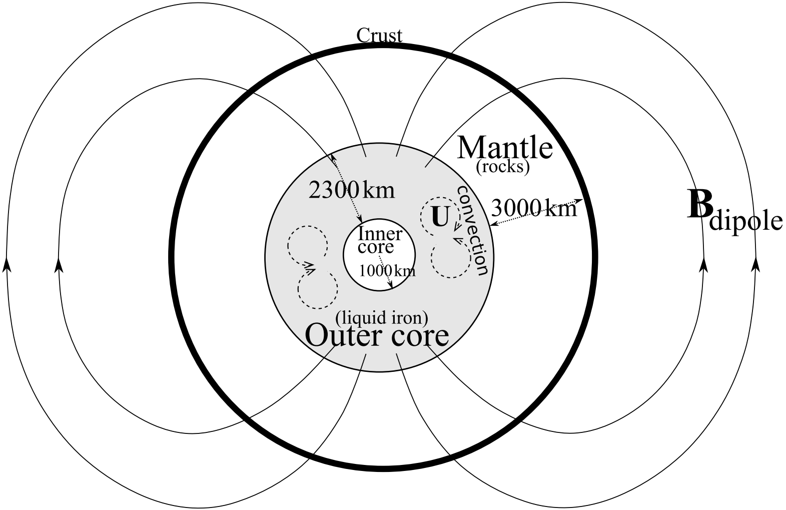
Geomagnetic reversals and excursions as an outcome of turbulence in the Earth's liquid core

K. A. Mizerski

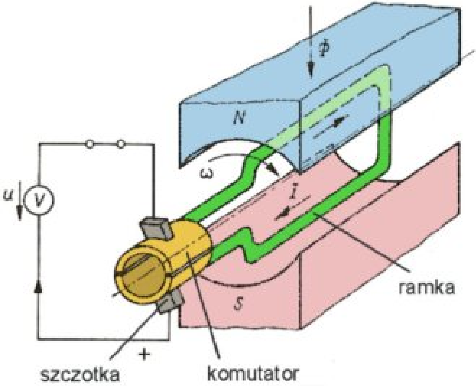
Department of Magnetism, Institute of Geophysics, Polish Academy of Sciences, Warsaw

IPPT PAN, May 2026

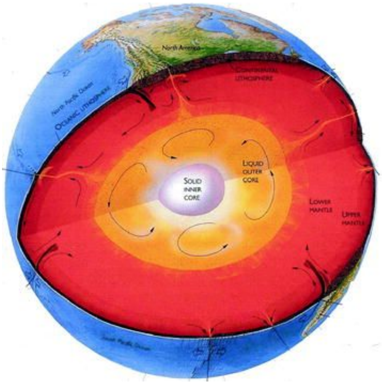
The Earth

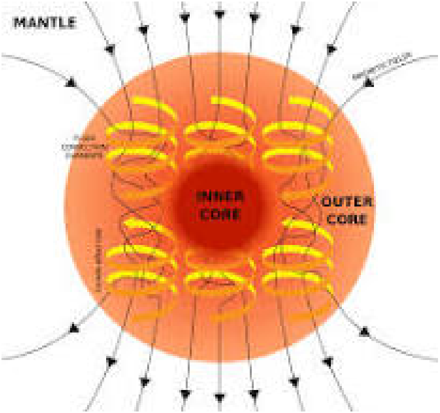

Crust
Mantle
(rocks)
2300 km
3000 km
U
convection
Inner
core
1000 km
(liquid iron)
Outer core
$\mathbf{B}_{\text{dipole}}$

Dynamo mechanism

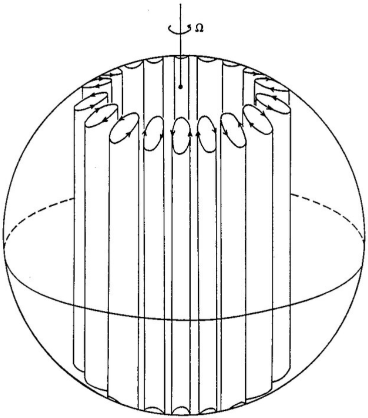


The core


North America
Continental lithosphere
North Pacific Ocean
Oceanic lithosphere
Liquid outer core
Solid inner core
Lower mantle
Upper mantle


MANTLE
INNER CORE
OUTER CORE

Convection cells

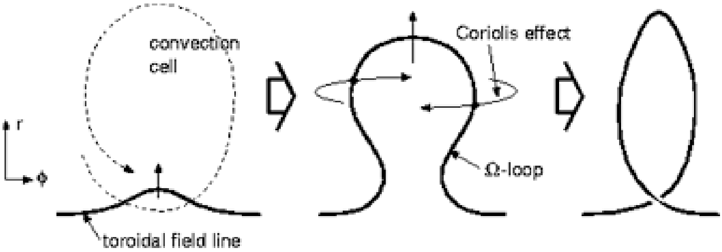


Taylor-Proudman theorem

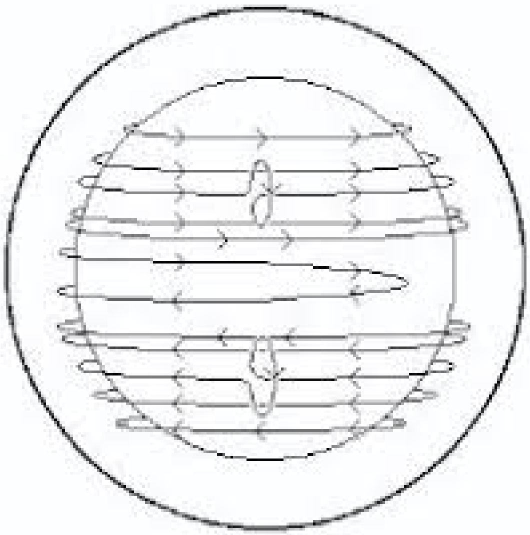
$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla p,$$

$$(\boldsymbol{\Omega} \cdot \nabla)\,\boldsymbol{u} = 0,$$

α-effect



Earth-dynamo



MHD equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} = \boldsymbol{f} - \nabla\Pi + \frac{1}{\mu_0\rho}(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} + \nu\nabla^2\boldsymbol{U},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{U} + \eta\nabla^2\boldsymbol{B},$$

$$\nabla\cdot\boldsymbol{U} = 0 \qquad \nabla\cdot\boldsymbol{B} = 0,$$

where $\Pi = p/\rho + B^2/2\mu_0\rho - (\boldsymbol{\Omega}\times\boldsymbol{x})^2/2$.

Means

We separate the variables into means and fluctuating parts,

$$\boldsymbol{U} = \langle \boldsymbol{U} \rangle + \boldsymbol{u}', \quad \boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{b}', \quad p = \langle p \rangle + p',$$

$$\frac{\partial \langle \boldsymbol{B} \rangle}{\partial t} = \nabla \times (\langle \boldsymbol{U} \rangle \times \langle \boldsymbol{B} \rangle) + \nabla \times \left\langle \boldsymbol{u}' \times \boldsymbol{b}' \right\rangle + \eta \nabla^2 \langle \boldsymbol{B} \rangle ,$$

and we put forward the assumption of scale separation between the slowly evolving means and the typical spatial scale of turbulent eddies

The EMF: $\qquad \mathscr{E} = \left\langle \boldsymbol{u}' \times \boldsymbol{b}' \right\rangle$

Equations for fluctuations

$$\frac{\partial \boldsymbol{u}'}{\partial t} - \nu\nabla^2 \boldsymbol{u}' + (\langle \boldsymbol{U}\rangle \cdot \nabla)\,\boldsymbol{u}' - (\langle \boldsymbol{B}\rangle \cdot \nabla)\,\boldsymbol{b}' + \nabla\Pi' = \boldsymbol{f} - \nabla\cdot\left(\boldsymbol{u}'\boldsymbol{u}' - \boldsymbol{b}'\boldsymbol{b}'\right) + \nabla\cdot\left(\left\langle \boldsymbol{u}'\boldsymbol{u}'\right\rangle - \left\langle \boldsymbol{b}'\boldsymbol{b}'\right\rangle\right) - \left(\boldsymbol{u}'\cdot\nabla\right)\langle \boldsymbol{U}\rangle + \left(\boldsymbol{b}'\cdot\nabla\right)\langle \boldsymbol{B}\rangle,$$

$$\frac{\partial \boldsymbol{b}'}{\partial t} - \eta\nabla^2 \boldsymbol{b}' + (\langle \boldsymbol{U}\rangle \cdot \nabla)\,\boldsymbol{b}' - (\langle \boldsymbol{B}\rangle \cdot \nabla)\,\boldsymbol{u}' = \nabla\times\left(\boldsymbol{u}'\times\boldsymbol{b}' - \left\langle \boldsymbol{u}'\times\boldsymbol{b}'\right\rangle\right) + \left(\boldsymbol{b}'\cdot\nabla\right)\langle \boldsymbol{U}\rangle - \left(\boldsymbol{u}'\cdot\nabla\right)\langle \boldsymbol{B}\rangle,$$

$$\nabla\cdot\boldsymbol{b}' = 0, \quad \nabla\cdot\boldsymbol{u}' = 0.$$

Relax stationarity

$$\boldsymbol{u}'(\mathrm{x},t) = \left(\hat{\boldsymbol{u}}^{(1)}\mathrm{e}^{\mathrm{i}\omega_1 t} + \hat{\boldsymbol{u}}^{(2)}\mathrm{e}^{\mathrm{i}\omega_2 t}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\mathrm{x}},$$

$$\boldsymbol{b}'(\mathrm{x},t) = \left(\hat{\boldsymbol{b}}^{(1)}\mathrm{e}^{\mathrm{i}\omega_1 t} + \hat{\boldsymbol{b}}^{(2)}\mathrm{e}^{\mathrm{i}\omega_2 t}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\mathrm{x}},$$

$$\begin{aligned}
\mathscr{E}(\boldsymbol{k}) &= \left\langle \boldsymbol{u}' \times \boldsymbol{b}' \right\rangle \\
&= \frac{1}{2}\Re\mathfrak{e}\left[\left(\hat{\boldsymbol{u}}^{(1)}\mathrm{e}^{\mathrm{i}\omega_1 t} + \hat{\boldsymbol{u}}^{(2)}\mathrm{e}^{\mathrm{i}\omega_2 t}\right) \times \left(\hat{\boldsymbol{b}}^{(1)*}\mathrm{e}^{-\mathrm{i}\omega_1 t} + \hat{\boldsymbol{b}}^{(2)*}\mathrm{e}^{-\mathrm{i}\omega_2 t}\right)\right] \\
&= \frac{1}{2}\Re\mathfrak{e}\left[\hat{\boldsymbol{u}}^{(1)} \times \hat{\boldsymbol{b}}^{(2)*}\mathrm{e}^{\mathrm{i}\Delta\omega t} + \hat{\boldsymbol{u}}^{(2)} \times \hat{\boldsymbol{b}}^{(1)*}\mathrm{e}^{-\mathrm{i}\Delta\omega t}\right] \\
&\quad + \frac{1}{2}\Re\mathfrak{e}\left[\hat{\boldsymbol{u}}^{(1)} \times \hat{\boldsymbol{b}}^{(1)*} + \hat{\boldsymbol{u}}^{(2)} \times \hat{\boldsymbol{b}}^{(2)*}\right],
\end{aligned}$$

The non-equilibrium effect

$$\mathscr{E}(\boldsymbol{k}) = \frac{1}{2}\mathfrak{Re}\left[\hat{\boldsymbol{u}}^{(1)} \times \hat{\boldsymbol{b}}^{(2)*} e^{i\Delta\omega t} + \hat{\boldsymbol{u}}^{(2)} \times \hat{\boldsymbol{b}}^{(1)*} e^{-i\Delta\omega t}\right]$$

$\Delta\omega$ - small, so that $2\pi/\Delta\omega \gg \tau$ where τ is the mean field evolution time scale - EFFECT OF BEATING WAVES.

More generally

Introducing Fourier transforms

$$\boldsymbol{u}'(\mathrm{x},t) = \int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}\omega \hat{\boldsymbol{u}}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\mathrm{x}-\omega t)},$$

We apply the first-order smoothing approximation and neglect the nonlinearities. In such a case one obtains

$$\hat{\boldsymbol{b}} = \mathrm{i}\frac{\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle}{\gamma_\eta}\hat{\boldsymbol{u}} - \frac{1}{\gamma_\eta}\boldsymbol{\Gamma}\cdot\hat{\boldsymbol{u}} + \frac{1}{\gamma_\eta}\boldsymbol{G}\cdot\hat{\boldsymbol{b}},$$

$$\hat{\boldsymbol{u}} = \frac{1}{\gamma_u}\hat{\boldsymbol{f}} - \frac{1}{\gamma_u}\mathsf{P}\cdot\boldsymbol{G}\cdot\hat{\boldsymbol{u}} - \frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\gamma_u\gamma_\eta^2}\boldsymbol{G}\cdot\hat{\boldsymbol{u}} - \frac{\mathrm{i}\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle}{\gamma_u\gamma_\eta}\boldsymbol{\Gamma}\cdot\hat{\boldsymbol{u}} + \mathrm{i}\frac{\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle}{\gamma_u\gamma_\eta}\mathsf{P}\cdot\boldsymbol{\Gamma}\cdot\hat{\boldsymbol{u}}$$

where

$$G_{ij} = \frac{\partial\langle U\rangle_i}{\partial x_j}, \qquad \Gamma_{ij} = \frac{\partial\langle B\rangle_i}{\partial x_j}$$

$$\gamma_u = -\mathrm{i}(\omega - \boldsymbol{k}\cdot\langle\boldsymbol{U}\rangle) + \nu k^2 + \frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\gamma_\eta}, \qquad \gamma_\eta = -\mathrm{i}(\omega - \boldsymbol{k}\cdot\langle\boldsymbol{U}\rangle) + \eta k^2$$

The electromotive force

$$\varepsilon_{ijk}\left\langle \hat{u}_j(\omega,\boldsymbol{k})\,\hat{b}_k(\omega',\boldsymbol{k}')\right\rangle = \mathrm{i}\frac{k'_n\langle B\rangle_n}{\gamma'_\eta}\varepsilon_{ijk}\langle\hat{u}_j\hat{u}'_k\rangle - \varepsilon_{ijk}\frac{\partial\langle B\rangle_k}{\partial x_p}\frac{1}{\gamma'_\eta}\langle\hat{u}_j\hat{u}'_p\rangle + \varepsilon_{ijk}\frac{\partial\langle U\rangle_k}{\partial x_p}\frac{1}{\gamma'_\eta}\left\langle\hat{u}_j\hat{b}'_p\right\rangle$$

Substituting for $\hat{\boldsymbol{b}}$ and treating the gradients of means in a perturbational manner

$$\varepsilon_{ijk}\left\langle \hat{u}_j(\omega,\boldsymbol{k})\,\hat{b}_k(\omega',\boldsymbol{k}')\right\rangle = \mathrm{i}\frac{k'_n\langle B\rangle_n}{\gamma'_\eta}\varepsilon_{ijk}\langle\hat{u}_j\hat{u}'_k\rangle - \varepsilon_{ijk}\frac{\partial\langle B\rangle_k}{\partial x_p}\frac{1}{\gamma'_\eta}\langle\hat{u}_j\hat{u}'_p\rangle + \mathrm{i}\varepsilon_{ijk}\langle B\rangle_n\frac{\partial\langle U\rangle_k}{\partial x_p}\frac{k'_n}{\gamma'^2_\eta}\langle\hat{u}_j\hat{u}'_p\rangle$$

$$\left\langle \hat{u}_i(\boldsymbol{k},\omega)\hat{u}_j(\boldsymbol{k}',\omega')\right\rangle = \left[E\left(\omega,\omega',k\right)P_{ij}(\boldsymbol{k}) + \mathrm{i}H\left(\omega,\omega',k\right)\varepsilon_{ijk}k_k\right]\delta(\boldsymbol{k}+\boldsymbol{k}')$$

Then

$$\varepsilon_{ljk}\int^{\Lambda_\nu}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\mathrm{x}-\omega t)}\int^{\Lambda_\nu}\mathrm{d}^4q'\,\mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}'\cdot\mathrm{x}-\omega' t\right)}\left\langle \hat{u}_j\left(\omega,\boldsymbol{k}\right)\hat{b}_k\left(\omega',\boldsymbol{k}'\right)\right\rangle =$$

$$\frac{8\pi}{3}\langle B\rangle_l\int\mathrm{d}k\int\mathrm{d}\omega\int\mathrm{d}\omega'\mathrm{e}^{-\mathrm{i}(\omega+\omega')t}\frac{k^4}{\gamma'_\eta}H\left(\omega,\omega',k\right)$$

$$-\frac{8\pi}{3}\left[\nabla\times\langle\boldsymbol{B}\rangle\right]_l\int\mathrm{d}k\int\mathrm{d}\omega\int\mathrm{d}\omega'\mathrm{e}^{-\mathrm{i}(\omega+\omega')t}\frac{k^2}{\gamma'_\eta}E\left(\omega,\omega',k\right)$$

$$-\frac{4\pi}{3}\left[\left(\nabla\langle\boldsymbol{U}\rangle\right)^T\cdot\langle\boldsymbol{B}\rangle\right]_l\int\mathrm{d}k\int\mathrm{d}\omega\int\mathrm{d}\omega'\mathrm{e}^{-\mathrm{i}(\omega+\omega')t}\frac{k^4}{\gamma'^2_\eta}H\left(\omega,\omega',k\right)$$

Homogeneous, isotropic, nonstationary turbulence

$$E\left(\omega,\omega',k\right)=e\left(\omega^2,k\right)\Delta\left(\omega,\omega';\tilde{\omega},\Gamma\right),$$

$$H\left(\omega,\omega',k\right)=h\left(\omega^2,k\right)\Delta\left(\omega,\omega';\tilde{\omega},\Gamma\right),$$

$$\Delta\left(\omega,\omega';\tilde{\omega},\Gamma\right)=\delta(\omega+\omega')+\Xi\frac{\delta(\omega+\omega'+\tilde{\omega})-\delta(\omega+\omega'-\tilde{\omega})}{2\mathrm{i}},$$

$$\left\langle u_i(\mathrm{x},t)u_i(\mathrm{x}',t)\right\rangle\sim 1+\Xi\sin(\tilde{\omega}t).$$

The final result

$$\mathscr{E} = \bar{\alpha}\langle \boldsymbol{B}\rangle - \bar{\eta}\nabla\times\langle \boldsymbol{B}\rangle,$$

where

$$\bar{\alpha} = \eta\, I_1^{(\alpha)} + \frac{\eta\,\Xi I_2^{(\alpha)}}{\cos\phi_\alpha}\sin\left(\tilde{\omega}t + \phi_\alpha\right)$$

$$\bar{\eta} = \eta + \eta\, I_1^{(\eta)} + \frac{\eta\,\Xi I_2^{(\eta)}}{\cos\phi_\eta}\sin\left(\tilde{\omega}t + \phi_\eta\right)$$

$$\tan\phi_\alpha = \frac{\tilde{\omega} I_3^{(\alpha)}}{\eta\, I_2^{(\alpha)}}, \qquad \tan\phi_\eta = \frac{\tilde{\omega} I_3^{(\eta)}}{\eta\, I_2^{(\eta)}},$$

The corrections from nonstationarity can be dominant.

Force-free solution

$$\nabla \times \langle \boldsymbol{B} \rangle = \kappa \langle \boldsymbol{B} \rangle,$$

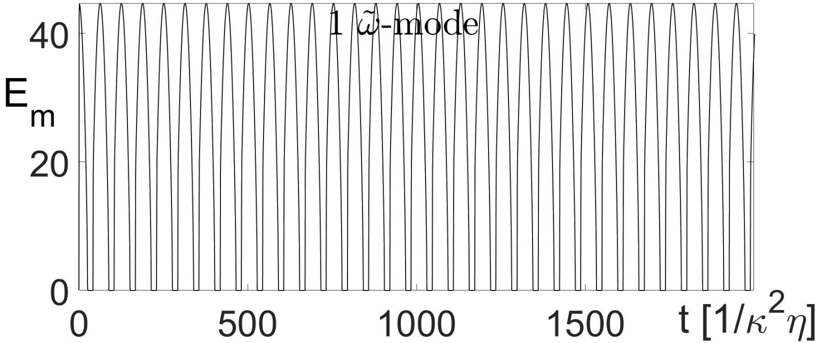
where κ is a scalar constant; this ensures vanishing of the Lorentz force $\nabla \times \langle \boldsymbol{B} \rangle \times \langle \boldsymbol{B} \rangle = 0$. On defining the magnetic energy and phase shifts

$$E_m = \frac{1}{2} \langle B \rangle^2, \qquad \phi_\alpha = 0, \qquad \phi_\eta = \pi,$$

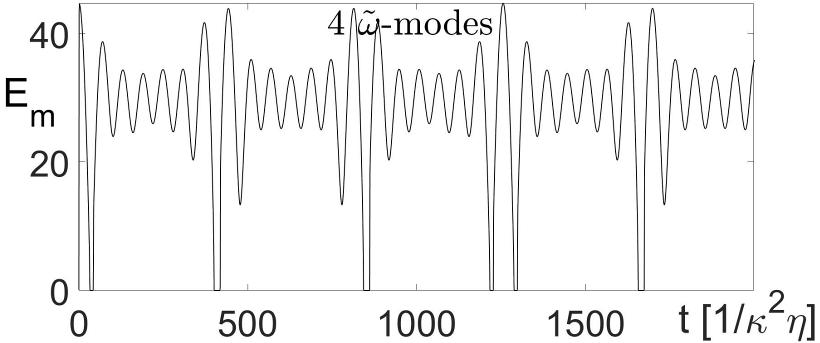
we get a well suited case study - the maximal enhancement of the field takes place when the effect of the resistive decay is the weakest (STABLE FIELD) and vice versa - the strongest resistive decay of the mean field is associated with the weakest amplification by the α-effect (EXCURSIONS). In such a case we get

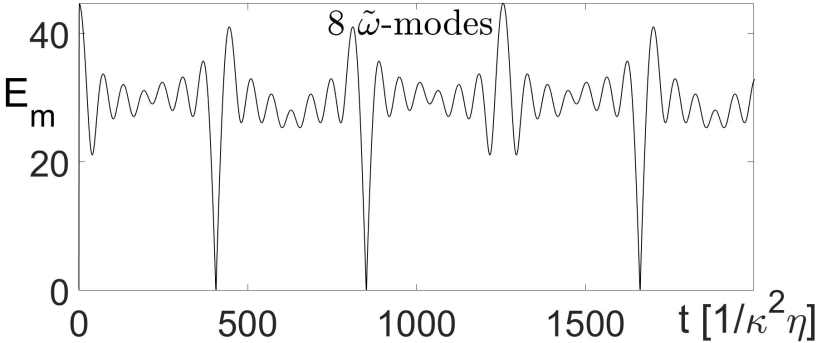
$$\begin{aligned} \frac{\partial E_m}{\partial t} =& 2\bar{\alpha}_0 \kappa (1 + \cos \tilde{\omega} t) \frac{E_m}{1 + \bar{A} E_m^2}, \\ & - 2\kappa^2 \left[\eta \left(1 + \bar{A} E_m^2 \right) + \bar{\eta}_0 - \bar{\eta}_0 \cos (\tilde{\omega} t) \right] \frac{E_m}{1 + \bar{A} E_m^2}, \end{aligned}$$

Force-free mode

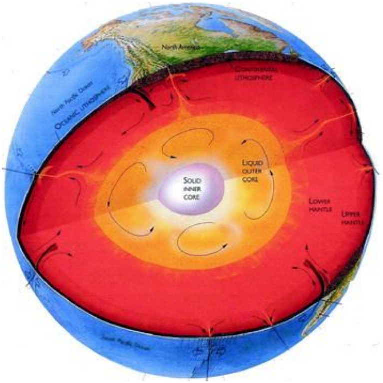


Force-free modes




8 $\tilde{\omega}$-modes
40
20
0
E_m
0
500
1000
1500
t [$1/\kappa^2\eta$]

The core


North America
North Pacific Ocean
Oceanic lithosphere
Continental lithosphere
Solid inner core
Liquid outer core
Lower mantle
Upper mantle

The dispersion relation for the *M*agnetic-*A*chimedean-*C*oriolis waves

$$\hat{\omega}^2_{1,2,3,4} = \frac{1}{2}\left\{\omega_C^2 + 2\omega_M^2 - \omega_A^2 \pm \sqrt{\left(\omega_C^2 - \omega_A^2\right)^2 + 4\omega_C^2\omega_M^2}\right\},$$

$$\omega_M = \frac{\overline{\boldsymbol{B}}\cdot\boldsymbol{k}}{\sqrt{\mu_0\rho}}, \qquad \omega_C = 2\Omega\cdot\frac{\boldsymbol{k}}{k}, \qquad \omega_A = \sqrt{g\alpha_T\theta}\frac{k_h}{k},$$

The eigenmodes

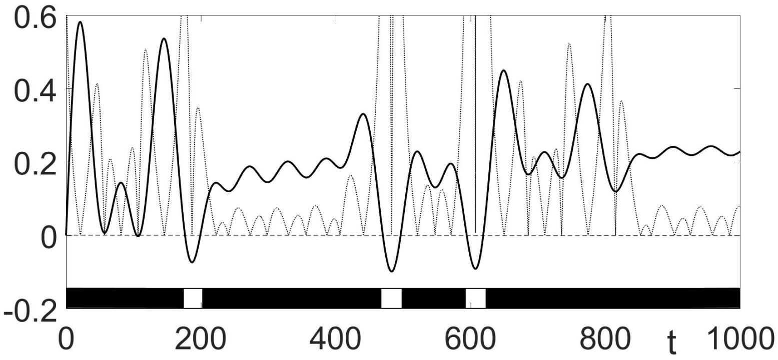
$$\boldsymbol{u}'_i = \mathscr{U}_i\left[\hat{\boldsymbol{e}}_z - \frac{k_z\boldsymbol{k}_h}{k_h^2} - \mathrm{i}\frac{\omega_K\hat{\omega}_i}{\hat{\omega}_i^2 - \omega_A^2}\frac{k\boldsymbol{k}_h}{k_h^2}\times\hat{\boldsymbol{e}}_z\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \hat{\omega}_i t)},$$

$$\boldsymbol{b}'_i = -\sqrt{\mu_0\rho}\frac{\omega_A}{\hat{\omega}_i}\boldsymbol{u}'_i - \frac{\mathrm{i}}{\hat{\omega}_i}\boldsymbol{\Gamma}\cdot\boldsymbol{u}'_i,$$

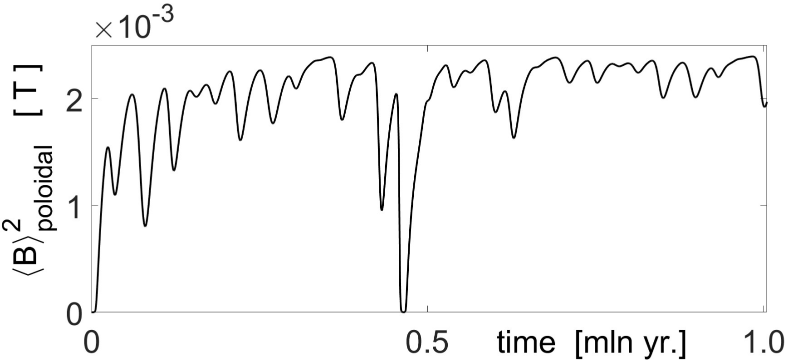
And the EMF

$$\mathscr{E}(\boldsymbol{k}) = \sum_{i=1}^{4}\sum_{j=1}^{4}\left\langle\boldsymbol{u}'_i\times\boldsymbol{b}'_j\right\rangle_{\boldsymbol{x}} = \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\Re\mathfrak{e}\left(\boldsymbol{u}'_i\times\boldsymbol{b}'^{*}_j\right)$$

8 beating modes



Geomagnetic reversals from beating MAR waves within bulk



Dynamo number and time-scales

$$\mathscr{D} = \frac{\left|\bar{\alpha}\partial_r b^2\right|}{\bar{\eta} j^2}, \tag{1}$$

Short, sign-change time scale

$$\tau_{\text{fast}} = \frac{1}{j\bar{\eta}\partial_r b^2 \left[2 + \mathscr{D}\left(\frac{j}{\partial_r b^2}\right)^2\right]} \ll \tau_E, \tag{2}$$

is much sorter than the typical time scale of energy evolution.

- Small value of the dynamo number $\mathscr{D}(t)$ at a given time t, followed by its rapid increase seems to be a good indicator of an approaching strong decay of the dipole field, i.e. a reversal or a strong excursion.
- A short time-scale $\tau_{\text{fast}}(t)$, decreasing as the dynamo number increases, suggests forthcoming occurrence of the sign change, i.e. that the excursion should become a reversal.

Summary

- Relaxed stationarity of turbulence - interactions of distinct waves lead to non-vanishing EMF, even in the absence of diffusion.
- Both $\bar{\alpha}$ and $\bar{\eta}$ depend slowly on time, with different phase shifts, leading to an Earth-like long-time behaviour of the magnetic energy, including 'random' reversals and excursions.
- Small value of the dynamo number $\mathscr{D}(t)$ defined in at a given time t, followed by its rapid increase seems to be a good indicator of an approaching strong decay of the dipole field, i.e. a reversal or a strong excursion.
- A short time-scale $\tau_{\mathrm{fast}}(t)$, decreasing as the dynamo number increases, suggests forthcoming occurrence of the sign change, i.e. that the excursion should become a reversal.

Renormalization is based on Taylor expansions in a parameter, which is initially assumed small (treated perturbationally) and identification of a recursion scheme which allows to contract the series.

$$f = \varepsilon f_1 + \varepsilon^2 f_2 + \ldots .$$

If we are able to formulate the asymptotic problem in the following way

$$\frac{\mathrm{d} f_k}{\mathrm{d} k} = \mathscr{F}(k, f_k),$$

and we can solve this equation, then we contract the series.

Simplest example

$$0 < x < 1, \qquad f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = 1 + xf(x)$$

Inclusion of nonlinearities - renormalization, $\tilde{\omega} t \ll 1, \|\langle \mathrm{B} \rangle\|$-weak

Nonlinear evolution of fluctuations, i.e. we include the terms $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ in the fluctuational equations in a perturbational manner (cf. Yakhot and Orszag 1986, *J. Sci. Comput.* **1**, pp. 3-51, Mizerski 2020, Astroph. J. Suppl. Ser. **251**:21)

$$\left\langle \hat{f}_i(\boldsymbol{k}, \omega) \hat{f}_j(\boldsymbol{k}', \omega') \right\rangle = \left[\frac{D_0}{k^{\sigma_0}} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) + \mathrm{i} \frac{D_1}{k^{\sigma_1}} \varepsilon_{ijk} k_k \right] \delta(\boldsymbol{k} + \boldsymbol{k}') \Delta(\omega, \omega')$$

Leading order,

$$\bar{\alpha} = \bar{\alpha}_A - \bar{\alpha}_S \langle B \rangle^2,$$

$$\bar{\alpha}_A \approx -\frac{\langle h_k \rangle}{6(\nu + \eta) K_e^2} \left[1 - \frac{9\nu + \eta}{12(\nu + \eta)\nu K_e^2} \frac{\partial \langle e_k \rangle}{\partial t} - \frac{\langle e_k \rangle}{6(\nu + \eta)^2 K_e^2} \right],$$

$$\bar{\alpha}_S \approx -\frac{\langle h_k \rangle}{15\nu\eta(\nu + \eta) K_e^4} \left[1 - \frac{3\nu^2 + 13\nu\eta + 4\eta^2}{8\eta(\nu + \eta)\nu K_e^2} \frac{\partial \langle e_k \rangle}{\partial t} \right].$$

The sign of the non-equilibrium corrections depends on $\partial_t \langle e_k \rangle$.

TSDIA - non-equilibrium cross-helicity effect

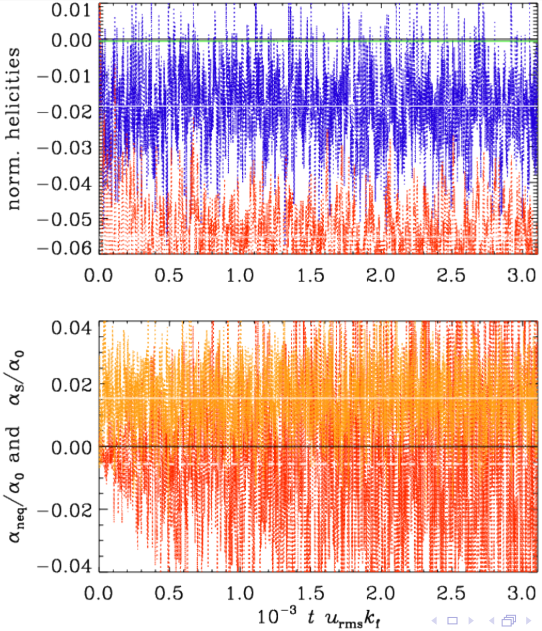
Two-Scale Direct-Interaction Approximation, Mizerski, Yokoi, Brandenburg 2023, submitted to *J. Plasma. Phys.* It is based on introduction of point-force Green's response functions of the MHD turbulence.

Leading order,

$$\bar{\alpha} \approx -\frac{1}{3}\tau_t\left(\langle\boldsymbol{u}'\cdot\boldsymbol{w}'\rangle - \langle\boldsymbol{b}'\cdot\boldsymbol{j}'\rangle\right), \\ -\frac{1}{3}\frac{\langle\boldsymbol{u}'\cdot\boldsymbol{b}'\rangle}{\sqrt{\langle u'^2\rangle\langle b'^2\rangle}}\int_{-\infty}^{\tau}\mathrm{d}\tau'\left[\langle\boldsymbol{u}'(\mathrm{x},\tau)\cdot\boldsymbol{j}'(\mathrm{x},\tau')\rangle - \langle\boldsymbol{u}'(\mathrm{x},\tau')\cdot\boldsymbol{j}'(\mathrm{x},\tau)\rangle\right],$$

where the second effect is clearly a non-equilibrium effect, associated with the cross-response functions between the magnetic and velocity perturbations. It can be argued that $\langle\boldsymbol{u}'(\mathrm{x},\tau)\cdot\boldsymbol{j}'(\mathrm{x},\tau')\rangle$ is proportional to the kinetic helicity, hence this effect relies on 3 factors - non-stationarity and co-existence of the cross and kinetic helicities.

numerics - non-equilibrium cross-helicity effect



Helicity

$$\langle \mathbf{A} \cdot \mathbf{B} \rangle = \langle \mathbf{A} \rangle \cdot \langle \mathbf{B} \rangle + \left\langle \mathbf{a}' \cdot \mathbf{b}' \right\rangle . \tag{3}$$

The total magnetic helicity $\mathscr{H}_m = \int_V \mathbf{A} \cdot \mathbf{B} \mathrm{d}V$ in the entire fluid volume V is always conserved in the absence of fluid's resistivity,

$$\frac{D}{Dt} \int_V \mathbf{A} \cdot \mathbf{B} \mathrm{d}V = 0 \qquad \Longrightarrow \qquad \int_V \langle \mathbf{A} \rangle \cdot \langle \mathbf{B} \rangle \mathrm{d}V = \mathscr{H}_m - \int_V \langle \mathbf{a}' \cdot \mathbf{b}' \rangle \mathrm{d}V, \tag{4}$$

The general evolution of the fluctuational magnetic helicity is governed by the following equation

$$\begin{aligned} \frac{D}{Dt} \left\langle \mathbf{a}' \cdot \mathbf{b}' \right\rangle = & -2\mathscr{E} \cdot \langle \mathbf{B} \rangle \\ & + \nabla \cdot \left[\left\langle \mathbf{u}' \cdot \mathbf{a}' \right\rangle \langle \mathbf{B} \rangle + \left\langle \left[\left(\langle \mathbf{u} \rangle + \mathbf{u}' \right) \cdot \mathbf{a}' - \phi' \right] \mathbf{b}' \right\rangle - \left\langle \left(\mathbf{a}' \cdot \langle \mathbf{B} \rangle \right) \mathbf{u}' \right\rangle \right] \\ & + \eta \left\langle \nabla^2 \mathbf{a}' \cdot \mathbf{b}' \right\rangle + \eta \left\langle \mathbf{a}' \cdot \nabla^2 \mathbf{b}' \right\rangle . \end{aligned} \tag{5}$$