}(\omega)\right)$		$E\left[C^{(\mathrm{eff})}_{1212}(\omega)\right]$		$\alpha\left(C^{(\mathrm{eff})}_{1212}(\omega)\right)$	
	SFEM	MCS	SFEM	MCS	SFEM	MCS	SFEM	MCS	SFEM	MCS	SFEM	MCS
1	14.92	14.77	0.084	0.087	5.06	4.94	0.092	0.094	18.21	18.07	0.092	0.096
2	14.95	14.78	0.043	0.045	5.08	4.95	0.044	0.047	18.21	18.07	0.092	0.096
3	14.95	14.78	0.082	0.086	5.08	4.95	0.092	0.094	18.17	18.06	0.045	0.048
4	14.99	14.79	0.040	0.043	5.08	4.95	0.044	0.047	18.17	18.06	0.045	0.048

randomness on the respective probabilistic moments of the homogenized elasticity tensor (simplified probabilistic sensitivity studies).

The homogenization of the same composite has been previously considered in Ref. [27] (deterministic analysis) and [23] (probabilistic analysis). The area of the fiber cross-section is about 50% of the total periodicity cell area. The results in the form of expected values and coefficients of variation of the homogenized tensor components obtained from four computational tests are shown in Table 3 and compared against the corresponding values obtained by using the MCS technique. The direct simulation method has been used in Monte-Carlo experiments with Box–Müller randomization technique where the total number of random trials is taken as 10^3; the numerical illustration of sufficient statistical estimators convergence can be found in Ref. [25].

It is seen that all the SFEM-based expected values are slightly higher than those obtained by MCS while the coefficients of variation show an opposite property. This effect may be caused by the fact that the expected values of homogenization function as well as the effective elasticity tensor include the second order terms of the respective random fields in the SFEM approach, while the Monte-Carlo results do not include any higher order terms and at the same time vary on the total number of assumed random trials only.

However, the main reason for numerical implementation of the SFEM equations modeling the homogenization problem was the decisive decreasing of computation time in comparison to that needed by MCS technique. It should be noticed that the time of Monte-Carlo sampling can be approximated here as a multiplication of the following times: (a) single deterministic cell problem, (b) the total number of the homogenization required (three functions $\chi_{(11)}$, $\chi_{(12)}$ and $\chi_{(22)}$ in plane strain carried out), (c) the total number of random trials performed. There are some time consuming procedures in the MCS programs as random numbers generation [7], post-processing estimation procedure [5] and the subroutines for averaging of needed parameters within the RVE. However, their times are negligible small in comparison with the routines described previously.

On the other hand, the time of stochastic finite element analysis can be approximated by multiplication of the following procedures times: (a) the SFEA of the cell problem (with the same order of the cost considered as the deterministic analysis) and the total number of homogenization functions. Taking into account the remarks posed above, the difference in computational time between MCS and SFEM approaches to the homogenization problem is of about 10^{n-1} order assuming 10^n as the total number of MCS random samples. Observing this and considering negligible differences between the results of both of these methods, the stochastic second order and second moment computational analysis of composite materials should be preferred in most of engineering problems. The only one disadvantage is the complexity of problem equations which have to be implemented in the respective program as well as the bounds dealing with randomness of input variables (the coefficients of variation should be generally smaller than about 0.15).

5. Conclusions

It has been demonstrated that the first and the second order probabilistic moments of the homogenized elasticity tensor components resulting from the SFEM technique proposed in the paper may be quite useful in assessing the overall elastic properties of elastic composites. The results match quite well those obtained by using the Monte-Carlo simulation.

The next developments of the approach presented will be to improve another thermal and elastic constants included usually in the elasticity tensor components – the Poisson's ratios and thermal expansion coefficients. It will complicate some of the equations – there will appear the second order partial derivatives of the elasticity tensor; however, it will enable the full thermoelastic homogenization of the two-phase composite in terms of the second order perturbation second central probabilistic moment approach.

As it can be seen (cf. Refs. [19,20]), the stochastic computations of the effective elasticity tensor homogenized are very important considering the technical applications of the composite materials. The probabilistic approach worked out is important taking into account the fact that most of the elastic characteristics are measured experimentally where the mean values and the standard deviations are estimated as resulting values. The approach proposed above, as well as the Monte-Carlo simulation technique implemented previously [25], allow the engineers to include these parameters into the FEM analysis and to obtain the second order variations of the effective elasticity tensor as the output.

Considering the possibilities of modeling of the stochastic interface defects phenomena appearing frequently in the fiber-reinforced composite materials, it should be mentioned that the starting point to the relevant formulation is Eq. (A.2) with the appropriate stress boundary conditions on the $\partial\Omega_{12}$ (in the form of the presence or the lack of friction, for instance). On the other hand, the stochastic formulation of the structural defects problem introduced in Refs. [25–27] can be successively linked with the stochastic finite element formulation of the homogenization problem approach proposed above and may be used even for viscoplastic localization problem in composites [19].

Finally, it should be mentioned that the approach proposed should turn out to be useful in generating material data for the efficient FEM analysis of various composites – because of its relatively low computational cost it should also find applications in the optimization area (see Ref. [33] for instance) as well as probabilistic numerical modeling of composite materials non-linear behavior [8], strength, fracture and fatigue [13,37].

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Appendix A

To derive the expressions for the effective elasticity tensor components let us rewrite the principle of virtual work for the boundary problem defined on the periodicity cell Ω as follows

$$\int_{\Omega} C_{ijkl}\varepsilon_{kl}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\Omega = \int_{\Omega} f_i v_i\, \mathrm{d}\Omega, \tag{A.1}$$

where $\mathbf{v}$ is any kinematically admissible periodic displacement function and $\mathbf{u}$ is the periodic displacement field to be determined. The left-hand side of the Eq. (A.1) can be divided in the terms for regions Ω_1 and Ω_2 as well as for the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, which in the general case need not be coherent, however both must be sufficiently smooth. Hence, there holds

$$\int_{\Omega_1} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\Omega + \int_{\Omega_2} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\Omega - \int_{\partial\Omega_1} \sigma_{ij}^{(1)} n_j v_i\, \mathrm{d}(\partial\Omega) + \int_{\partial\Omega_2} \sigma_{ij}^{(2)} n_j v_i\, \mathrm{d}(\partial\Omega) = \int_{\Omega} f_i v_i\, \mathrm{d}\Omega, \tag{A.2}$$

where $\sigma_{ij}^{(1)}$, $\sigma_{ij}^{(2)}$ describe the stresses on the contours $\partial\Omega_1$ and $\partial\Omega_2$. This formulation is adequate for the composite structures where the components have discontinuous bonds along the respective interfaces. Considering the simplification $\partial\Omega_1 = \partial\Omega_2 = \partial\Omega_{12}$ and neglecting body forces it is obtained that

$$\int_{\Omega_1} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\Omega + \int_{\Omega_2} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\Omega = \int_{\partial\Omega_{12}} \left(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}\right) n_j v_i\, \mathrm{d}(\partial\Omega). \tag{A.3}$$

As it has been described in Section 2.2, the following stress boundary conditions are applied

$$\sigma_{ij}(\chi_{(pq)}) n_j = \left[C_{ijpq}\right]\big|_{\partial\Omega_{12}} n_j = F_{(pq)i}\big|_{\partial\Omega_{12}}; \quad \mathbf{x} \in \partial\Omega_{12}, \tag{A.4}$$

where n_j is the component of the unit vector normal to the fiber–matrix interface contour and directed to the fiber interior while $\chi_{(pq)}$ denotes the components of the homogenization function. Moreover, function $[C_{ijpq}]|_{\partial\Omega_{12}}$ denotes the difference of the elasticity tensor values for fiber and matrix. Thus, Eq. (A.3) becomes

$$\int_{\Omega_1} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega + \int_{\Omega_2} C_{ijkl}\varepsilon_{kl}(\chi_{(pq)})\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega = -\int_{\partial\Omega_{12}} \left[C_{ijpq}\right]\big|_{\partial\Omega_{12}}\, n_j v_i\,\mathrm{d}(\partial\Omega) \tag{A.5}$$

which makes it possible to compute the $\chi_{(11)}$, $\chi_{(12)}$ and $\chi_{(22)}$ displacement fields. Next, it will be shown that

$$\int_{\partial\Omega_{12}} \left[C_{ijpq}\right]\big|_{\partial\Omega_{12}}\, n_j v_i\,\mathrm{d}(\partial\Omega) = \int_{\Omega} C_{ijpq}\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega. \tag{A.6}$$

There holds

$$\begin{aligned}\int_{\Omega} C_{ijpq}\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega &= \int_{\Omega}\left(C_{ijpq}v_i\right)_{,j}\mathrm{d}\Omega - \int_{\Omega}\left(C_{ijpq}\right)_{,j} v_i\,\mathrm{d}\Omega \\ &= \int_{\partial\Omega} C_{ijpq} n_j v_i\,\mathrm{d}(\partial\Omega) - \int_{\partial\Omega_{12}} \left[C_{ijpq}\right]\big|_{\partial\Omega_{12}} n_j v_i\,\mathrm{d}(\partial\Omega) - \int_{\Omega}\left(C_{ijpq}\right)_{,j} v_i\,\mathrm{d}\Omega.\end{aligned} \tag{A.7}$$

Considering the periodicity of the elasticity tensor, it can be written that

$$\int_{\partial\Omega} C_{ijpq} n_j v_i\,\mathrm{d}(\partial\Omega) = 0, \tag{A.8}$$

$$\int_{\Omega}\left(C_{ijpq}\right)_{,j} v_i\,\mathrm{d}\Omega = 0, \tag{A.9}$$

which gives as a result Eq. (A.6). Including this in the formulation (A.5) it is obtained that

$$\int_{\Omega} C_{ijkl}\varepsilon_{kl}\left(\chi_{(pq)}\right)\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega = -\int_{\Omega} C_{ijpq}\,\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega, \tag{A.10}$$

hence,

$$\int_{\Omega}\left(C_{ijpq} + C_{ijkl}\varepsilon_{kl}\left(\chi_{(pq)}\right)\right)\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\Omega = 0. \tag{A.11}$$

Thus, the effective elasticity tensor components are equal to

$$\int_{\Omega} C^{(\mathrm{eff})}_{ijpq}\,\mathrm{d}\Omega = \int_{\Omega}\left(C_{ijpq} + C_{ijkl}\varepsilon_{kl}\left(\chi_{(pq)}\right)\right)\mathrm{d}\Omega \tag{A.12}$$

which gives Eq. (29) as a result.

Appendix B

The components of the covariance matrix of the effective elasticity tensor components are calculated below. First, the covariance of the first component in Eq. (80) is derived as

$$\begin{aligned}\mathrm{cov}\left(C_{ijkl}, C_{mnpq}\right) &= \int_{-\infty}^{+\infty}\left(C_{ijkl} - E\left[C_{ijkl}\right]\right)\left(C_{mnpq} - E\left[C_{mnpq}\right]\right)p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &= \int_{-\infty}^{+\infty}\left(C^0_{ijkl} + \Delta b_r C^{,\mathrm{r}}_{ijkl} - C^0_{ijkl}\right)\left(C^0_{mnpq} + \Delta b_s C^{,\mathrm{s}}_{mnpq} - C^0_{mnpq}\right)p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &= C^{,\mathrm{r}}_{ijkl}C^{,\mathrm{s}}_{mnpq}\int_{-\infty}^{+\infty}\Delta b_r \Delta b_s p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} = C^{,\mathrm{r}}_{ijkl}C^{,\mathrm{s}}_{mnpq}\cdot\mathrm{cov}(b^{\mathrm{r}}, b^{\mathrm{s}}).\end{aligned} \tag{B.1}$$

Next, the covariances of the second component are calculated. There holds

$$\operatorname{cov}(C_{ijtw}\chi_{(kl)t,w}, C_{mnuv}\chi_{(pq)u,v}) = \int_{-\infty}^{+\infty} (C_{ijtw}\chi_{(kl)t,w} - E[C_{ijtw}\chi_{(kl)t,w}])(C_{mnuv}\chi_{(pq)u,v} - E[C_{mnuv}\chi_{(pq)u,v}])p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} = \quad (B.2)$$

which, by introducing the simplifying notation, becomes

$$\begin{aligned} &= \int_{-\infty}^{+\infty} (\mathbf{C}^0\boldsymbol{\chi}^0 + \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0 + \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u + \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^{,u}\,\Delta b_u + \tfrac{1}{2}\mathbf{C}^0\boldsymbol{\chi}^{,uv}\,\Delta b_u\,\Delta b_v - \{\mathbf{C}^0\boldsymbol{\chi}^0 + (\mathbf{C}^{,r}\boldsymbol{\chi}^{,s} + \tfrac{1}{2}\mathbf{C}^0\boldsymbol{\chi}^{,rs})\cdot\operatorname{cov}(b^r,b^s)\}) \\ &\quad\cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &\quad\times \int_{-\infty}^{+\infty} (\mathbf{D}^0\boldsymbol{\varphi}^0 + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 + \mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^{,c}\,\Delta b_c + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\varphi}^{,cd}\,\Delta b_c\,\Delta b_d - \{\mathbf{D}^0\boldsymbol{\varphi}^0 + (\mathbf{D}^{,a}\boldsymbol{\varphi}^{,c} + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\varphi}^{,ac}) \\ &\qquad\cdot\operatorname{cov}(b^a,b^c)\}) \cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} = \end{aligned} \quad (B.3)$$

and, finally, it is obtained

$$\begin{aligned} &= \int_{-\infty}^{+\infty} (\mathbf{C}^0\boldsymbol{\chi}^0 + \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0 + \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u + \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^{,u}\,\Delta b_u + \tfrac{1}{2}\mathbf{C}^0\boldsymbol{\chi}^{,uv}\,\Delta b_u\,\Delta b_v \\ &\quad - \{\mathbf{C}^0\boldsymbol{\chi}^0 + (\mathbf{C}^{,r}\boldsymbol{\chi}^{,s} + \tfrac{1}{2}\mathbf{C}^0\boldsymbol{\chi}^{,rs})\cdot\operatorname{cov}(b^r,b^s)\}) \cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &\quad\times \int_{-\infty}^{+\infty} (\mathbf{D}^0\boldsymbol{\varphi}^0 + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 + \mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^{,c}\,\Delta b_c + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\varphi}^{,cd}\,\Delta b_c\,\Delta b_d \\ &\quad - \{\mathbf{D}^0\boldsymbol{\varphi}^0 + (\mathbf{D}^{,a}\boldsymbol{\varphi}^{,c} + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\varphi}^{,ac})\cdot\operatorname{cov}(b^a,b^c)\}) \cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &= \int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0\mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0\mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c p_R(\mathbf{b}(\mathbf{x}))\mathrm{d}\mathbf{b} \\ &\quad + \int_{-\infty}^{+\infty} \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u\mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u\mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b}. \end{aligned} \quad (B.4)$$

By integrating over the probability domain, it can be written that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0\mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\boldsymbol{\chi}^0\mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &\quad + \int_{-\infty}^{+\infty} \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u\mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\varphi}^0 p_R(\mathbf{b}(\mathbf{x}))\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \mathbf{C}^0\boldsymbol{\chi}^{,u}\,\Delta b_u\mathbf{D}^0\boldsymbol{\varphi}^{,c}\,\Delta b_c p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &\quad = \left\{\mathbf{C}^{,r}\mathbf{D}^{,s}\boldsymbol{\chi}^0\boldsymbol{\varphi}^0 + \mathbf{C}^{,r}\boldsymbol{\chi}^0\mathbf{D}^0\boldsymbol{\varphi}^{,s} + \mathbf{C}^0\boldsymbol{\chi}^{,r}\mathbf{D}^{,s}\boldsymbol{\varphi}^0 + \mathbf{C}^0\boldsymbol{\chi}^{,r}\mathbf{D}^0\boldsymbol{\varphi}^{,s}\right\}\cdot\operatorname{cov}(b^r,b^s) \end{aligned} \quad (B.5)$$

or, in a more explicit way, as

$$\begin{aligned} \operatorname{cov}(C_{ijtw}\chi_{(kl)t,w}, C_{mnuv}\chi_{(pq)u,v}) = \Big\{ & C_{ijtw}^{,r}C_{mnuv}^{,s}(\chi_{(kl)t,w})^0(\chi_{(pq)u,v})^0 + C_{ijtw}^{,r}C_{mnuv}^{0}(\chi_{(kl)t,w})^0(\chi_{(pq)u,v})^{,s} \\ & + C_{ijtw}^{0}C_{mnuv}^{,r}(\chi_{(kl)t,w})^{,s}(\chi_{(pq)u,v})^0 + C_{ijtw}^{0}C_{mnuv}^{0}(\chi_{(kl)t,w})^{,r}(\chi_{(pq)u,v})^{,s}\Big\}\cdot\operatorname{cov}(b^r,b^s). \end{aligned} \quad (B.6)$$

Finally, the third component of Eq. (80) is calculated as follows:

$$\begin{aligned} \operatorname{cov}(C_{ijkl}; C_{mnuv}\chi_{(pq)u,v}) &= \operatorname{cov}(\mathbf{C};\mathbf{D}\boldsymbol{\chi}) \\ &= \int_{-\infty}^{+\infty} (\mathbf{C}^0 + \mathbf{C}^{,r}\,\Delta b_r - \mathbf{C}^0)\cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b}\int_{-\infty}^{+\infty} \big(\mathbf{D}^0\boldsymbol{\chi}^0 + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\chi}^0 + \mathbf{D}^0\boldsymbol{\chi}^{,c}\,\Delta b_c \\ &\qquad + \mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\chi}^{,c}\,\Delta b_c + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\chi}^{,cd}\,\Delta b_c\,\Delta b_d - \{\mathbf{D}^0\boldsymbol{\chi}^0 + (\mathbf{D}^{,a}\boldsymbol{\chi}^{,c} + \tfrac{1}{2}\mathbf{D}^0\boldsymbol{\chi}^{,ac})\cdot\operatorname{cov}(b^a,b^c)\}\big) \\ &\quad\cdot p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &= \int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\mathbf{D}^{,a}\,\Delta b_a\boldsymbol{\chi}^0 p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} + \int_{-\infty}^{+\infty} \mathbf{C}^{,r}\,\Delta b_r\mathbf{D}^0\boldsymbol{\chi}^{,c}\,\Delta b_c p_R(\mathbf{b}(\mathbf{x}))\,\mathrm{d}\mathbf{b} \\ &= \left\{\mathbf{C}^{,r}\mathbf{D}^{,s}\boldsymbol{\chi}^0 + \mathbf{C}^{,r}\mathbf{D}^0\boldsymbol{\chi}^{,s}\right\}\cdot\operatorname{cov}(b^r,b^s). \end{aligned} \quad (B.7)$$

Introducing the symbolic summation notation for the tensor function considered above, it can be written that

$$\begin{aligned} \operatorname{cov}(C_{ijkl}; C_{mnuv}\chi_{(pq)u,v}) &= \operatorname{cov}(\mathbf{C}; \mathbf{D}\boldsymbol{\chi}) = \left\{ \mathbf{C}^{,r}\mathbf{D}^{,s}\boldsymbol{\chi}^{0} + \mathbf{C}^{,r}\mathbf{D}^{0}\boldsymbol{\chi}^{,s} \right\} \cdot \operatorname{cov}(b^{r}, b^{s}) \\ &= \left\{ C^{,r}_{ijkl} C^{,s}_{mnuv} (\chi_{(pq)u,v})^{0} + C^{,r}_{ijkl} C^{0}_{mnuv} (\chi_{(pq)u,v})^{,s} \right\} \cdot \operatorname{cov}(b^{r}, b^{s}). \end{aligned} \tag{B.8}$$

By the analogous way it can be derived that

$$\begin{aligned} \operatorname{cov}(C_{ijtw}\chi_{(kl)t,w}; C_{mnpq}) &= \operatorname{cov}(\mathbf{C}\boldsymbol{\chi}; \mathbf{D}) = \left\{ \mathbf{C}^{,r}\boldsymbol{\chi}^{0}\mathbf{D}^{,s} + \mathbf{C}^{0}\boldsymbol{\chi}^{,r}\mathbf{D}^{,s} \right\} \operatorname{cov}(b^{r}, b^{s}) \\ &= \left\{ C^{,r}_{ijtw} (\chi_{(kl)t,w})^{0} C^{,s}_{mnpq} + C^{,r}_{ijtw} (\chi_{(kl)t,w})^{,s} C^{0}_{mnpq} \right\} \cdot \operatorname{cov}(b^{r}, b^{s}). \end{aligned} \tag{B.9}$$

Next, covariances of the effective elasticity tensor components are to be found. Starting from the classical definition it is shown that

$$\begin{aligned} \operatorname{cov}\left(C^{(\mathrm{eff})}_{ijkl}, C^{(\mathrm{eff})}_{mnpq}\right) &= \operatorname{cov}(C_{ijkl} + C_{ijtw}\chi_{(kl)t,w}, C_{mnpq} + C_{mnuv}\chi_{(pq)u,v}) \\ &= \int_{-\infty}^{+\infty} (C_{ijkl} + C_{ijtw}\chi_{(kl)t,w} - E[C_{ijkl}] - E[C_{ijtw}\chi_{(kl)t,w}]) \\ &\quad \times (C_{mnpq} + C_{mnuv}\chi_{(pq)u,v} - E[C_{mnpq}] - E[C_{mnuv}\chi_{(pq)u,v}]) \cdot p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b}. \end{aligned} \tag{B.10}$$

Transforming the respective integrands and using Fubini's theorem applied to the integrals of random functions, we further obtain that

$$\begin{aligned} &\int_{-\infty}^{+\infty} (C_{ijkl} - E[C_{ijkl}])(C_{mnpq} - E[C_{mnpq}]) \cdot p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} \\ &\times \int_{-\infty}^{+\infty} (C_{ijkl} - E[C_{ijkl}])(C_{mnuv}\chi_{(pq)u,v} - E[C_{mnuv}\chi_{(pq)u,v}]) \cdot p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} \\ &\times \int_{-\infty}^{+\infty} (C_{ijtw}\chi_{(kl)t,w} - E[C_{ijtw}\chi_{(kl)t,w}])(C_{mnpq} - E[C_{mnpq}]) \cdot p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b} \\ &\times \int_{-\infty}^{+\infty} (C_{ijtw}\chi_{(kl)t,w} - E[C_{ijtw}\chi_{(kl)t,w}])(C_{mnuv}\chi_{(pq)u,v} - E[C_{mnuv}\chi_{(pq)u,v}]) \cdot p_R(\mathbf{b}(\mathbf{x}))\, \mathrm{d}\mathbf{b}, \end{aligned} \tag{B.11}$$

which, using the classical definition of the covariance, is equal to

$$\operatorname{cov}(C_{ijkl}, C_{mnpq}) + \operatorname{cov}(C_{ijkl}, C_{mnuv}\chi_{(pq)u,v}) + \operatorname{cov}(C_{ijtw}\chi_{(kl)t,w}, C_{mnpq}) + \operatorname{cov}(C_{ijtw}\chi_{(kl)t,w}, C_{mnuv}\chi_{(pq)u,v}). \tag{B.12}$$

Introducing the statements (B.1), (B.6) and (B.8) into the last one, it can be finally written that

$$\begin{aligned} \operatorname{cov}\left(C^{(\mathrm{eff})}_{ijkl}, C^{(\mathrm{eff})}_{mnpq}\right) = \Big\{ & C^{,r}_{ijkl} C^{,s}_{mnpq} + C^{,r}_{ijtw} (\chi_{(kl)t,w})^{0} C^{,s}_{mnpq} + C^{,r}_{ijtw} (\chi_{(kl)t,w})^{,s} C^{0}_{mnpq} + C^{,r}_{ijkl} C^{,s}_{mnuv} (\chi_{(pq)u,v})^{0} \\ & + C^{,r}_{ijkl} C^{0}_{mnuv} (\chi_{(pq)u,v})^{,s} + C^{,r}_{ijtw} C^{,s}_{mnuv} (\chi_{(kl)t,w})^{0} (\chi_{(pq)u,v})^{0} + C^{,r}_{ijtw} C^{0}_{mnuv} (\chi_{(kl)t,w})^{0} (\chi_{(pq)u,v})^{,s} \\ & + C^{0}_{ijtw} C^{,r}_{mnuv} (\chi_{(kl)t,w})^{,s} (\chi_{(pq)u,v})^{0} + C^{0}_{ijtw} C^{0}_{mnuv} (\chi_{(kl)t,w})^{,r} (\chi_{(pq)u,v})^{,s} \Big\} \operatorname{cov}(b^{r}, b^{s}) \end{aligned} \tag{B.13}$$

which completes the considerations.

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