On applications of topology in the theory of differential equations

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Problems on nonlinear differential equations include:

1. Existence and multiplicity of solutions (initial value problems, boundary value problems, etc.)
2. Qualitative properties of solutions (stationary points, periodic points, chaotic dynamics, invariant sets, etc.)

$f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth (i.e. of C^1-class) map.

Definition

$x \in \mathbb{R}^n$ is a *regular point* iff $d_x f$ is an epimorphism (i.e. the differential is surjective). x is a *critical point* iff it is not regular.

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Particular cases:

If $m = n$ then x is regular iff $d_x f$ is an isomorphism.

If $m = 1$ then x is regular iff $d_x f$ (equivalently: $\nabla f(x)$) is nonzero.

Definition

y is a *regular value* iff each point of $f^{-1}(y)$ is regular. y is a *critical value* if there exists a critical point in $f^{-1}(y)$.

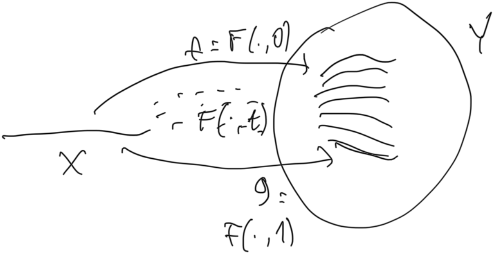
X and Y are topological spaces, $f, g\colon X \to Y$.

Definition

f is *homotopic to* g (written as $f \simeq g$) iff there exists a continuous map

$$F\colon X \times [0,1] \to Y$$

such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$.



$v\colon \mathbb{R}^n \to \mathbb{R}^n$ a smooth vector-field,

$$\dot{x} = v(x). \qquad (*)$$

$t \mapsto \phi_t(x_0)$ is the solution of $(*)$ with initial value x_0 at 0, i.e.

$$\begin{cases} \dfrac{d}{dt}\phi_t(x_0) = v(\phi_t(x_0)), \\ \phi_0(x_0) = x_0. \end{cases}$$

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Properties:

$$\phi_t \circ \phi_s = \phi_{t+s}, \quad \phi_0 = \operatorname{id}. \tag{$**$}$$

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Definition

Let X be a topological space. A continuous map

$$\phi\colon X \times \mathbb{R} \ni (x,t) \to \phi_t(x) \in X$$

which satisfies $(**)$ is called a *dynamical system*.

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If ϕ is given by $(*)$, we call it the *dynamical system generated by* v or the *dynamical system generated by* $(*)$.

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Definition

x_0 is a *stationary point* of ϕ iff $\phi_t(x_0) = x_0$ for all $t \in \mathbb{R}$.

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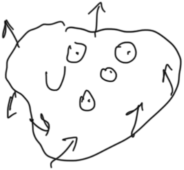
Remark

x_0 is a stationary point of ϕ generated by v iff $v(x_0) = 0$, i.e. x_0 is a *zero* of v.

$v\colon \mathbb{R}^n \to \mathbb{R}^n$ a continuous vector-field
U open and bounded in $\mathbb{R}^n$
$v(x) \neq 0$ for all $x \in \partial U$



Definition (Brouwer degree, 1910)

If v is smooth and 0 is a regular value of v (i.e. $d_x v$ is an isomorphism for every $x \in v^{-1}(0)$),

$$\deg(v, U) := \sum_{x \in v^{-1}(0) \cap U} \operatorname{sgn} \det d_x v \in \mathbb{Z}.$$

In general,

$$\deg(v, U) := \deg(w, U),$$

where w is smooth, 0 is a regular value of w, and w is close enough to v.

Properties of the Brouwer degree:

Solvability. If $\deg(v, U) \neq 0$ then there exists $x \in U$ such that $v(x) = 0$.

Excision. If $U' \subset U$ and $v(x) \neq 0$ for all $x \in U \setminus U'$ then

$$\deg(v, U) = \deg(v, U').$$

Homotopy invariance. If $\mathbb{R}^n \times [0,1] \ni (x,t) \to v_t(x) \in \mathbb{R}^n$ is continuous and $v_t(x) \neq 0$ for all $(x,t) \in \partial U \times [0,1]$ then

$$\deg(v_0, U) = \deg(v_1, U).$$

Additivity. If $U_0 \cap U_1 = \emptyset$ then

$$\deg(v, U_0 \cup U_1) = \deg(v, U_0) + \deg(v, U_1).$$

Multiplicativity. If $v \colon \mathbb{R}^n \to \mathbb{R}^n$ and $v' \colon \mathbb{R}^{n'} \to \mathbb{R}^{n'}$ then

$$\deg(v \times v', U \times U') = \deg(v, U) \cdot \deg(v', U').$$

Definition

Z is an *isolated set of zeros of* v if it is compact and there exists U, a neighborhood of Z, such that $Z = \{x \in U \colon v(x) = 0\}$.

For such Z and U set

$$d(v, Z) := \deg(v, U)$$

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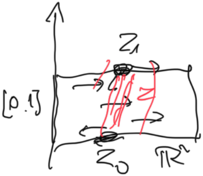
Let Z_0 and Z_1 are isolated sets of zeros of v_0 and, respectively, v_1.

Definition

$(v_0, Z_0) \simeq (v_1, Z_1)$ iff there exists a vector-field

$$V \colon \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \times \mathbb{R}$$



such that

$$V(\cdot, 0) = (v_0, 0), \quad V(\cdot, 1) = (v_1, 0)$$

and an isolated set Z of zeros of V in $\mathbb{R}^n \times [0, 1]$ such that

$$Z_0 = \{x \colon (x, 0) \in Z\}, \quad Z_1 = \{x \colon (x, 1) \in Z\}.$$

An equivalent description of properties of the Brouwer degree:

An equivalent description of properties of the Brouwer degree:

Solvability. If $d(v, Z) \neq 0$ then $Z \neq \emptyset$.

Homotopy invariance. If $(v, Z) \simeq (v', Z')$ then

$$d(v, Z) = d(v', Z').$$

Additivity. If $Z \cap Z' = \emptyset$ then

$$d(v, Z \cup Z') = d(v, Z) + d(v, Z').$$

Multiplicativity. If $v \colon \mathbb{R}^n \to \mathbb{R}^n$ and $v' \colon \mathbb{R}^{n'} \to \mathbb{R}^{n'}$ then

$$d(v \times v', Z \times Z') = d(v, Z) \cdot d(v', Z').$$

A generalization of the Brouwer degree to normed linear spaces is called the *Leray-Schauder degree* (1934). It is used, in particular, to prove the existence of solutions of boundary value problems, by representing those solutions as zeros $v(x) = 0$ for a suitable vector-field v in some normed space.

Let $n \geq 2$ (the case $n = 1$ is trivial) and let ∂U be smooth.

if $n = 2$, the Brouwer degree is equal to the *winding number* of $v|_{\partial U}$, i.e.

$$\deg(v, U) = \frac{1}{2\pi} \int_{\partial U} v^* d\theta,$$

where

$$d\theta := \frac{-y dx + x dy}{x^2 + y^2}$$

is the *angular form*; the right-hand side is the *Cauchy index* (ca. 1830), Let $n \geq 2$ (the case $n = 1$ is trivial) and let ∂U be smooth.

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more general, for $n \geq 2$,

$$\deg(v, U) = \frac{1}{\mu_{n-1}} \int_{\partial U} v^* \left(\frac{\sigma}{\|x\|^n} \right),$$

where $\sigma = \sum_{i=1}^{n} (-1)^i x_i dx_1 \ldots \widehat{dx_i} \ldots dx_n$ and μ_{n-1} is the volume of the $(n-1)$-dimensional unit sphere; the right-hand side is the *Kronecker index* (1869).

$\chi(\ ) = V - E + F - \ldots$

Theorem (Poincaré-Hopf formula, ca. 1925)

If ∂U is smooth and $v(x)$ is directed outward of U for each $x \in \partial U$ then

$$\deg(v, U) = \chi(U),$$

χ

where χ denotes the Euler-Poincaré characteristic.

Remark

Since the Euler-Poincaré characteristic of an odd-dimensional manifold is equal to 0,

$$\chi(U) = (-1)^n(\chi(U) - \chi(\partial U)).$$

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$S \subset X$

Definition

The set

$$\operatorname{Inv} S := \{x \in S \colon \phi_t(x) \in S \text{ for all } t \in \mathbb{R}\}$$

is called the *invariant part of* S.

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S is called *isolated invariant* iff it is compact and there exists U, a neighborhood of S, such that

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S is called *invariant* iff $A = \operatorname{Inv} A$.
S is called *isolated invariant* iff it is compact and there exists U, a neighborhood of S, such that
$$S = \operatorname{Inv} U.$$
Such an U is called an *isolating neighborhood*.

Example

Let A be a matrix with no eigenvalues on the imaginary axis.

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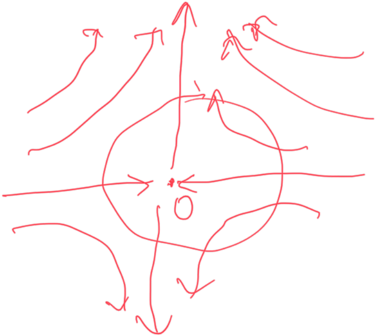
The set $\{0\}$ is an isolated invariant set for the dynamical system generated by

$$\dot{x} = Ax.$$



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x_0 is a critical point of f iff $\nabla f(x_0) = 0$.

A finite set of isolated critical points of f is an isolated invariant set of ϕ.

$B \subset X$.

Definition

The *exit set* of B is defined as

$$B^- := \{x \in B \colon \phi_{\epsilon_n}(x) \notin B \text{ for some } 0 < \epsilon_n \to 0\}.$$

B is called an *isolating block* iff B and B^- are compact and

$$\operatorname{Inv} B \subset \operatorname{int} B.$$

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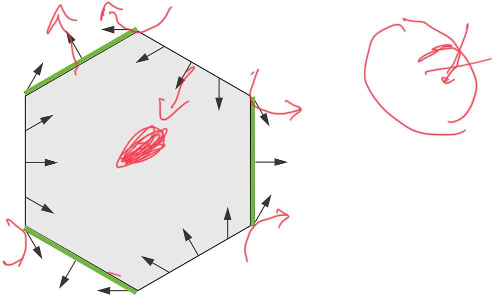

It follows $\operatorname{Inv} B$ is an isolated invariant set and B is its isolating neighborhood.

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A retraction r is called a *strong deformation retraction* iff

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where $i\colon A \hookrightarrow X$ is the inclusion map.

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A retraction r is called a *strong deformation retraction* iff

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where $i\colon A \hookrightarrow X$ is the inclusion map. A is called a *strong deformation retract* of X if there exist a strong deformation retraction $X \to A$.

Theorem (Ważewski, 1947)
If B is an isolating block and B^{-} is not a strong deformation retract of B then

$$\operatorname{Inv} B \neq \emptyset.$$

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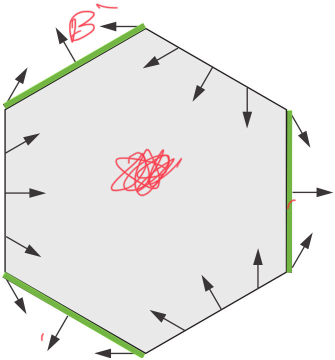

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Definition

The *quotient space* X/A is defined as

$$X/A := (X \setminus A) \cup \{*\}$$

endowed with the following topology:

> if $A \neq \emptyset$ obtained from the topology of X/A is such that the neighborhoods of $*$ are induced from the neighborhoods of A in X (intuitively: X/A is obtained by "squeezing" A to one point $*$);

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In particular, X/X is a one-point space $\{*\}$.
(In consequence, $\emptyset/\emptyset = \{*\}$.)

Definition

A *pointed topological space* (X, x_0) is a topological space X together with a distinguished point $x_0 \in X$ (called the *base point*).

Example

We treat X/A as the pointed space

$$((X \setminus A) \cup \{*\}, *).$$

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A homotopy between $f, g\colon (X, x_0) \to (Y, y_0)$ is a continuous map $F\colon X \times [0,1] \to Y$ such that

$$F(\cdot, 0) = f, \quad F(\cdot, 1) = g, \quad F(x_0, t) = y_0 \text{ for all } t \in [0,1].$$

If such a homotopy exists, f and g are called homotopic (written as $f \simeq g$).

Definition

(X, x_0) and (Y, y_0) are of the same *homotopy type* iff there exist maps $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (X, x_0)$ such that

$$g \circ f \simeq \mathrm{id}_X \; f \circ g \simeq \mathrm{id}_Y.$$

The homotopy type of (X, x_0) is denoted as $[(X, x_0)]$.

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The homotopy type of (X, x_0) is denoted as $[(X, x_0)]$.
The homotopy type of one-point space $(\{*\}, *)$ is denoted $\overline{0}$.

Remark

If A is a strong deformation retract of X then $[X/A]$ is equal to $\overline{0}$.

Remark (Corollary from Theorem of Ważewski)

If B is an isolating block and $[B/B^-] \neq \overline{0}$ then $\mathrm{Inv}(B) \neq \emptyset$.

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Theorem (Conley, Easton 1971)

For every neighborhood U of S there exists an isolating block B such that

$$S = \operatorname{Inv} B, \quad B \subset U.$$

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Theorem (Conley 1972)

If B and B_* are isolating blocks such that

$$S = \operatorname{Inv} B = \operatorname{Inv} B_*$$

then

$$[B/B^-] = [B_*/B_*^-].$$

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Definition

The homotopy type $h(\phi, S) := [B/B^-]$ is called the *Conley index* of S.

S^k denotes the k-dimensional unit sphere and $* \in S^k$ is an arbitrary point.

$\mathbb{R}^{k+1}$

S^k denotes the k-dimensional unit sphere and $* \in S^k$ is an arbitrary point.

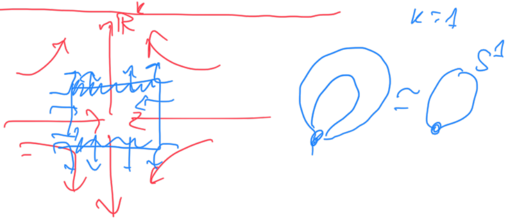
$$\Sigma^k := [(S^k, *)].$$

Example

Let ϕ be generated by $\dot{x} = Ax$, where A has no eigenvalues on the imaginary axis. Then

$$h(\phi, \{0\}) = \Sigma^k$$

where k is the number of eigenvalues with the real part positive.



Let x_0 be a critical point of $f\colon \mathbb{R}^n \to \mathbb{R}$.

Definition

x_0 is *non-degenerate* if the Hessian of f at x_0 (i.e. the linearization of ∇f at x_0) is an isomorphism.

Theorem (Morse, 1929)

If x_0 is a non-degenerate critical point then in a suitable coordinate system in a neighborhood of x_0,

$$f(x) = f(x_0) - \sum_{i=1}^{k}(x_i - x_{0i})^2 + \sum_{i=k+1}^{n}(x_i - x_{0i})^2.$$

The number $i(x_0) := k$ is independent of the choice of a coordinate system and is called the *Morse index of x_0*

Example

If ϕ is generated by $\dot{x} = -\nabla f(x)$ and x_0 is non-degenerated then

$$h(\phi, \{x_0\}) = \Sigma^{i(x_0)}.$$

Let S_0 and S_1 are isolated invariant sets of ϕ^0 and, respectively, ϕ^1.

Definition

$(\phi^0, S_0) \simeq (\phi^1, S_1)$ iff there exists a dynamical system

$$\Phi \colon X \times [0,1] \times \mathbb{R} \ni (x, \lambda, t) \to (\phi_t^\lambda(x), \lambda) \in X \times [0,1]$$

and an isolated set S of zeros of Φ in $X \times [0,1]$ such that

$$S_0 = \{x \colon (x, 0) \in S\}, \quad S_1 = \{x \colon (x, 1) \in S\}.$$

Properties of the Conley index:

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Ważewski property. If $h(\phi, S) \neq \overline{0}$ then $S \neq \emptyset$.

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Additivity. If $S \cap S' = \emptyset$ then

$$h(\phi, S \cup S') = h(\phi, S) \vee h(\phi, S').$$

Multiplicativity. If S is an isolated invariant set for ϕ and ϕ' is an isolated invariant set for ϕ' then

$$h(\phi \times \phi', S \times S') = h(\phi, S) \wedge h(\phi', S').$$

Properties of the Conley index vs. properties of the Brouwer degree:

$$h(\phi, S) \neq \bar{0} \Rightarrow S \neq \emptyset \quad \text{vs.} \quad d(v, Z) \neq 0 \Rightarrow Z \neq \emptyset,$$

$$(\phi, S) \simeq (\phi', S') \Rightarrow h(\phi, S) = h(\phi', S')$$
$$\text{vs.} \quad (v, Z) \simeq (v', Z') \Rightarrow d(v, Z) = d(v', Z').$$

$$h(\phi, S \cup S') = h(\phi, S) \vee h(\phi, S')$$
$$\text{vs.} \quad d(v, Z \cup Z') = d(v, Z) + d(v, Z'),$$

$$h(\Phi, S \times S') = h(\phi, S) \wedge h(\phi', S')$$
$$\text{vs.} \quad d(v \times v', Z \times Z') = d(v, Z) \cdot d(v', Z').$$

The Euler-Poincaré characteristics of a pointed space (X, x_0) differs from the E-P characteristic of a single space $\chi(X)$ by one, i.e.

$$\chi(X, x_0) = \chi(X) - 1.$$

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Theorem (R.S. 1985)

If ϕ is generated by a vector-field v, S is an isolated invariant set of ϕ,

$$Z = \{x \in S \colon v(x) = 0\}$$

then

$$d(v, Z) = (-1)^n \chi(h(\phi, S)).$$

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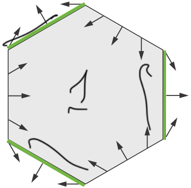
Example



1 − 3

$$d(v, Z) = \overbrace{\chi(B/B^-) = \chi(B) - \chi(B^-)} = -2$$

Some generalized versions of the Conely index:

K.Rybakowski 1987; for dynamical systems generated by parabolic equations,

K.Gęba, M.Izydorek, A.Pruszko 1999; for gradient systems in Hilbert spaces,

J.Robbin, D.Salamon 1988; for discrete-time dynamical systems generated by diffeomorphisms,

M.Mrozek 1990; for discrete-time systems generated by continuous maps,

M.Mrozek 1988; for multi-valued dynamical systems,

M.Izydorek 2000; for dynamical systems with group symmetries.

Applications of the Conley index include problems on:

existence and multiplicity of critical points of functionals,

existence of bifurcations in parametrized dynamical systems,

existence of connecting trajectories between stationary points,

existence of periodic orbits,

existence of symbolic dynamics.

Other topological tools in the theory of differential equations include:

- applications of the Lusternik-Schnirelman category,
- applications of the Krasnoselski's genus and, more general, so-called index theories,
- applications of Nielsen fixed point classes theory,
- modifications of the Leray-Schauder degree (e.g. the Gaines-Mawhin's coincidence index),
- further generalizations of the Leray-Schauder degree (e.g. Skrypnik's degree),
- equivariant degrees (e.g. K.Gęba's G-∇-degree),
- the Fuller index on periodic orbits of dynamical systems.