Barenblatt profiles for a nonlocal porous medium equation

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in collaboration with

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$$\nabla^{\beta} U(x) = C_{d,\beta} \int (U(x+z) - U(x)) \frac{z}{|z|^{d+\beta+1}} \, \mathrm{d}z$$

$C_{d,\beta} > 0$, U – smooth

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general porous medium equation

$$\partial_t u = \nabla \cdot (u \nabla p)$$

$m = 2$, $p = I_{2s} u$, I_{2s} – the Riesz potential, $2s = 2 - \alpha$
L. Caffarelli, J. L. Vázquez (2010)

$p = I_{2-\alpha}(f(u))$

Particular cases and related equations

the **porous medium equation**: $u \geq 0$, $\alpha = 2$, $m > 1$

$$\partial_t u = \frac{1}{m-1} \nabla \cdot (u \nabla u^{m-1}) = \Delta(u^m), \quad t > 0, \ x \in \mathbb{R}^d$$

$m = 2$ – the **Boussinesq equation**

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$m = 2$ – the **Boussinesq equation**

the (inviscid) **aggregation equation** (or granular media equation)

$$\partial_t u = \nabla \cdot (u(\nabla K * u)).$$

a fractal version of the classical **thin film equation**: $\alpha = m = 3$

$$\partial_t u = \nabla \cdot (u^3 \nabla (-\Delta)^{1/2} u)$$

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the **evolution of the dislocation density** in crystals
$(u = w_x,\ x \in \mathbb{R})$
$\alpha = 1$, A. K. Head, N. Louat (1955)

$$u_t = \nabla \cdot \left(|u| \nabla^{\alpha-1} u \right), \quad \alpha \in (0,2]$$

P. Biler, G. Karch, R. Monneau (Comm. Math. Phys. 294, 145–168 (2010))

Equation for the primitive

$d = 1$, $w_x = u$, $\alpha \in (0, 2)$,

$$w_t = -|w_x| \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} w \qquad \mathbb{R} \times (0, +\infty)$$

$$w(x, 0) = w_0(x) \qquad x \in \mathbb{R}$$

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Lévy–Khintchine formula

$$\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} w(x) = C(\alpha) \int_{\mathbb{R}} \left(w(x+z) - w(x) - z w'(x) \mathbf{1}_{\{|z| \le 1\}}\right) \frac{\mathrm{d}z}{|z|^{1+\alpha}}$$

invariant scaling

$$w^{\lambda}(x,t) = w(\lambda x, \lambda^{\alpha+1} t)$$

$$w_{\alpha}(x,t) = \Psi_{\alpha}(y) \quad \text{with} \quad y = \tfrac{x}{t^{1/(\alpha+1)}}$$

$$-(\alpha+1)^{-1}\, y\, \Psi_{\alpha}'(y) = -\left(\left(-\partial^2/\partial x^2\right)^{\alpha/2} \Psi_{\alpha}(y)\right) \Psi_{\alpha}'(y) \quad \text{for} \quad y \in \mathbb{R}$$

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existence of self-similar solutions

For $\alpha \in (0,2)$ there exists a nondecreasing function $\Psi_\alpha \in C^{1+\alpha/2}$, analytic in $(-y_\alpha, y_\alpha)$:

$$\Psi_\alpha = \begin{cases} 0 & \text{on} \quad (-\infty, -y_\alpha), \\ 1 & \text{on} \quad (y_\alpha, +\infty), \end{cases}$$

$$w_0(x) = H(x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

Stability of self-similar solutions

$\alpha \in (0,2)$, $w_0 \in BUC(\mathbb{R})$:

$$\lim_{x\to-\infty} w_0(x) = 0 \qquad \lim_{x\to+\infty} w_0(x) = 1$$

viscosity solutions $w = w(x,t)$

$w^\lambda = w^\lambda(x,t) \equiv w(\lambda x, \lambda^{\alpha+1} t)$

$K \subset (\mathbb{R} \times [0,+\infty)) \setminus \{(0,0)\}$ – compact

$$w^\lambda(x,t) \to \Psi_\alpha\left(\frac{x}{t^{1/(\alpha+1)}}\right) \quad \text{in} \quad L^\infty(K) \quad \text{for} \quad \lambda \to +\infty$$

March 2023, F. del Teso, E. Jakobsen: Finite differences approximations

An existence result for the Cauchy problem - fpme

$$\partial_t u - \nabla \cdot (u \nabla^{\alpha-1}(|u|^{m-1})) = 0.$$

L. Caffarelli, J. L. Vázquez (2010)

$m = 2$, $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$:

$$0 \leq u_0(x) \leq A\mathrm{e}^{-a|x|} \quad \text{for some } A, a > 0.$$

Then there exists a weak solution u satisfying
$\int u(t,x)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x$.

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Then there exists a weak solution u satisfying
$\int u(t,x)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x$.
$u : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ is a **weak solution** in $Q_T = (0,T) \times \mathbb{R}^d$, $u(0,x) = u_0(x)$ if $u \in L^1(Q_T)$, $I_{2s}(u) \in L^1(0,T; W^{1,1}_{loc}(\mathbb{R}^d))$, $u \nabla I_{2s}(u) \in L^1(Q_T)$

$$\iint u(\varphi_t - \nabla I_{2s}(u) \cdot \nabla \varphi) dxdt + \int u_0(x)\varphi(x)\, dx = 0$$

for all continuous functions $\varphi : Q_T \to \mathbb{R}$, $\nabla_x \varphi$ continuous, φ has compact support in the space variable x, and vanishes near $t = T$.

approximations:

bounded domain, nondegenerate equation, regularized kernel

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Alternative approach: **Construction of weak solutions — approximations via parabolic regularization**

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Duhamel formula in $W_p^{\min\{\alpha-1,0\}}$, $p \gg 1$,
$u(t) = \mathrm{e}^{\delta t \Delta} u_0 + \int_0^t \nabla \mathrm{e}^{\delta(t-s)\Delta} \cdot |u| \nabla^{\alpha-1} G(u) \, \mathrm{d}s$
in $\mathcal{C}([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$

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Intermediate asymptotics, entropy estimates
mass conservation

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mass conservation
positivity preserving property

the speed of propagation of solutions is proved to be **finite** using comparison with suitable supersolutions (C. Imbert)

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comparison principle ?
regularity of solutions (C. Imbert, R. Tarhini, F. Vigneron)

Decay of the L^p norms – hypercontractivity

$m > 1$, $1 \leq p < \infty$,

$$\|u(t)\|_p \leq Ct^{-\beta}$$

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$$C = C(d, \alpha, m, p)\|u_0\|_1^{\frac{\frac{m-1}{p}+\frac{\alpha}{d}}{m-1+\frac{\alpha}{d}}}, \ \beta = \frac{p-1}{p\left(m-1+\frac{\alpha}{d}\right)}$$

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These estimates are sharp.

Kato and Stroock–Varopoulos inequalities

$$\int (-\Delta)^{\frac{\alpha}{2}} w \,\mathrm{sgn}\, w \,\mathrm{d}x \geq 0,$$

$$\int (-\Delta)^{\frac{\alpha}{2}} w\, w^{+} \,\mathrm{d}x \geq 0, \quad \int (-\Delta)^{\frac{\alpha}{2}} w\, w^{-} \,\mathrm{d}x \leq 0$$

$w^{+} = \max\{0, w\}$, $w^{-} = \max\{0, -w\}$

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$w^+ = \max\{0, w\}, \; w^- = \max\{0, -w\}$

$$\int (-\Delta)^{\frac{\alpha}{2}} w |w|^{p-2} w \,\mathrm{d}x \geq \frac{4(p-1)}{p^2} \int \left| \nabla^{\frac{\alpha}{2}} |w|^{\frac{p}{2}} \right|^2 \mathrm{d}x$$

$w \in L^p(\mathbb{R}^d)$: $(-\Delta)^{\frac{\alpha}{2}} w \in L^p(\mathbb{R}^d)$

Proof of hypercontractivity estimates

u^{p-1}, integrate by parts

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p \mathrm{d}x &= -(p-1)\int u u^{p-2}\nabla^{\alpha-1}(u^{m-1})\cdot\nabla u \,\mathrm{d}x \\
&= -\frac{p-1}{p}\int u^p(-\Delta)^{\frac{\alpha}{2}}(u^{m-1})\,\mathrm{d}x \\
&\le -\frac{4(p-1)(m-1)}{(p+m-1)^2}\left\|\nabla^{\frac{\alpha}{2}}\left(u^{\frac{p+m-1}{2}}\right)\right\|_2^2
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Nash inequality

$$\|v\|_2^{2(1+\frac{\alpha}{d})} \leq C_N\|\nabla^{\frac{\alpha}{2}}v\|_2^2\|v\|_1^{\frac{2\alpha}{d}}$$

v with $v \in L^1(\mathbb{R}^d)$, $\nabla^{\frac{\alpha}{2}}v \in L^2(\mathbb{R}^d)$ with a constant $C_N = C(d,\alpha)$

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the **Gagliardo–Nirenberg** type inequality

$$\|u\|_p^a \leq C_N\left\|\nabla^{\frac{\alpha}{2}}|u|^{\frac{r}{2}}\right\|_2^2\|u\|_1^b$$

$a = \frac{p}{p-1}\frac{d(r-1)+\alpha}{d}$, $b = \frac{p\alpha+d(r-p)}{d(p-1)}$

Interpolating

$$\|u\|_p \le \|u\|_r^{\gamma} \|u\|_1^{1-\gamma}, \qquad \|u\|_{\frac{r}{2}} \le \|u\|_p^{\delta} \|u\|_1^{1-\delta},$$

$\gamma = \frac{p-1}{r-1}\frac{r}{p}$, $\delta = \frac{r-2}{p-1}\frac{p}{r}$

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$$\frac{\mathrm{d}}{\mathrm{d}t} \int |u|^p \, \mathrm{d}x \le -K \|u\|_p^a \|u\|_1^{-b}$$

some $K > 0$

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$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \le -KM^{-b}f(t)^{\frac{a}{p}}$$

$f(t) = \|u(t)\|_p^p$, $M = \|u_0\|_1$, $a/p > 1$,

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$f(t) \le \left(K\left(\frac{a}{p} - 1\right)M^{-b}\,t\right)^{-\frac{1}{\frac{a}{p}-1}}$

and one more iteration scheme

Self-similar solutions

$$u(t,x) = \frac{1}{(1+t)^{d\lambda}} U\left(\frac{x}{(1+t)^{\lambda}}\right)$$

$\lambda = \frac{1}{(m-1)d+\alpha}, \quad y = \frac{x}{(1+t)^{\lambda}}$

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$$-\lambda \nabla \cdot (yU) = \nabla \cdot (U \nabla^{\alpha-1}(U^{m-1}))$$

$U : \mathbb{R}^d \to \mathbb{R}, \; m > 1$

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$$\Phi_{m,\alpha}(y) = k(1-|y|^2)_+^{\frac{\alpha}{2(m-1)}}$$

then u defined with $U = \Phi_{m,\alpha}$ is a weak solution in $(a,T)\times\mathbb{R}^d$, $0 < a < T$, $\quad \frac{\alpha}{2(m-1)}$-Hölder at the interface.

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Mass of $u(t,.)$ is conserved, and by suitable scaling of $\Phi_{m,\alpha}$, u its mass can be prescribed as any $M \in [0,\infty)$.

$$\Phi_{m,\alpha}(y) = \left(k_{\alpha,d}(1-|y|^2)_+^{\frac{\alpha}{2}}\right)^{\frac{1}{m-1}}$$

$k_{\alpha,d} = \frac{d\Gamma(\frac{d}{2})}{(d+\alpha)2^{\alpha}\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{d+\alpha}{2})}$

$\alpha = 2$: classical Kompaneets–Zeldovich–**Barenblatt**–Pattle solutions

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$\alpha = 2$: classical Kompaneets–Zeldovich–**Barenblatt**–Pattle solutions

Self-similar solutions enjoy the optimal decay rates.

$$-\lambda y\Phi = \Phi\nabla^{\alpha-1}\Phi^{m-1}$$

Φ vanishing outside B_1: $\Phi \sim (1-|y|^2)_+^{\frac{\alpha}{2(m-1)}}$

$$-\lambda y = \nabla^{\alpha-1}\Phi^{m-1} \quad \text{in } B_1$$

the homogeneous Dirichlet condition should be understood under the form $\Phi \equiv 0$ outside B_1, and not only $\Phi = 0$ on ∂B_1.

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Getoor

$\alpha \in (0,2]$,

$$K_{\alpha,d}(-\Delta)^{\frac{\alpha}{2}}(1-|y|^2)_+^{\frac{\alpha}{2}} = -1 \text{ in } B_1$$

$$K_{\alpha,d} = \frac{\Gamma\left(\frac{d}{2}\right)}{2^\alpha\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}$$

more generally:
the **Weber–Schafheitlin** integrals for $0 < b \leq a$

$$\int_0^{+\infty} t^{-\lambda} J_\mu(at) J_\nu(bt)\,\mathrm{d}t = \frac{b^\nu 2^{-\lambda} a^{\lambda-\nu-1} \Gamma(\frac{\nu+\mu-\lambda+1}{2})}{\Gamma(\frac{-\nu+\mu+\lambda+1}{2})\Gamma(1+\nu)} \times {}_2F_1\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{b^2}{a^2}\right).$$

for the hypergeometric function ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$ complex numbers a, b, c and $|z| < 1$, where $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1)$ (and $(a)_0 = 1$)

Boundary obstacle problem for the fractional Laplacian

$$P \geq \Phi, \quad V = (-\Delta)^{\frac{\alpha}{2}} P \geq 0,$$
$$\text{either } \; P = \Phi \; \text{ or } \; V = 0,$$

with $\alpha \in (0, 2)$ and $\Phi(y) = C - a|y|^2$

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$P(y) = I_\alpha \left(\Phi_{\alpha,2,R}\right)(y)$,
I_α is the Riesz potential,

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$$\text{either} \quad P = \Phi \quad \text{or} \quad V = 0,$$

with $\alpha \in (0, 2)$ and $\Phi(y) = C - a|y|^2$

$P(y) = I_\alpha \left(\Phi_{\alpha,2,R} \right) (y)$,
I_α is the Riesz potential,

$\Phi_{\alpha,2}(y) = k_{\alpha,d}(R^2 - |y|^2)_+^{\frac{\alpha}{2}}$

Classical boundary obstacle problems:

given a smooth $\Omega \subset \mathbb{R}^d$, $d \geq 3$, seek a function u that:
– in the interior of Ω, u satisfies a nice, elliptic equation, say $\Delta u = f$,
– along the boundary of Ω , instead of giving Dirichlet or Neumann conditions we prescribe "complementary conditions":
as long as u is bigger than some prescribed function ϕ, there is no flux across $\partial\Omega$: $\partial u/\partial \nu = 0$. But as soon as u becomes equal to ϕ, boundary flux, $\partial u/\partial \nu$, is turned on ($\partial u/\partial \nu > 0$) to keep u above ϕ.

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This type of problem arises in elasticity (the Signorini problem) when an elastic body is at rest, partially lying on a surface,
– in optimal control of temperature across a surface,
– in the modelling of semipermeable membranes where some saline concentration can flow through the membrane only in one direction,
– and in financial math when the random variation of underlying asset changes in a discontinuous fashion (a Lévy process).

Another point of view:

$u \geq \phi$, $(-\Delta)^{\alpha/2} u = 0$ for $u > \phi$, $(-\Delta)^{\alpha/2} u \geq 0$ for $x \in \mathbb{R}^d$
– a variational problem in $\dot{H}^{\alpha/2}(\mathbb{R}^d)$,
– the least supersolution of $(-\Delta)^{\alpha/2} v \geq 0$ among $v \geq \phi$,
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Fractional Laplacian as "Dirichlet to Neumann" operator (as for $\alpha = 1$):
$u(x,0) \geq \phi(x)$ for $x \in \mathbb{R}^d$,
$\nabla \cdot (y^{1-\alpha} \nabla u(x,y)) = 0$ for $y > 0$
$\lim_{y \searrow 0} y^{1-\alpha} \partial_y u(x,y) = 0$ for $u(x,0) > \phi(x)$,
$\lim_{y \searrow 0} y^{1-\alpha} \partial_y u(x,y) \leq 0$ for $x \in \mathbb{R}^d$.

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