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Generalized Calogero-Moser-Sutherland systems, quantization, topological methods and relationships with quantum chaos

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Abstract

Calogero-Moser-Sutherland Hamiltonians describe one-dimensional many-body systems with inverse-square interactions. They can be generalized by substituting the coupling constants in the interaction terms with additional dynamical variables, often referred to as internal or spin degrees of freedom. Classical, completely integrable N-body systems of this type, with the internal degrees of freedom spanning the $\mathfrak{so}(N)$, $\mathfrak{su}(N)$ or $\mathfrak{sp}(N)$ Lie algebra, arise in the surprisingly opposite context, namely energy level repulsion in quantum chaotic systems. On the other hand the quantum Calogero-Moser-Sutherland models appear in the area of topological quantum matter, Quantum Hall Effect in particular, as their eigenfunctions are obtained as 1D representations of Laughlin states projected on a single Landau level.

The classical dynamics of the internal degrees of freedom were studied to some extent, but the details relevant to the problem of level repulsion, such as the existence of a finer classification of models within the orthogonal, unitary and symplectic classes, still need to be resolved. As for the quantum case, specific models with internal degrees of freedom have been investigated, but they typically loose this internal structure in the classical limit. Models whose spin degrees of freedom survive the quantum-classical transition, together with the potential presence of topological effects, are not fully understood.

The perhaps most remarkable characteristic of the classical Calogero-Moser-Sutherland systems, taking the interactions and internal degrees of freedom into account, is their complete integrability. This trait is a direct consequence of the fact that these systems can be obtained *via* Hamiltonian reduction of very simple (such as free or harmonic), integrable dynamics in spaces of matrices. The reduction procedure, though defined rigorously in the language of symplectic geometry, in this case simplifies to diagonalizing an $N \times N$ time-dependent matrix $X(t)$, assigning its eigenvalues to instantaneous positions of the N particles and eliminating the eigenvectors from the equations of motion. So obtained functions on a reduced phase space can be quantized canonically by the Dirac's prescription. On the other hand, simple dynamics in a matrix phase space can be quantized and then reparametrized by eigenvalues and eigenvectors.

Within the framework of Hamiltonian mechanics I have shown that the Lie-algebraic, matrix generalization of the Calogero-Moser model is equivalent to an alternative, vectorial formulation in which the internal state of each particle in encoded in a complex vector. The dimension of the subspace spanned by these vectors (equivalent to the rank of the matrix from the corresponding Lie algebra) is a constant of motion which I have used to classify the models. I have proven that the models with purely imaginary matrices encoding the initial values of internal variables have special properties: they approximate the phase space trajectories of the non-generalized Calogero-Moser system with distinct coupling constants and have the smallest reachable sets. Additionally, by combining the matrix and vector degrees of freedom and reducing it as described above, I have obtained a new integrable model with $1/x$ interactions. I have obtained two different Hamiltonians from the procedures of canonical quantization of the matrix model and reducing a free quan-

tum model. The reduction of the quantum model introduces an additional term $-\frac{\hbar^2}{4m}\sum_{i<j}\frac{\alpha(2-\alpha)}{(x_i-x_j)^2}$ where $\alpha = 1, 2$ for the $\mathfrak{so}(N)$ and $\mathfrak{su}(N)$ cases respectively, which is an attractive term in the orthogonal setting. I have identified some of the irreducible representations of the Lie algebra spanned by the internal observables, for which the internal degrees of freedom introduce diagonal matrices into the Hamiltonian, and I have found the spectrum and eigenfunctions for $N = 3$.

The classical results can be used in the further study of energy level repulsion. The expected level spacing distribution for the purely imaginary matrices is different from the one which is known for general $\mathfrak{su}(N)$ matrices. Similarily, one should expect differences in probability distributions depending on the rank of the matrices. The study of the quantum systems can be continued in two main directions: the quantum reduction can be done in the symplectic case, as well as the spectra and eigenfunctions can be found (if not exactly, then at least approximately) for higher numbers of particles and dimensions of the space of internal states. The finite-dimensional space of internal states, once well understood, can serve as an additional dimension, thus yield the system two-dimensional and prone to topological effects.

The original results presented in the thesis were published in the following papers:

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- K. Kowalczyk-Murynka, M. Kuś. Calogero-Moser models with internal degrees of freedom revisited, arXiv:2010.10215.

Streszczenie

Hamiltoniany Calogero-Mosera-Sutherlanda opisują jednowymiarowe układy wielu cząstek z oddziaływaniem proporcjonalnym do odwrotności kwadratu odległości. Można je uogólnić zastępując stałe sprzężenia w wyrazach określających oddziaływanie dodatkowymi zmiennymi dynamicznymi, które zwykle nazywane są spinowymi lub wewnętrznymi stopniami swobody. Klasyczne, całkowalne N-ciałowe układy tego typu, z wewnętrznymi stopniami swobody rozpinającymi algebrę $\mathfrak{so}(N)$, $\mathfrak{su}(N)$ lub $\mathfrak{sp}(N)$, pojawiają się w zaskakująco innym kontekście, mianowicie odpychania poziomów energetycznych w chaotycznych układach kwantowych. Z drugiej strony, kwantowe modele Calogero-Mosera-Sutherlanda pojawiają się w obszarze topologicznej materii kwantowej, kwantowego Efektu Halla w szczególności. Ich funkcje własne otrzymuje się jako jednowymiarowe reprezentacje stanów Laughlina zrzutowanych na pojedynczy poziom Landaua.

Klasyczna dynamika wewnętrznych stopni swobody została w pewnym stopniu zbadana, ale jej szczegóły istotne dla problemu odpychania poziomów, takie jak istnienie dokładniejszej klasyfikacji modeli wewnątrz klasy ortogonalnej, unitarnej i symplektycznej, wciąż wymagają doprecyzowania. W przypadku kwantowym konkretne modele z wewnętrznymi stopniami swobody były badane, ale typowo tracą one tę wewnętrzną strukturę w granicy klasycznej. Modele, których spinowe stopnie swobody są w stanie przetrwać przejście do granicy klasycznej, wraz z ich potencjałem związanym z efektami topologicznymi, nie zostały dokładnie poznane.

Być może najbardziej niezwykłą cechą klasycznych układów Calogero-Mosera-Sutherlanda, wziąwszy pod uwagę oddziaływania i wewnętrzne stopnie swobody, jest ich zupełna całkowalność. Cecha ta jest bezpośrednią konsekwencją tego, że modele te otrzymuje się poprzez redukcję hamiltonowską bardzo prostego (np. swobodnego lub harmonicznego) ruchu w przestrzeni macierzy. Proces redukcji, choć można go zdefiniować formalnie w języku geometrii symplektycznej, sprowadza się w tym przypadku do diagonalizacji N-wymiarowej, zależnej od czasu macierzy $X(t)$, przypisania jej wartościom własnym roli położeń i eliminacji wektorów własnych z równań ruchu. Tak otrzymane funkcje określone w zredukowanej przestrzeni fazowej można skwantować kanonicznie według przepisu Diraca. Z drugiej strony prosty układ w przestrzeni macierzy może zostać skwantowany, a następnie sparametryzowany przez wartości i wektory własne.

W ramach formalizmu hamiltonowskiego pokazałam, że macierzowe uogólnienie modelu Calogero-Mosera związane z algebrami Liego jest równoważne innemu, wektorowemu sformułowaniu, w którym wewnętrzny stan każdej cząstki opisany jest zespolonym wektorem. Wymiar podprzestrzeni rozpinanej przez te wektory (równoważny rzędowi macierzy z odpowiedniej algebry Liego) jest stałą ruchu, którą wykorzystałam do klasyfikacji modeli. Pokazałam, że modele z czysto urojonymi macierzami opisującymi wewnętrzne stopnie swobody w chwili początkowej mają specjalne własności: ich trajektorie w dobrym przybliżeniu opisują dynamikę modeli bez wewnętrznych stopni swobody ale z różnymi wartościami stałych sprzężenia, a także mają minimalne zbiory osiągalne. Dodatkowo, łącząc macierzowe i wektorowe stopnie swobody i przeprowadzając opisaną wcześniej redukcję otrzymałam nowy

całkowalny model z oddziaływaniami zależnymi od odwrotności odległości między cząstkami. Poprzez kanoniczną kwanyzację układu z wewnętrznymi stopniami swobody i redukcję kwanowego układu swobodnego otrzymałam dwa różne hamiltoniany. Hamiltonian otrzymany z redukcji układu swobodnego posiada dodatkowy wyraz $-\frac{\hbar^2}{4m}\sum_{i<j}\frac{\alpha(2-\alpha)}{(x_i-x_j)^2}$ gdzie $\alpha = 1, 2$ odpowiednio w przypadku stopni swobody należących do algebry $\mathfrak{so}(N)$ lub $\mathfrak{su}(N)$. Wyraz ten wprowadza przyciąganie w modelu ortogonalnym. Zidentyfikowałam niektóre z nieprzywiedlnych reprezentacji algebry rozpinanej przez obserwable związane z wewnętrznymi stopniami swodody, wprowadzających do hamiltonianu macierze diagonalne, oraz znalazłam wartości i wektory własne w przypadku $N = 3$.

Klasyczne wyniki mogą zostać zastosowane w dalszych badaniach nad problemem dynamiki poziomów energetycznych. Spodziewany rozkład prawdopodobieństwa odległości między poziomami dla czysto urojonych macierzy różni się od tego, który odpowiada wszystkim macierzom należącym do algebry $\mathfrak{su}(N)$. Podobnie, można się spodziewać różnic w rozkładach w zależności od rzędu macierzy opisującej wewnętrzne stopnie swobody. Praca nad układami kwantowymi może być kontynuowana w dwóch głównych kierunkach: redukcja kwantowa może zostać wykonana w przypadku symplektycznym, oraz widma i funkcje własne mogą zostać znalezione (jeśli nie ściśle, to przynajmniej w przybliżeniu) dla większej liczby cząstek i większych wymiarów przestrzeni stanów wewnętrznych. Skończenie wymiarowa przestrzeń stanów wewnętrznych, po dokładnym zbadaniu, może posłużyć za dodatkowy wymiar, czyniąc cały układ dwuwymiarowym i podatnym na efekty topologiczne.

Oryginalne wyniki zaprezentowane w niniejszej rozprawie zostały opublikowane w następujących pracach:

- K. Kowalczyk-Murynka, M. Kuś. Matrix and vectorial generalized Calogero–Moser models. *Physica D: Nonlinear Phenomena*, 440: 133491, 2022
- K. Kowalczyk-Murynka, M. Kuś. Calogero-Moser models with internal degrees of freedom revisited, arXiv:2010.10215.

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Contents

Chapter 1

Introduction

Integrability, let alone explicit solvability, of equations of motion should not be taken for granted. Apart from the case of one degree of freedom, it is in fact not common for a system to be symmetric enough to have completely integrable equations of motion. In the case of many-body interacting systems on a line, defined by the Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} \mathcal{V}(x_i - x_j)$$

the requirement of integrability imposes very stringent conditions on the interaction potential. These conditions can be transferred to the quantum context as well, and they limit $\mathcal{V}(x)$ to only few known cases:

- the Calogero system with $\mathcal{V}(x) = \frac{g}{x^2} + \frac{m\omega^2}{2}x^2$, usually called the Calogero-Moser system for $\omega = 0$,
- the Sutherland system defined by $\mathcal{V}(x) = \frac{g}{a^2 \sin^2(ax)}$,
- the inverse hyperbolic sine potential $\mathcal{V}(x) = \frac{g}{a^2 \sinh^2(ax)}$.
- the Weierstrass function $\mathcal{V}(x) = ga^2\mathcal{P}(x)$[1],
- the Toda chains with nearest neighbour exponential interactions.

The earliest work on this class of systems was done in the late sixties and early to mid seventies, both in the classical and quantum case. One of the first studies involved the normal modes of a classical chain with exponential interactions by M. Toda [1] and the general method of using linear operators to solve non-linear equations of motion associated with interacting potentials by P.D. Lax [2]. Shortly after F. Calogero found the eigenfunctions and energy spectra for the inverse square potential (with and without the harmonic confinement) [3–5] and for the hyperbolic sine potential [6], while B. Sutherland studied the inverse sine square model [7,8]. The systems combining different inter-particle and external potentials, their integrability and scattering properties were studied by M. Adler [9]. The classical integrability of the above systems was proven by J. Moser [10] with the method of isospectral matrix deformations and, on more formal mathematical grounds of Hamiltonian

[1] Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and $\Lambda = \{m\omega_1 + n\omega_2 : \quad m, n \in \mathbb{Z}$ be a lattice generated by them in $\mathbb{C}$. The Weierstrass function is then defined for $z \in \mathbb{C} \setminus \Lambda$ as:

$$\mathcal{P}(z) = \mathcal{P}(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

group actions, by Kazhdan, Kostant and Sternberg [11]. The results of this period were summarised by M.A. Olshanetsky and A.M. Perelomov in two comprehensive review articles: [12] on the classical and [13] on the quantum context. They both emphasise the connection of the different types of two-body interacting potentials listed above with Lie algebra representations.

Since the late seventies the Calogero-Moser-Sutherland (from now on referred to as CMS) systems were studied extensively, as they emerged in different areas of physics and mathematics. They entered the field of non-linear dynamical systems through their relation with certain solutions of the Korteweg-deVries[2], Kadomtsev-Petviashvili[3] and Benjamin-Ono[4] equations [14–22]. They also emerged in the area of energy level repulsion in quantum chaotic systems. The equations of motion related to this class of Hamilton functions were shown to describe the repulsion of the energy levels of quantum systems in a form $H(\lambda) = H_0 + \lambda V$, with the control parameter λ taking the role of time [23–26]. This in turn, if H_0 is identified with an integrable, and V with a chaotic contribution (both in the classical limit), can be related to the probability distribution of level spacing [27, 28], and compared with the predictions of random matrix theory [29]. Depending on the universality class of the V term, meaning if it is diagonalizable *via* (a) orthogonal, (b) unitary or (c) symplectic matrix, the level spacing distribution is $P(s) \propto s^{\beta}$ as $s \to 0$, where $\beta = 1, 2, 4$ respectively. This means the energy levels repel, as opposed to the level clustering indicated by the distribution $P(s) = e^{-s}$ in an integrable system.

The third field, in which CMS systems made a noticeable appearance is condensed matter theory, the Quantum Hall Effect [30–32],and fractional statistics [33–35] in particular.

The relationship of the CMS potentials with root systems and Coxeter groups pointed out by Olshanetsky and Perelomov and the notion of the Calogero-Moser spaces introduced by Kazkdan, Kostant and Sternberg sparked the interest in this class of systems among the mathematical community [36–42].

1.1 Generalizations of the CMS systems

The interaction potentials listed in the beginning of this chapter involved a common coupling constant for all interacting pairs. This set-up, to which I will be referring to as the family of ordinary CMS systems, can be generalized in various ways. For example, the positions and momenta can be upgraded to complex variables [43]. The common coupling constant can be altered to pair-dependent values, as to model multiple species of particles in the system [44], although this kind of modifications may in principle lead to the loss of integrability.

Perhaps the most prominent generalizations are the ones involving additional, internal degrees of freedom modifying the interaction potential. These are often called, despite the classical context, the spin generalizations and can be divided into two basic types: the one-particle properties encoded in complex vectors $\{|e_i) : i = 1, 2, ..., N\}$ [45, 46] and two-particle functions L_{ij} which stem from the Hamiltonian reduction procedure in a matrix phase space, and turn out to form the representation of the $\mathfrak{so}(N)$ or $\mathfrak{su}(N)$ Lie algebra [10,12,47]. Spin generalizations of the CMS systems in the quantum setting involve exchange operators acting on internal states [48–52].

[2] usually abbreviated to KdV, is a nonlinear PDE for a function $u = u(t, x)$ of the form $\partial_t u + \partial_x^3 u - 6u\partial_x u = 0$.

[3] often abbreviated to KP, is a non-linear PDE for a function $u = u(t, x, y)$, defined as $\partial_x(\partial_t u + \partial_x^3 u + 6u\partial_x u) + 3\sigma^2\partial_y^2 u = 0$, where $\sigma^2 = \pm 1$.

[4] it is a non-linear integro-differential equation of the form $\partial_t u + u\partial_x u + H(\partial_x^2 u)$=0, where $H(f)(\xi) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(\tau)}{\xi - \tau}$ is the Hilbert transform.

The vector and matrix degrees of freedom mentioned above are the main focus of this thesis. The matrix formulation arises naturally in the level dynamics of integrable and chaotic Hamiltonians. The two-particle $|L_{ij}|^2$ functions replace the coupling constants in the repulsive potential. Therefore a closer look at the classical dynamics of these functions, meaning the types of orbits and reachable sets in phase space, as well as their influence on the spatial degrees of freedom can be of use to the problem of level repulsion.

On the other hand, the internal properties of each individual particle, entering the repulsion potential, seem much less mysterious and much more natural as a model of physical interaction. The questions is how are the two formulations related or mapped onto one another.

Another problem is the quantization of these degrees of freedom. The classical scheme of Hamiltonian reduction as well as methods of mapping the ordinary quantum Calogero-Moser systems onto the systems of free particles [53] or decoupled harmonic oscillators [54] can be treated as guidelines towards this task. Independently of the purely operator-algebraic approach presented in [55], a direct derivation of a quantum Calogero-Moser Hamiltonian with internal degrees of freedom will be of much benefit. Such a system, with spin degrees of freedom that survive in the classical limit is expected to have a rich structure of eigenstates, and can be further used in the study of topological effects.

1.2 Overview of the thesis

The rest of the dissertation is organized as follows: in chapter 2 I introduce the mathematical tools, definitions and notation used in the subsequent chapters. These involve the fundamentals of group theory and differential geometry, as well as an overview of Hamiltonian formalism and non-relativistic quantum mechanics, and finally the method of Hamiltonian reduction.

In chapter 3 I review the key facts about the matrix and vector formulations of the classical CM systems. I also present the solution for the spectrum and wavefunctions of the ordinary quantum CM system, some properties of the systems with spin state exchange and the relationship of the CM system with the Quantum Hall Effect.

In chapters 4 and 5 I present my contribution. Chapter 4 involves the classical dynamics of the internal degrees of freedom: the relationship between the matrix and vectorial formulation, the uniqueness of fixed points, the reachable sets and the influence on the motion in physical space, as well as a new model which combines the matrix and vector degrees of freedom in an augmented phase space.

Chapter 5 is devoted to quantization schemes of the generalized Calogero-Moser system, one of them being the direct, canonical quantization of the spatial and internal degrees of freedom, and the other being the reduction of a canonically quantized free system in a larger phase space, using the classical procedure as a blueprint.

In chapter 6 I summarize the obtained results together with an outlook on future research, particularly in the areas of quantum chaos and topological effects.

The appendices A and B contain the detailed proofs of some of the results from chapter 4 and 5 respectively.

Chapter 2

Mathematical Tools

In this chapter I provide a review of the mathematical definitions and theorems used in the rest of the thesis. The topics covered here are:

- the basics of Lie groups, Lie algebras and group actions,
- the fundamentals of differential geometry
- symplectic geometry and Hamiltonian mechanics,
- canonical quantization,
- Marsden-Weinstein reduction theorem,
- quantum reduction.

My main goal and focus is to introduce the mathematical tools and justify the methods used throughout the thesis without delving into the depth of the underlying theories. The sources I have used are [56] for Lie algebras, [57, 58] for differential geometry, [59–61] for classical Hamiltonian mechanics, [62] for quantum Hamiltonian mechanics, [63] for canonical quantization, and [64, 65] for reduction procedures.

2.1 Lie groups, Lie algebras and important examples

2.1.1 Basic definitions

Definition 1. *A group is a set G with a composition operation $\cdot : G \times G \to G$ which satisfies the following axioms:*

- *there is an element $e \in G$, called the identity, such that $e \cdot g = g \cdot e = g$ for all $g \in G$,*
- *every element $g \in G$ has an inverse, $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$,*
- *the composition operation is associative, that is $(f \cdot g) \cdot h = f \cdot (g \cdot h) = f \cdot g \cdot h$.*

A group is called Abelian if all elements $g, h \in G$ commute, i.e. $g.h = h.g$. The symbol of the group composition can be omitted to simplify the notation: $fg = f \cdot g$.

Definition 2. *A subset $H \subset G$ is called a subgroup it is closed under the group operations, i.e. for all $g, h \in H$ both $g.h$ and g^{-1} belong to H.*

Groups are especially relevant to physics when they define some operations on a physical object: they translate,rotate or permute its constituents for example. This is captured by the notion of group actions:

Definition 3. *The action of the group G on a set X is a function $\varphi : G \times X \to X$ which satisfies the following axioms:*

- *the identity $e \in G$ acts as identity on every element of X:*

$$\forall_{x \in X} \varphi(e, x) = x$$

- *the actions compose through composition within the group G:*

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x)$$

Depending on what G and X consist of, it may make sense for an element g to act from the left, meaning that if x is itself a map of some type, then g acts on the result of x, that is:

$$\varphi_L(g, x)(;) = g.x(;) = g(x(;)).$$

Another possibility is to act from the right, that is to act on the argument of x, i.e. $x.g(;) = x(g(;))$, but in this case the composition takes place in the different order: $(x.g).h(;) = x.(hg)(;)$, this is why the right action of g which satisfies the second axiom must be defined with the inverse, g^{-1}:

$$\varphi_R(g, x)(;) = x.g^{-1}(;) = x(g^{-1}(;)).$$

Whenever both kinds of action make sense, for example when both G and X are matrices, they can be composed into conjugation:

$$\varphi_L(g, \varphi_R(g, x)) = g.x.g^{-1}.$$

Often the symbols of $\varphi_{L,R}$ and the dot can be omitted, and the usual notation is gx, xg^{-1}, gxg^{-1}.

The following important constructions arise when group actions are considered:

Definition 4. *A stabilizer of a point $x \in X$ is a subset of $G_x \subset G$ such that:*

$$G_x = \{g \in G : g.x = x\}.$$

Every such subset is a subgroup of G.

Definition 5. *An orbit of a point $x \in X$ is a subset $\mathcal{O}_x \subset X$ such that:*

$$\mathcal{O}_x = \{g.x : g \in G\}.$$

Definition 6. *Fixed points of X under the action of G are the following subset:*

$$X^G = \{x \in X : \forall g \in G : g.x = x\}.$$

Definition 7. *The action of a group G on a set X is called faithfull or effective, if*

$$\forall x \in X : \quad g.x = x \implies g = e.$$

Definition 8. *The action of a group G on a set X is called free, if*

$$\exists x \in X : \quad g.x = x \implies g = e.$$

A group can be discreet (like the permutation group) or a continuous meaning that its elements can be parametrised by d continuously varying real numbers. It locally looks like $\mathbb{R}^d$ thus it is a d-dimensional manifold. This leads us to the definition of a Lie group:

Definition 9. *A Lie group is a group, which is also a manifold, and the action of the group on itself by the composition operation is a smooth map.*

Definition 10. *A Lie algebra is a vector space L with a bilinear, antisymmetric operation $[\cdot,\cdot] : L \times L \to L$ which obeys the Jacobi identity:*

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

for all elements $A, B, C \in L$.

Definition 11. *For every Lie group G there is a uniquely defined Lie algebra, which is the tangent space at the identity $\mathfrak{g} = T_eG$.*

A group can act on itself, and a Lie algebra $\mathfrak{g}$, as a vector space is endowed with a dual space $\mathfrak{g}^*$. Due to these two facts, for every group there are two naturally defined actions:

Definition 12. *Every group G acts on itself by conjugation:*

$$g, h \in G : \Psi_g(h) = ghg^{-1}.$$

In case of a Lie group the adjoint action $Ad : G \to Aut(\mathfrak{g})$ can be defined for every $g \in G$ as the derivative of $\Psi_g : G \to G$ at the identity:

$$Ad_g = d\Psi_g : \mathfrak{g} \to \mathfrak{g}.$$

Definition 13. *The coadjoint action of Ad^* on the dual space $\mathfrak{g}^*$ is defined with the use of the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$:*

$$\langle Ad^*_g x, \xi\rangle = \langle x, Ad_{g^{-1}}\xi\rangle.$$

2.1.2 The $\mathfrak{so}(N)$ and $\mathfrak{su}(N)$ Lie algebras

The formulation of the various, both classical and quantum, Calogero-Moser models relies strongly on the properties of the $\mathfrak{so}(N)$ and $\mathfrak{su}(N)$ Lie algebras. Their respective Lie groups, $SO(N)$ and $SU(N)$, are also an important ingredient. The general notion of the Lie algebra was already introduced (def. 10), together with its relation to Lie groups (def. 11). The specific examples which are the subject of this section are introduced through the following definitions:

Definition 14. *The general linear group $G(N;\mathbb{C}(\mathbb{R}))$ is the group of all invertible $N \times N$ matrices with complex (real) entries.*

Definition 15. *A matrix Lie group G is a closed subgroup of $G(N;\mathbb{C})$. It is said to be compact if it is compact as a subset of complex $N \times N$ matrices $M_N(\mathbb{C}) \equiv \mathbb{R}^{2N^2}$. It is said to be connected, if for every two elements $g, h \in G$ there is a continuous path $g : [0,1] \to G$, such that $g(0) = g$ and $g(1) = h$.*

Definition 16. *A Lie algebra $\mathfrak{g}$ of a matrix Lie group G is a set of matrices V such that $\exp(tV) \in G$ for all $t \in \mathbb{R}$.*

The definition 12 of the adjoint action is much easier to grasp in the case of matrix Lie groups, as the derivative can be performed explicitly:

$$Ad_g(V) = \frac{d}{dt}\left(ge^{tV}g^{-1}\right)|_{t=0} = gVg^{-1}.$$

The definition 16 implies that $\mathfrak{g}$ is a vector space over the real numbers and it is closed under the commutation operation:

$$\forall X, Y \in \mathfrak{g}, \forall a, b \in \mathbb{R} \quad aX + bY \in \mathfrak{g}, \quad XY - YX = [X,Y] \in \mathfrak{g}$$

Definition 17. *The special orthogonal and special unitary groups and their underlying Lie algebras are defined as follows:*

$$\begin{aligned} SO(N) &= \left\{O \in G(N;\mathbb{R}) : OO^T = \mathbb{1}, \quad detO = 1\right\} \\ SU(N) &= \left\{U \in G(N;\mathbb{C}) : UU^\dagger = \mathbb{1}, \quad detU = 1\right\} \\ \mathfrak{so}(N) &= \left\{V \in M_N(\mathbb{R}) : V^T = -V, \quad \mathrm{Tr}V = 0\right\} \\ \mathfrak{su}(N) &= \left\{V \in M_N(\mathbb{C}) : V^\dagger = -V, \quad \mathrm{Tr}V = 0\right\} \end{aligned}$$

Groups, and their underlying algebras are abstract sets of objects obeying certain multiplication rules. These objects and rules can be realised with appropriately chosen matrices and a certain realisation is called a representation. The $N \times N$ matrix representation of these algebras, the so called defining representation defined in 17 is of particular importance, since the $(ij)^{th}$ matrix element encodes the repulsion between the i^{th} and j^{th} particle in an N-particle Calogero-Moser system. This is the reason why I am using pairs of numbers as labels for the basis elements defined below.

Using the natural basis of $\mathbb{R}^N$, that is $(\hat{e}_i)_j = \delta_{ij}$, I introduce:

Definition 18. *The basis of $\mathfrak{su}(N)$ consists of $N^2 - 1$ traceless, anti-Hermitian $N \times N$ matrices:*

$$\begin{aligned} \tau_{ij} &= \hat{e}_i\hat{e}_j^T - \hat{e}_j\hat{e}_i^T \\ \sigma_{ij} &= i\left(\hat{e}_i\hat{e}_j^T + \hat{e}_j\hat{e}_i^T\right) \\ d_k &= i\sqrt{\frac{2}{k(k+1)}}\left(\sum_{l=1}^{k}\hat{e}_l\hat{e}_l^T - k\hat{e}_{k+1}\hat{e}_{k+1}^T\right) = \sum_{l=1}^{k+1} i\alpha_{kl}\hat{e}_l\hat{e}_l^T \end{aligned}$$

where $1 \leqslant i < j \leqslant N$ and $1 \leqslant k < N$. The τ_{ij} matrices span $\mathfrak{so}(N)$. The above matrices are orthogonal with respect to $(A, B) = \mathrm{Tr}A^\dagger B$. The only pairs of matrices with a nonvanishing scalar product are:

$$(\tau_{ij}, \tau_{kl}) = (\sigma_{ij}, \sigma_{kl}) = -2\delta_{ik}\delta_{jl}, \qquad (d_k, d_l) = -2\delta_{kl}.$$

All vector spaces of the same finite dimension are isomorphic, but an algebra is more than just a vector space. It has a commutation operation built in, and the commutation relations between the basis elements are what distinguishes one algebra from another:

Definition 19. *The commutation relations between basis elements of $\mathfrak{su}(N)$ are the following:*

$$\begin{aligned} [\tau_{ij}, \tau_{kl}] &= \delta_{jk}\tau_{il} - \delta_{ik}\tau_{jl} - \delta_{jl}\tau_{ik} + \delta_{il}\tau_{jk} \\ [\tau_{ij}, \sigma_{kl}] &= \delta_{jk}\sigma_{il} - \delta_{ik}\sigma_{jl} + \delta_{jl}\sigma_{ik} - \delta_{il}\sigma_{jk} \\ [\sigma_{ij}, \sigma_{kl}] &= -\delta_{jk}\tau_{il} - \delta_{ik}\tau_{jl} - \delta_{jl}\tau_{ik} - \delta_{il}\tau_{jk} \\ [d_k, d_l] &= 0 \\ [d_k, \tau_{ij}] &= \sum_{l=1}^{k+1}\alpha_{kl}(\delta_{il}\sigma_{jl} - \delta_{jl}\sigma_{il}) \\ [d_k, \sigma_{ij}] &= \sum_{l=1}^{k+1}\alpha_{kl}(\delta_{il}\tau_{jl} + \delta_{jl}\tau_{il}) \end{aligned}$$

The subspace spanned by τ_{ij}, the $\mathfrak{so}(N)$ algebra, is closed under the commutation operation.

Definition 20. *Commutation relations between basis elements* $(\tau_1, \tau_2, ..., \tau_d)$ *of a Lie algebra can be presented with the use of structure constants* $f_{ab}^c \in \mathbb{R}$*:*

$$[\tau_a, \tau_b] = \sum_{c=1}^{d} f_{ab}^c \tau_c, \qquad f_{ab}^c = \frac{([\tau_a, \tau_b], \tau_c)}{(\tau_c, \tau_c)}$$

The $d \times d$ *matrices* $(f_1, f_2, ..., f_d)$*, such that* $(f_a)_{bc} = -f_{ab}^c$ *obey the same commutation relations as* $(\tau_1, \tau_2, ..., \tau_d)$*:*

$$[f_a, f_b] = \sum_{c=1}^{d} f_{ab}^c f_c$$

which means they themselves form a representation of the same algebra, the so called adjoint representation.

2.2 Fundamentals of differential geometry

Definition 21. *A real differentiable manifold* M *is a topological space*[1] *with a family of pairs* $\{(U_i, \varphi_i)\}$*, where* U_i *are open sets covering* M*, i.e.* $M = \cup_i U_i$ *and* $\varphi_i : U_i \to U_i' \subset \mathbb{R}^m$ *are homeomorphisms, i.e. continuous bijections with continuous inverses. For* $U_i \cap U_j \neq \emptyset$ *the function* $\varphi_j \circ \varphi_i^{-1}$ *in infinitely differentiable. Each pair* (U_i, φ_i) *is called a chart.*

In simple words, a differentiable manifold defined above resembles, at least locally, the familiar m-dimensional real space $\mathbb{R}^m$. The maps φ_i describe every point $p \in M$ in terms of m real coordinates $\varphi_i(p) = (x_1, ..., x_m)(p)$. On a non-empty overlap $U_i \cap U_j$ of two sets, there are two valid coordinate systems, φ_i and φ_j and the transition between them, that is $\varphi_j \circ \varphi_i^{-1}$ is required to be smooth.

A map $f : M \to N$ between two manifolds can be expressed with the use of charts (U, φ) and (V, ψ), where $p \in U$ and $f(p) \in V$ as:

$$\mathbf{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n.$$

If $\mathbf{f}$ is invertible and both $\mathbf{f}$ and $\mathbf{f}^{-1}$ are smooth, we call the map $f : M \to N$ a diffeomorphism and the manifolds M and N diffeomorphic to each other, which means they are essentially the same. The set of diffeomorphisms $f : M \to M$ of a manifold onto itself, $Diff(M)$, is known to form a group.

As special cases of maps applied to manifolds we distinguish: functions, $f : M \to \mathbb{R}$ and curves, $\gamma : (a, b) \subset \mathbb{R} \to M$.

It is not difficult to imagine a line, which is tangent to a curve, or a plane which is tangent to a curved surface at a single point p. For a general differentiable manifold this idea is generalized by the notion of the tangent space:

Definition 22. *Let* $f, g \in C_p^\infty$*, that is functions which are smooth on a neighbourhood of a point* $p \in M$*. The set of all maps* $v : C_p^\infty \to \mathbb{R}$*, such that:*

- $v(af + bg) = a(vf) + b(vg)$
- $v(fg)(p) = (vf)g(p) + (vg)f(p)$
- $(av + bw)f = a(vf) + b(wf)$

[1] i.e. a most general space on which neighbourhoods, open sets, closed sets and continuity are possible to define

is a vector space called the tangent space to M at the point p, denoted by $T_p(M)$, and its elements are called tangent vectors at p. The set theoretical union of all points with their tangent spaces, $TM = \cup_p(p, T_p(M))$, is called the tangent bundle.

Relative to a chart (U, φ), with $\varphi(p) = (x_1, x_2, ...x_m)(p)$ every element $v \in T_p(M)$ can be expressed as:

$$v = \sum_{i=1}^{m} v_i \frac{\partial}{\partial x_i}, \qquad v(f) = \sum_{i=1}^{m} v_i \frac{\partial (f \circ \varphi^{-1})}{\partial x_i},$$

where the coefficients v_i are uniquely defined by $v_i = v(x_i)$. The coordinate vectors $\left\{ \frac{\partial}{\partial x_i} : i = 1, 2, ..., m \right\}$ form a (so called coordinate) basis of $T_p(M)$ at each point p.

Definition 23. *A cotangent space $T_p^*(M)$ is the dual space to $T_p(M)$. It consists of linear functions $\alpha : T_p(M) \to \mathbb{R}$ and the image in $\mathbb{R}$ is denoted $\alpha(v)$ or $\langle \alpha, v \rangle$. For any function $f : C_p^\infty \to \mathbb{R}$ we define a unique element $df \in T_p^*(M)$ by requiring:*

$$\langle df, v \rangle = v(f)$$

for all $v \in T_p(M)$. From this it can be shown that every $\alpha \in T_p^*(M)$ may be expressed as

$$\alpha = \sum_{j=1}^{m} \alpha^j dx_j, \qquad \langle dx_j, \frac{\partial}{\partial x_i} \rangle = \delta_j^i,$$

in particular $df = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j$. The elements of $T_p^*(M)$ are called 1-forms or covectors at $p \in M$. Again the union of all $(p, T_p^*(M))$, denoted by T^*M is called the cotangent bundle.

Definition 24. *A tangent vector $v \in T_p(M)$ assigned smoothly to every point $p \in M$ defines a vector field, which can be described in the coordinate basis as $V = v_i(p) \frac{\partial}{\partial x_j}$, where the components are now functions of coordinates $(x_1, ..., x_m) = \varphi(p)$. The set of vector fields on M is denoted by $\mathcal{X}(M)$ and is a linear space itself. Moreover, with the commutator $[V, W] = VW - WV$ it forms a Lie algebra.*

Analogously, assigning a 1-form to every point $p \in M$ defines a covector field.

The idea of vector and covector fields can be generalized to tensor fields of arbitrary type (r, s), where $r, s \in \{0, 1, 2...\}$, defined by the transformation rules under the change of coordinate system from $(x_1, ..., x_m)$ to $(y_1, ..., y_m)$:

$$(T')^{l_1 ... l_s}_{j_1 ... j_r} = \left(\frac{\partial y_{j_1}}{\partial x_{h_1}} ... \frac{\partial y_{j_r}}{\partial x_{h_r}} \right) \left(\frac{\partial x_{k_1}}{\partial y_{l_1}} ... \frac{\partial x_{k_s}}{\partial y_{l_s}} \right) T^{k_1 ..., k_s}_{h_1 ... h_r},$$

which makes the vector fields fall into the $(1, 0)$ type, and the 1-forms into the $(0, 1)$ type. In simple words, an (r, s) tensor field has r slots for covector fields and s slots for vector fields and is linear in all $r + s$ arguments. Given these $r + s$ objects it will result in a function.

Definition 25. *A k-form is a $(0, k)$ type completely antisymmetric tensor:*

$$\alpha^{(k)} : T_p(M)^{\times k} \to \mathbb{R}, \qquad \alpha^{(k)}(v_{(1)}, ...v_{(k)}) = (-1)^{\text{sign}(\sigma)} \alpha^{(k)}(v_{(\sigma(1))}, ...v_{(\sigma(k))}.$$

A k-form field assigns a k-form at every point $p \in M$.

The wedge (exterior) product, $\wedge$, is a product of forms, which preserves the antisymmetry condition. A wedge product of a k-form and an l-form results in a (k+l)-form if $k + l \leqslant m$. If $k + l > m$ the exterior product vanishes identically. The direct sum of the spaces $\Lambda^k(T_p(M))$ of k-forms, where $k = 0, 1, ...m$ constitutes an algebra with the wedge product. The coordinate basis of k-forms on an m-dimensional manifold has $\binom{m}{k}$ elements given by:

$$dx_{l_1} \wedge dx_{l_2} \wedge ... \wedge dx_{l_k} : \qquad 1 \leqslant l_1 < l_2 < ... < l_k \leqslant m.$$

Definition 26. *The map called the exterior derivative, $d : \Lambda^k \to \Lambda^{k+1}$ is defined by the following conditions:*

- *for $f \in C^\infty(M) = \Lambda^0$: $df(v) = v(f)$*
- *for $\alpha, \beta \in \Lambda^k$: $d(a\alpha + b\beta) = ad\alpha + bd\beta$*
- *for $\alpha \in \Lambda^k$ and $\beta \in \Lambda^l$: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$*
- *for any $\alpha \in \Lambda^k$: $d^2\alpha = d(d\alpha) = 0$.*

There is also a map from Λ^k to Λ^{k-1}, which requires the existence of a smooth vector field V, at least on the neighbourhood $p \in U \subset M$:

Definition 27. *The interior multiplication of $\alpha \in \Lambda^k$ is denoted by $\iota_V\alpha, \iota(V)\alpha$ or $V \lrcorner \alpha$. It satisfies the following conditions:*

- *for a function (0-form) $f : U \to \mathbb{R}$: $\iota_V f = 0$*
- *for any 1-form $\alpha \in \Lambda^1$: $\iota_V\alpha = \langle \alpha, V \rangle$.*
- *for $\alpha, \beta \in \Lambda^k$: $\iota_V(a\alpha + b\beta) = a\iota_V\alpha + b\iota_V\beta$*
- *for $\alpha \in \Lambda^k$ and $\beta \in \Lambda^l$: $\iota_V(\alpha \wedge \beta) = \iota_V(\alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_V(\beta)$*

The final important operation is the Lie derivative. For an arbitrary tensor field it returns a tensor field of the same type. It also requires a vector field, and may be understood as derivation along the flow of the vector field, i.e. a function $\sigma : (-a, a) \times U \to M$, that describes the curves passing through every $p \in U \subset M$ at $t = 0$ with the velocity given by $V(p) \in T_pM$:

$$\sigma(0, p) = p, \qquad \frac{d}{dt}\sigma(t, p) = V(\sigma(t, p)).$$

It has the following properties:

- when it acts on a function: $\mathcal{L}_V f = V(f)$,
- when it acts on a vector field $\mathcal{L}_V W = [V, W]$,
- when it act on a 1-form $\mathcal{L}_V\alpha = \iota_V d\alpha + d(\iota_V\alpha)$.
- for any two tensors of the same type $\mathcal{L}_V(aT_1 + bT_2) = a\mathcal{L}_V T_1 + b\mathcal{L}_V T_2$,
- for two tensors of arbitrary type $\mathcal{L}_V(T_1 \otimes T_2) = \mathcal{L}_V(T_1) \otimes T_2 + T_1 \otimes \mathcal{L}_V(T_2)$,

from which the expressions in coordinate basis can be derived.

Lastly, for a smooth map $\Phi : M \to N$ between two manifolds two induced maps can be defined:

Definition 28. *Let $f \in C^\infty_q$, where $q = \Phi(p)$ for some $p \in M$. Then $F = f \circ \Phi \in C^\infty_p$. An induced map Φ_* called the push-forward is defined as follows:*

$$\Phi_* : T_pM \to T_qN, \qquad \forall_{V \in T_pM}(\Phi_* V)(f) = V(f \circ \Phi).$$

Definition 29. *An induced map called the pull-back Φ^* is defined in the following way:*

$$\Phi^* : T^*_q N \to T^*_p M, \qquad \forall_{\alpha \in T^*_q N} \forall_{V \in T_pM}(\Phi^*\alpha)(V) = \alpha(\Phi_* V).$$

It can be generalized to k-forms in the following way:

$$(\Phi^*\alpha^{(k)})(V_1, ..., V_k) = \alpha^{(k)}(\Phi_* V_1, ..., \Phi_* V_k).$$

2.3 Hamiltonian mechanics

The generalised Calogero-Moser systems studied in this thesis are formulated within the Hamiltonian formalism. The Hamiltonian formulation provides us with the notion of a phase space, in which we can look at the trajectories traced by the coordinates (usually the positions and momenta) from a purely geometrical point of view. This point of view, expressed in the language of symplectic geometry, is reviewed below. The formalism of Hamiltonian dynamics is also necessary for us to transition to the (non-relativistic) quantum world. The quantum Hamiltonian formulation and the canonical quantization procedure are also reviewed in this section.

2.3.1 Classical Hamiltonian mechanics

The phase space, that is the space of all states of a mechanical system is an example of a symplectic manifold.

Definition 30. *Let M^{2n} be a differentiable manifold of even dimension. A symplectic structure on M^{2n} is a closed, nondegenerate 2-form ω on M^{2n}:*

$$d\omega = 0, \quad \forall v \neq 0 \exists w : \omega(v,w) \neq 0, \quad (v, w \in T_x M) \tag{2.1}$$

The pair (M^{2n}, ω) is called a symplectic manifold.

The space R^{2n} with coordinates $(\bar{q}, \bar{p}) = (q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ and $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ is an example of a symplectic manifold. The coordinates $\bar{q} \in \mathbb{R}^n$ are the available positions, and $\bar{p} \in \mathbb{R}^n$ are the available momenta. Yet the available positions, that is the configuration space, need not to be the full Euclidean space. They may be constrained to a different n-dimensional manifold (like a circle or sphere in the case of a pendulum). Let us call it V, the configuration space. Then the cotangent bundle T^*V with coordinates $(\bar{q}, \bar{p})$, where $\bar{q} \in V$, $\bar{p} \in T^*_{\bar{q}} V$ and with the form $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ is a symplectic manifold. It is also the phase space of a system with the configuration space V. The cotangent bundle of the configuration space covers most of the cases relevant in physics. Yet for a completely general case of a symplectic manifold there is the Darboux Theorem which states that locally there always exists a coordinate system $(\bar{q}, \bar{p})$ in which the symplectic form can be written as $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. From now on I will express every introduced object in (q, p) coordinates to stay in touch with its physical meaning.

Definition 31. *There is an isomorphism of tangent vectors and 1-forms defined by the symplectic structure (M^{2n}, ω). To every $v \in T_x M$ we assign a 1-form $\omega_v \in T^*_x M$ such that:*

$$\forall w \in T_x M : \omega_v(w) = \omega(w, v). \tag{2.2}$$

We denote the inverse of this isomorphism as $I : T^*_x M \to T_x M$.

Definition 32. *Consider a function $H : M \to R$. Then dH is a 1-form on M, and $I(dH)$ is a vector field on M. We call $V_H = I(dH)$ a Hamiltonian vector field, and H itself - the Hamilton function.*

Let us express the Hamiltonian vector field in coordinates $(\bar{q}, \bar{p})$. For $\omega = d\bar{p} \wedge$

$d\bar{q}$,and $H : M \to R$ we have:

$$\begin{aligned} dH &= \frac{\partial H}{\partial \bar{q}} d\bar{q} + \frac{\partial H}{\partial \bar{p}} d\bar{p} \\ V_H &= h_{\bar{q}} \frac{\partial}{\partial \bar{q}} + h_{\bar{p}} \frac{\partial}{\partial \bar{p}}, \\ v &= v_{\bar{q}} \frac{\partial}{\partial \bar{q}} + v_{\bar{p}} \frac{\partial}{\partial \bar{p}}, \\ \omega(v, I(dH)) &= dH(v) \\ -h_{\bar{p}} v_{\bar{q}} + h_{\bar{q}} v_{\bar{p}} &= \frac{\partial H}{\partial \bar{q}} v_{\bar{q}} + \frac{\partial H}{\partial \bar{p}} v_{\bar{p}} \\ V_H &= \frac{\partial H}{\partial \bar{p}} \frac{\partial}{\partial \bar{q}} - \frac{\partial H}{\partial \bar{q}} \frac{\partial}{\partial \bar{p}}. \end{aligned}$$

Having a vector field, we may ask what are its integral curves $\gamma : \mathbb{R} \to M$. In the case of a Hamiltonian vector field:

$$\begin{aligned} \gamma(t) &= (\bar{q}(t), \bar{p}(t)) \\ \frac{d\gamma(t)}{dt} &= V_H(\gamma(t)) \\ \frac{d}{dt} \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} &= \begin{pmatrix} \frac{\partial H}{\partial \bar{p}} \\ -\frac{\partial H}{\partial \bar{q}} \end{pmatrix} \end{aligned}$$

they are the solutions of the Hamilton canonical equations.

Definition 33. *Equivalently to the integral curves we can think of a group of one-parameter diffeomorphisms $\gamma^t : M \to M$ defined by the Hamiltonian vector field:*

$$\frac{d}{dt} \gamma^t(x)|_{t=0} = V_H(x).$$

This group is called the Hamiltonian phase flow with Hamilton function H, and will be denoted as γ_H^t.

Clearly, the value of H is, by definition, constant along the integral curves of V_H, in other words H is the co called first integral of the system of ordinary differential equations defined by the Hamiltonian vector field. But there can be other, nontrivially different first integrals of V_H, and we need to know how to identify them. This motivates the introduction of the Poisson bracket:

Definition 34. *Let $F, H : M \to M$ be two functions on a symplectic manifold M. The Poisson bracket of F and H is defined as the derivative of F along the Hamiltonian flow of γ_H^t:*

$$\{F, H\}(x) = \frac{d}{dt} F(\gamma_H^t(x))|_{t=0}$$

which is again a function on M.

This definition implies that the Poisson bracket obeys the Leibniz rule:

$$\{F_1 F_2, H\} = F_1 \{F_2, H\} + F_2 \{F_1, H\}.$$

It also indicates that a function F is a first integral of the Hamiltonian flow γ_H^t if and only if the Poisson bracket $\{F, H\}$ vanishes. Using the definitions of γ_H^t and the isomorphism I, we may write two other, equivalent definitions of the Poisson bracket:

$$\{F, H\} = dF(V_H) = \omega(V_H, V_F).$$

The latter makes it apparent that the Poisson bracket is billinear and skew-symmetric in F and H. It allows for an efficient calculation of the Poisson bracket in (q,p) coordinates:

$$\{F,H\} = \omega\left(\frac{\partial H}{\partial \bar{p}}\frac{\partial}{\partial \bar{q}} - \frac{\partial H}{\partial \bar{q}}\frac{\partial}{\partial \bar{p}}, \frac{\partial F}{\partial \bar{p}}\frac{\partial}{\partial \bar{q}} - \frac{\partial F}{\partial \bar{q}}\frac{\partial}{\partial \bar{p}}\right) = \frac{\partial F}{\partial \bar{q}}\frac{\partial H}{\partial \bar{p}} - \frac{\partial F}{\partial \bar{p}}\frac{\partial H}{\partial \bar{q}}$$

The above expression (or some more general results about Lie derivatives) can be used to prove, that the Poisson bracket obeys the Jacobi identity:

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0.$$

Recalling the definiton of a Lie algebra, 10 we clearly see, that the Hamilton functions on a symplectic manifold form a Lie algebra,and the operation in this algebra is given by the Poisson bracket. Moreover, the first integrals of the Hamiltonian flow form its subalgebra (that is a subset, which is also an algebra and is closed under the algebra operation).

It is important to mention, that vector fields on a manifold also form a Lie algebra, with the operation called a Lie bracket or commutator. They act as derivatives of functions along the integral curves of the field. In particular, we have the Hamiltonian vector fields on a symplectic manifold, and they act in the following way:

$$[V_F, V_H] = L_{V_H}(F)(x) = \frac{d}{dt}F(\gamma_H^t(x))|_{t=0} = \{F,H\}(x), \qquad x \in M.$$

Integrability

The standard physical interpretation of the constructions presented above is that the function $H : M \to \mathbb{R}$ represents the total energy of the system. The equations for integral curves $\gamma : \mathbb{R} \to M$ of the Hamiltonian vector field associated with H parametrised with time,

$$\gamma(t) = (\bar{q}(t), \bar{p}(t)), \qquad \dot{\gamma}(t) = V_H(\gamma(t)), \tag{2.3}$$

can be rewritten equivalently as the Hamilton canonical equations[2]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.4}$$

Their solutions tell us what happens with the positions and momenta of the bodies the system defined by H consists of, given the initial conditions. This, among many other reasons, may motivate us to ask, is the single H function giving us enough data to solve the equations (2.4)? Or, more generally, we may ask, how many first integrals do we need to know in order to solve the system of differential equations by quadratures[3]? In other words: what are the conditions for integrability of (2.4)? Let us first consider a general dynamical system:

$$\dot{x}_i = f_i(x_1, x_2, ..., x_N), \qquad \bar{x}(0) = (x_{1,0}, x_{2,0}, ..., x_{N,0}). \tag{2.5}$$

The existence of r constants of motion $(F_1, F_2, ..., F_r)$, which are functionally independent, that is $dF_1 \wedge dF_2 \wedge ... \wedge dF_r \neq 0$ reduces the dimensionality of the problem from N to $N-r$. The way it works is that each constant of motion has a defined value $F_j = F_j(\bar{x}(0))$ at the initial $\bar{x}(0)$, defining an $N-1$ dimensional surface (so

[2] In the case of $H = \frac{1}{2m}\sum p_i^2 + V(x_1, ..., x_N)$ they may be directly transformed into Newton's equations of motion.

[3] i.e. through algebraic operations, as well as taking integrals and partial derivatives

called leaf) of points the integral curve of (2.5) is allowed to pass through. Intersecting r such leafs decreases the dimensionality of the allowed submanifold to $N-r$. In particular if $r = N-1$ this results in a curve which is in fact the solution of (2.5), perhaps up to reparametrisation. One can also think of the vector field $V = \sum_{i=1}^{N} f_i \frac{\partial}{\partial x_i}$ (the solutions of (2.5) are its integral curves) and search for vector fields X_a, where $a = 1, ..., s$, such that $[X_a, V] = [X_a, X_b] = 0$. Having such vector fields, one may change the coordinates to $(y_1, ..., y_N)$, where:

$$X_a = \frac{\partial}{\partial y_a}, \qquad a = 1, ..., s, \qquad V = \sum_{i=s+1}^{N} g_i(\bar{y}) \frac{\partial}{\partial y_i}. \tag{2.6}$$

and (2.5) in these coordinates turns out to automatically have s constants of motion:

$$\dot{y}_i = 0, \qquad i = 1, ..., s, \qquad \dot{y}_i = g_i(y_1, ..., y_N) \qquad i = s+1, ..., N \tag{2.7}$$

Having r constants of motion and $s = N-r-1$ vector fields (such that $X_i(F_j) = 0$),that is all in all $N-1$ pieces of information (or N if we are counting V as well) makes the system (2.5) integrable.

Does this mean that the integrability of (2.4), which is a special case of (2.5) with $N = 2n$ needs indeed $2n-1$ pieces of information (i.e. independent constants of motion or commuting vector fields)? The Arnold-Liouville theorem [60] states, that n functionally independent constants of motion, $(F_1 = H, F_2, ..., F_n)$ are enough for the integrability of (2.4), but the additional condition is, that they have to be in involution, that is their mutual Poisson brackets must vanish:

$$\{F_i, F_j\} = 0, \qquad i, j = 1, 2, ..., n. \tag{2.8}$$

In case of a system with many degrees of freedom this is a significant improvement. The reason for it lies in the symplectic structure. Having n constants of motion we may of course limit the search of the integral curve of (2.4) to an $2n - n = n$ dimensional submanifold of M^{2n} given by the intersection of level sets containing the initial $(\bar{q}, \bar{p})(0)$. But thanks to the symplectic structure M is endowed with, we may also construct the Hamiltonian vector fields V_{F_i} corresponding to each constant of motion. The tangency condition is fulfilled due to the vanishing Poisson brackets $V_{F_i}(F_j) = \{F_i, F_j\} = 0$. The $n-1$ Hamiltonian vector fields define the coordinates $V_{F_i} = \frac{\partial}{\partial y_i}$ in which $n-1$ equations have the form $\dot{y}_i = 0$ and a simple solution $y_i(t) = y_i(0)$. These $n-1$ constants of motion carve out an integral curve in the n-dimensional submanifold of M we got from the intersection of level sets $F_i(\bar{q}(t), \bar{p}(t)) = F_i(\bar{q}(0), \bar{p}(0))$.

2.3.2 Quantum Hamiltonian mechanics

The classical trajectories in phase space provide us with very precise information about the physical objects: where they are and how fast they move in every instant of time. Quantum mechanics, on the other hand, offers us probability distributions of observing different results,together with their time evolution.[4] The description of the quantum world is realised through normalized state vectors $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space defined by the degrees of freedom present in the system.

Definition 35. *A Hilbert space $\mathcal{H}$ is a complex vector space with a Hermitian product. Finite dimensional Hilbert spaces used to describe the spin degree of freedom are isomorphic to $\mathbb{C}^n$. Infinite dimensional Hilbert spaces used to describe particles in a configuration space $\mathcal{M}$ are the $L^2(\mathcal{M})$ spaces of square-integrable functions.*

[4] These probability distributions converge to classical trajectories in the limit of the Planc constant $h \approx 6.6 \cdot 10^{-34} Js$ being negligible in comparison with the values of action typical for the system.

Physical quantities are represented by linear Hermitian operators, so called observables acting on vectors from $\mathcal{H}$. Each such operator has a diagonalizing orthonormal basis:

$$\hat{A} = \sum_{i=1}^{n} a_i |\alpha_i\rangle\langle\alpha_i|, \quad \langle\alpha_i|\alpha_j\rangle = \delta_{ij}, \quad \sum_{i=1}^{n} |\alpha_i\rangle\langle\alpha_i| = \mathbb{1} \tag{2.9}$$

$$|\psi\rangle = \sum_{i=1}^{n} |\alpha_i\rangle\langle\alpha_i|\psi\rangle = \sum_{i=1}^{n} \psi_{a,i}|\alpha_i\rangle, \quad \hat{A}|\psi\rangle = \sum_{i=1}^{n} a_i\psi_{a,i}|\alpha_i\rangle \tag{2.10}$$

A measurement of the quantity A represented by $\hat{A}$ is realized as a projection of the state vector on one of the eigenspaces of $\hat{A}$. Each coefficient $\psi_{a,i} \in \mathbb{C}$ is the probability amplitude, and $p_{a,i} = |\psi_{a,i}|^2$ is the probability of obtaining the result a_i in a measurement of the quantity A taken for a system in the state $|\psi\rangle$. This probabilistic interpretation allows for calculating the expectation value of the quantity A in the state $|\psi\rangle$, that is $\langle\psi|\hat{A}|\psi\rangle = \sum_{i=1}^{n} a_i p_{a,i}$, as well as its higher moments. An experiment, on the other hand would involve preparing the quantum system in the quantum state $|\psi\rangle$, repeating the measurement of A many times and gathering a sample of $\{a_1, ..., a_n\}$ values comparable with the theoretical predictions. Probabilities must sum to unity (we are certain obtain one of the results) and this imposes the requirement for the state vectors to be normalized to unity:

$$\langle\psi|\psi\rangle = \sum_{i,j=1}^{n} \psi^*_{a,j}\psi_{a,i}\langle\alpha_j|\alpha_i\rangle = \sum_{i=1}^{n} p_{a,i} = 1 \tag{2.11}$$

It is important to stress, that even if we are interested in a single observable $\hat{A}$, it may happen that the eigenspaces of $\hat{A}$ are more than one-dimensional and within each such eigenspace we may want to know a complete decomposition of the state $|\psi\rangle$ into orthogonal projections on one-dimensional subspaces. What is needed is a complete set of commuting observables, $\hat{A}_i, i = 1, 2, ..., I$ which may be seen as a quantum analogy of the complete set of first integrals mentioned in section 2.3.1.

All the expansions above were discrete and finite, but they can be easily generalised to infinite sums. The generalization to the continuous case, that is to the observables with a continuous spectrum is worth taking a closer look.

The continuous spectrum

The most important examples of observables with a continuous spectrum are position and momentum. In the simplest case of a single spinless particle on a real line the state is represented by the wavefunction $\psi(x) = \langle x|\psi\rangle$:

$$\hat{x} = \int x|x\rangle\langle x|dx, \quad \langle x'|x\rangle = \delta(x' - x), \quad \int |x\rangle\langle x| = \mathbb{1} \tag{2.12}$$

$$|\psi\rangle = \int |x\rangle\langle x|\psi\rangle = \int \psi(x)|x\rangle dx, \quad \psi(x) = \langle x|\psi\rangle, \tag{2.13}$$

$$\hat{x}\psi(x) = x\psi(x), \quad \hat{p}\psi(x) = i\hbar\frac{d\psi(x)}{dx}, \quad \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{2.14}$$

The wavefunctions are elements of the infinite dimensional Hilbert space $L^2(\mathbb{R})$, that is the space of square-integrable functions. The value $|\psi(x)|^2$ is the probability distribution, therefore the probability of finding the particle in the segment $[a, b] \subset \mathbb{R}$ (for example some screen or detector) is given by $\int_a^b |\psi(x)|^2 dx$. The relationship between the position and momentum operator:

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{1} \tag{2.15}$$

lies at the heart of quantum uncertainty on one hand, and is the sibling of the classical conjugation relation $\{x, p\} = 1$ on the other.

Many-body quantum systems on a real line

This thesis is focused on many-body systems confined to a real line. Usually in the many-body case the problem of indistinguishability of quantum particles arises. In the Calogero-Moser systems the particles repel with a $1/x^2$ potential which makes the tunneling between states with exchanged pairs of particles impossible, and allows us to consider the wavefunctions on sectors of configuration space with fixed ordering of particles, for example $x_1 < x_2 < ... < x_N$.

The time evolution of states and observables

The time evolution of the state is given by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \tag{2.16}$$

where the operator $\hat{H}$ is the Hamiltonian, the operator associated with the energy of the system. In the case of a time independent $\hat{H}$ the solution of this equation is straightforward:

$$|\psi(t)\rangle = e^{-i\frac{t\hat{H}}{\hbar}}|\psi(0)\rangle = \hat{U}(t)|\psi(0)\rangle, \qquad \hat{H} = \sum_{i=1}^{n} E_i |\psi_i\rangle\langle\psi_i| \tag{2.17}$$

$$|\psi(0)\rangle = \sum_{i=1}^{n} \psi_{E,i}|\psi_i\rangle, \qquad |\psi(t)\rangle = \sum_{i=1}^{n} \psi_{E,i} e^{-i\frac{tE_i}{\hbar}}|\psi_i\rangle, \tag{2.18}$$

where the eigenstates of the Hamiltonian are called stationary states, since they only acquire a phase $|\psi_{E,i}\rangle \to e^{-i\frac{E_i t}{\hbar}}|\psi_{E,i}\rangle$ as time t passes. The task is more complicated when the Hamiltonian depends explicitly on time, but in essence it is still the problem of finding the unitary operator $\hat{U}(t)$ which takes the state $|\psi(0)\rangle$ to the state $|\psi(t)\rangle$.

The information about the quantum system is not accessed directly from the state vector $|\psi(t)\rangle$, but from the expectation values of observables calculated in this state. This is why we may just as well as treat the states vectors as evolving in time and operators as constant, attribute the time evolution to operators:

$$\langle\hat{A}\rangle(t) = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi|\hat{U}^\dagger(t)\hat{A}\hat{U}(t)|\psi\rangle = \langle\psi|\hat{A}(t)|\psi\rangle. \tag{2.19}$$

The description of state vectors as changing in time is called Schrödinger picture, while the one with evolving operators carries the name of Heisenberg. Moreover, as we have the Schrödinger equation for time evolution of state vectors, we have the Heisenberg equation for evolving operators:

$$\frac{d}{dt}\hat{A}(t) = \frac{1}{i\hbar}\left[\hat{A}(t), \hat{H}\right]. \tag{2.20}$$

2.3.3 Canonical quantization

The equation (2.20) resembles the equation for the time dependence of phase space functions along the integral curves generated by the Hamilton function:

$$\frac{df}{dt} = \{f, H\} \tag{2.21}$$

and as it was stated in section 2.3.1, functions defined on the phase space form a Lie algebra with the Poisson bracket as the algebra multiplication. This motivates the following approach towards quantizing a classical Hamiltonian system: let us take the algebra of relevant functions and define its representation Γ on a Hilbert space.

For two functions f, g this would mean a possibility of defining operators $\hat{f}, \hat{g}$, such that:

$$\Gamma(f) = \hat{f}, \quad \Gamma(g) = \hat{g}, \quad [\Gamma(f), \Gamma(g)] = i\hbar\Gamma(\{f, g\}). \tag{2.22}$$

The correspondence with classical mechanics should serve as a guide to obtain a canonical definition of the map Γ. The simplest example of an algebra of functions on phase space M of a system with n degrees of freedom is spanned by $(q_i, p_j, 1)$, where $i, j = 1, 2, ..., n$, that is the Heisenberg algebra with commutation relations $\{q_i, p_j\} = \delta_{ij} \cdot 1, \{q_i, 1\} = \{p_j, 1\} = 0$. Its representation is proven to be unique (up to unitary equivalence) by the Stone-von Neumann theorem [66] and given by operators acting on $L^2(\mathbb{R}^n)$ as:

$$\Gamma(q_i) = \hat{q}_i = q_i, \quad \Gamma(p_j) = \hat{p}_j = -i\hbar\frac{\partial}{\partial q_j}, \tag{2.23}$$

$$\Gamma(\{q_i, p_j\}) = \Gamma(\delta_{ij} \cdot 1) = [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbb{1}. \tag{2.24}$$

This algebra is clearly too small to model any physical system: it does not contain any reasonable Hamilton functions, not even the one of a free system. The question is how can this set be extended so that the map Γ is still well defined. Degree two polynomials of q_i and p_j may be included:

$$\Gamma(q_i q_j) = \hat{q}_i\hat{q}_j, \quad \Gamma(p_i p_j) = \hat{p}_i\hat{p}_j, \quad \Gamma(q_i p_j) = \frac{1}{2}\left(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\right). \tag{2.25}$$

If $i \neq j$ the operators commute and their order does not matter, but in case of $i = j$ the symmetrisation of $\Gamma(q_i p_i)$ is important, as one can check for example by comparison of $\{q_i^2, p_i^2\}$ and $[\hat{q}_i^2, \hat{p}_i^2]$. This allows the inclusion of the Hamilton functions of n free particles or harmonic oscillators. In fact Hamilton functions in a form:

$$H(\bar{q}, \bar{p}) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(q_1, ..., q_n) \tag{2.26}$$

allow for a consistent Γ representation which meets the condition (2.22). As stated by the Groenewold-van Hove theorem [67], higher order terms including both conjugate variables, such as $q_i p_i^2$ do not admit an unambiguous result of the Γ map. The problem lies in the ordering of operators in such a way, that (2.22) would be met for all the functions in the algebra. The discrepancies which arise due to this ambiguity are of order of $\hbar^k$, where $k > 1$, so they may be considered small of higher order than the commutators, and they clearly vanish faster than the commutators in the classical limit. Yet we must bear in mind, that the quantization procedure is hardly ever unambiguous. After all it is an attempt to guess a richer model from its limit, while some information was lost when the limit was taken. In fact different plausible quantum models could in principle have the same classical limit, and some examples of this fact are present in this thesis.

2.4 Symplectic reduction

Reduction procedures serve as a method of obtaining complicated dynamical systems from simple ones. They are vital to obtaining many-body interacting systems, which are also completely integrable, such as the Calogero-Moser system. The necessary ingredients for such a procedure are an equation of motion on a carrier space M, such as the phase space for example, giving rise to a solution $\Phi : \mathbb{R} \times M \to M$, an invariant submanifold $\Sigma \subset M$, i.e. $\Phi(R \times \Sigma) \subset \Sigma$ and an equivalence relation between points of Σ, such that if $m \equiv m'$, then $\Phi(\mathbb{R}, m) \equiv \Phi(\mathbb{R}, m')$. The reduced dynamics is defined on the manifold of equivalence classes. The submanifold Σ introduces

nonlinearities and couplings between the remaining degrees of freedom. The simplest prototype of this procedure involves a free particle moving in $\mathbb{R}^3$. Its motion is of course $\bar{r}(t) = \bar{r}(0) + t\dot{\bar{r}}(0)$, but instead of looking at the Cartesian components of $\bar{r}(t)$, we may consider the equations of motion in the spherical coordinate system, where $\bar{r} = r\bar{n}$, $\bar{n} \cdot \bar{n} = 1$ and $r > 0$. The equations of motion simplify to $\ddot{r} = r\left(\dot{\bar{n}}\right)^2$, the equivalence relation is given by the action of the rotation group $SO(3)$ on initial positions and momenta, and the invariant submanifold can be chosen to have a fixed value of angular momentum. This results in a condition $l^2 = r^4\left(\dot{\bar{n}}\right)^2$ and an equation $\ddot{r} = \frac{l^2}{r^3}$ for $r(t)$.

In this section I am reviewing the special case of Hamiltonian reduction, when the equivalence relation between points of the phase space is given by the action of a compact and connected Lie group. The quantum counterpart of this procedure is briefly summarised as well.

2.4.1 Group action on a symplectic manifold

The group action (def. 3), together with the notions of a stabilizer subgroup,orbit and fixed points (def.4-6) were defined for an unspecified set. When this set is a manifold, a vector field can be associated with every element of the Lie algebra $\mathfrak{g}$:

Definition 36. *For an action $a : G \times M \to M$ of a Lie group G on a manifold M, we may consider a map $a_m : G \to M$ and its derivative $da_m : \mathfrak{g} \to T_mM$, which maps an element of the Lie algebra to a vector attached at $m \in M$. Applying this definition at every point m we obtain a map from $\mathfrak{g}$ to vector fields $\mathcal{X}(M)$, and for every ξ the resulting vector field ξ_M is called the fundamental vector field.*

The fundamental vector field ξ_M has the following geometrical interpretation: an element $\xi \in \mathfrak{g}$ generates trajectories in M trough the action a. These trajectories are the integral curves of the fundamental vector field ξ_M.

When a group is acting on two manifolds, a map between the manifolds can be compatible with the group action:

Definition 37. *A map $f : M \to N$ between two manifolds a group G is acting on is called equivarant if:*

$$\forall g \in G, m \in M : f(g.m) = g.f(m).$$

One of the central notions which arise when a Lie group G is acting on a symplectic manifold (M, ω) is the moment map:

Definition 38. *A moment map $\mu : M \to \mathfrak{g}^*$ is a map such that:*

$$\forall \xi \in \mathfrak{g} : \omega(;, \xi_M) = d\langle \mu, \xi \rangle$$

A tautological example of a moment map would be simply the Hamilton function for $G = \mathbb{R}$ acting on M by translations in time, that is

$$t.(\bar{q}(t_0), \bar{p}(t_0)) = (\bar{q}(t_0 + t), \bar{p}(t_0 + t)).$$

Other examples are the momentum $\bar{p}$ for $G = \mathbb{R}^n$ acting by translations $\bar{a}.(\bar{q}, \bar{p}) = (\bar{q} + \bar{a}, \bar{p})$ and angular momentum for $G = SO(3)$ acting on $M = \mathbb{R}^6$ by rotations on positions and momenta: $O.(\bar{q}, \bar{p}) = (O\bar{q}, O\bar{p})$. It has to be stressed that not every action admits a moment map. The above definition of μ could be rephrased: the fundamental vector field ξ_M is the Hamiltonian vector field of the function $\langle \mu(;), \xi \rangle : M \to \mathbb{R}$, and not all vector fields are Hamiltonian. This distinguishes a special type of actions together with the symplectic structures they are applied to:

Definition 39. *A Hamiltonian G-space is a symplectic structure (M, ω) with an equivariant moment map, i.e.*

$$\forall g \in G, m \in M : \mu(g.m) = Ad_g^*(\mu(m)).$$

The equivariance of the moment map implies that for two elements $\xi, \eta \in \mathfrak{g}$ of the Lie algebra:

$$\{\langle \mu, \xi \rangle, \langle \mu, \eta \rangle\} = \langle \mu, [\eta, \xi] \rangle,$$

which means that μ gives rise to a Lie algebra homomorphism between $\mathfrak{g}$ and functions on M under the Poisson bracket.

In fact the moment map can be understood as a Hamiltonian version of the Noether theorem. Let us consider a Hamilton function H which is invariant under a group action:

$$H(g.(\bar{q}, \bar{p})) = H(e^{\lambda \xi}.(\bar{q}, \bar{p})) = H(\bar{q}, \bar{p}).$$

This means that:

$$0 = \frac{d}{d\lambda} H(e^{\lambda \xi}.(\bar{q}, \bar{p}))|_{\lambda=0} = dH(\xi_M) = \omega(\xi_M, V_H) = -(d\langle \mu, \xi \rangle)(V_H) = \{H, \langle \mu, \xi \rangle\},$$

that is $\langle \mu, \xi \rangle$ is a constant of motion.

2.4.2 The Marsden-Weinstein Theorem

One of the central applications of the Hamiltonian G-spaces and moment maps is the Marsden-Weinstein reduction theorem [68]:

Theorem. *Let $(G \circlearrowright M, \omega, \mu)$ be a Hamiltonian G-space. For a regular value $x \in \mathfrak{g}^*$ assume the action of the coadjoint stabilizer G_x on $\mu^{-1}(x)$ is free. Define the embedding*

$$i_x : \mu^{-1}(x) \hookrightarrow M,$$

and the projection

$$\pi_x : \mu^{-1}(x) \to M_x = \mu^{-1}(x)/G.$$

Then (M_x, ω_x) is a symplectic structure, where ω_x defined as:

$$i_x^* \omega = \pi_x^* \omega_x$$

is unique.

This theorem fits the general scheme described in the beginning of this section: having $\Phi(t, (\bar{q}_0, \bar{p}_0))$ as the integral curves of V_H passing through $(\bar{q}_0, \bar{p}_0) \in M$ and a group action which leaves the Hamilton function unchanged, we choose the invariant submanifold by choosing $x \in \mathfrak{g}^*$ and we have the equivalence relation between points in M given by the group action. What is special about the reduction applied to a Hamiltonian G-space is that the resulting dynamics takes place in a symplectic structure. This is especially fruitful for the physical interpretation of the reduced systems.

Quantum reduction

The simplistic picture of the quantum Hamiltonian reduction is that the functions $\mu_\xi = \langle \mu, \xi \rangle$ can be quantized as discussed in the previous section, together with the Hamilton function, and that the Poisson brackets will translate to commutators, as usual. Then the domain on which the quantum operators act is restricted to a quotient domain with respect to the group action. On the other hand a classically

reduced system could also be quantized, although the reduced symplectic manifold M_x is rarely a familiar $\mathbb{R}^{2n}$ phase space, for which it is clear how to define a Hilbert space. The conjecture that reduction and quantization commute i.e. result in the same space of quantum states no matter the order in which they are carried out, was first presented in [69] and proven in [70].

Chapter 3

Generalised Calogero-Moser system preliminaries

The simplest case of the ordinary CM system

$$\mathcal{H}_{CM,g} = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{g^2}{(x_i - x_j)^2}. \tag{3.1}$$

can be extended to arbitrary values of coupling constants

$$\mathcal{H}_{CM,g} = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{g_{ij}^2}{(x_i - x_j)^2} \tag{3.2}$$

mimicking some artificial charge carried by the particles for example. A further generalisation, which is the focus of this thesis consists of additional dynamical variables, sort of internal degrees of freedom instead of coupling constants:

$$\mathcal{H}_{CM,g} = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{\Lambda_{ij}(t)}{(x_i - x_j)^2} \tag{3.3}$$

In this chapter I am reviewing the various formulations of the latter generalisation and some results which were the foundation for my research.

3.1 Classical generalizations of Calogero-Moser systems

3.1.1 Matrix Calogero-Moser system

One of the ways of obtaining a system described by (3.3) is the reduction of a linear Hamiltonian system on the space of matrices [10]. The configuration space in this case is the linear space of $N \times N$ complex Hermitian matrices: $\mathcal{M} = \{X \in M_N(\mathbb{C}) : X^\dagger = X\}$. The corresponding symplectic structure consists of the space $M = T^*\mathcal{M} = \mathcal{M} \times \mathcal{M} = \{(X, Y) : X^\dagger = X, Y^\dagger = Y\}$, and the canonical symplectic form:

$$\omega = \sum_{i,j} dY_{ij} \wedge dX_{ji} = \mathrm{Tr}(dY \wedge dX). \tag{3.4}$$

The Poisson brackets are constructed as stated in section 2.3.1:

$$\{f, g\} = \sum_{i,j}\frac{\partial f}{\partial X_{ij}}\frac{\partial g}{\partial Y_{ji}} - \frac{\partial g}{\partial X_{ij}}\frac{\partial f}{\partial Y_{ji}} = \mathrm{Tr}\left(\frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial g}{\partial X}\frac{\partial f}{\partial Y}\right), \tag{3.5}$$

and the equation of motion for an arbitrary phase-space function f generated by a Hamilton function $\mathcal{H} : M \to \mathbb{R}$ reads, in terms of the Poisson bracket, as

$$\dot{f} = \{f, \mathcal{H}\}. \tag{3.6}$$

Let us consider the following Hamilton function on M:

$$\mathcal{H} = \frac{1}{2}\mathrm{Tr}YY^\dagger = \frac{1}{2}\mathrm{Tr}Y^2. \tag{3.7}$$

The corresponding equations of motion,

$$\dot{X} = Y, \quad \dot{Y} = 0, \tag{3.8}$$

have a simple solution,

$$X(t) = X_0 + t \cdot Y_0, \quad Y(t) = Y_0, \tag{3.9}$$

which is just a straight line in the space of Hermitian matrices. I will now express $X(t)$ and $Y(t)$ in the diagonalising basis of of $X(t)$ (assuming $X(0) = X_0$ is diagonal):

$$U(t)\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} U(t)^\dagger = \begin{pmatrix} D(t) \\ V(t) \end{pmatrix}, \tag{3.10}$$

with $D(t)$ diagonal. Denoting $A := \dot{U}U^\dagger \in \mathfrak{u}(N)$ and $L := [D, V] = U(t)[X_0, Y_0]U(t)^\dagger \in \mathfrak{su}(N)$ leads to the following equations of motion

$$\dot{D} = V + [A, D], \quad \dot{V} = [A, V], \quad \dot{L} = [A, L]. \tag{3.11}$$

Relabeling $x_i = X_{ii}$, and $p_i = V_{ii}$ and eliminating the entries of A as $A_{ij} = L_{ij}/(D_i - D_j)^2$ results in a system of nonlinear differential equations:

$$\dot{x}_i = p_i \tag{3.12}$$

$$\dot{p}_i = \sum_{k\neq i} \frac{-2L_{ik}L_{ki}}{(x_i - x_k)^3} \tag{3.13}$$

$$\dot{L}_{ij} = \sum_{k\neq i,j} L_{ik}L_{kj}\left(\frac{1}{(x_i - x_k)^2} - \frac{1}{(x_j - x_k)^2}\right), \tag{3.14}$$

which could be written shortly as:

$$\dot{v}_i = f_i(v), \qquad v = (x_1, .., x_N, p_1, ..., p_N, L_{12}, ..., L_{N,N-1}). \tag{3.15}$$

The pairs of matrices (V, A) and (L, A) are the examples of the so called Lax pairs defined by the equations $\dot{B} = [B, A]$. The existence of such linear operator equations associated to a nonlinear differential equation is very useful in the construction of the first integrals [2]. By applying (3.11) and the cyclic property of the trace one can check that

$$I_{k_1,k_2...,k_M} = \mathrm{Tr}(L^{k_1}V^{k_2}\cdots L^{k_{M-1}}V^{k_M}), \tag{3.16}$$

are the integrals of motion of the system (3.12)-(3.14) where $k_1, ...k_M$ are arbitrary natural numbers and V is expressed in terms of x_i, p_i, and L_{ij}. [71]. Of course only a limited subset of the integrals of motion is independent. Which and how many of the integrals belong to this subset can be found in [72].

Equations (3.12)-(3.14) are still Hamilton equations of motion derived from the old Hamilton function, $\mathcal{H} = \frac{1}{2}\mathrm{Tr}Y^2 = \frac{1}{2}\mathrm{Tr}V^2$ expressed in the new parametrisation,

$$\mathcal{H}_{CM,L} = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{L_{ij}L_{ji}}{(x_i - x_j)^2}. \tag{3.17}$$

They describe the dynamics of N interacting particles on a line. Note that the L matrix is anti-Hermitian and $L_{ij}L_{ji} = -|L_{ij}|^2$,thus the interaction in (3.17) is always repulsive.

Energy level repulsion

The task of diagonalising an $N \times N$ Hermitian matrix is of course familiar to every physicist in the quantum context. As a special case, one can consider a perturbed Hamiltonian in the form:

$$\hat{H} = \hat{H}_0 + \lambda \hat{V} \tag{3.18}$$

where the $\hat{H}_0$ is diagonal. The eigenvalues $\epsilon_i(\lambda)$ of the perturbed Hamiltonian depend on the parameter λ as the positions of a one-dimensional gas of repulsive particles described by (3.17) [73].

The harmonic and inverse sine squared potential

The same change of parametrisation applied to $H(X,Y) = \frac{1}{2}\mathrm{Tr}(\omega^2 X^2 + Y^2)$ leads to a similar system of repelling particles, but in an external harmonic potential [12],

$$\mathcal{H}_{C,L} = \frac{1}{2}\sum_i p_i^2 + \omega^2 x_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{L_{ij}L_{ji}}{(x_i - x_j)^2}. \tag{3.19}$$

As an intermediate step, the equations of motion in the big phase space will be:

$$\dot{X} = Y, \quad \dot{Y} = -\omega^2 X, \tag{3.20}$$

and their solution:

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}, \tag{3.21}$$

an ellipse in the phase space tends to the linear solution as $\omega \to 0$.

Applying the same procedure to a Hamilton function $H(X,Y) = \frac{1}{2}\mathrm{Tr}(XY)^2$ results in a unitary matrix $X(t) = e^{-itY_0}e^{-iX_0}$, and its eigenphases repel like particles in the Sutherland potential:

$$\mathcal{H}_{S,L} = \frac{1}{2}\sum_i p_i^2 - \frac{1}{8}\sum_{i\neq j}\frac{L_{ij}L_{ji}}{\sin^2((\phi_i - \phi_j)/2)}. \tag{3.22}$$

The symplectic form and the Poisson brackets

The symplectic structure on the big phase space M is of course modified by this change of parametrisation. The tautological one-form $\tau = \mathrm{Tr}(Y dX)$ is much easier to translate to the (x,p,L,U) variables, than $\omega = d\tau = \mathrm{Tr}(dY \wedge dX)$. Once τ is reexpressed, the expression for ω is straightforward:

$$\begin{aligned}
\tau &= \mathrm{Tr}(Y dX) = \mathrm{Tr}(U^\dagger V U d(U^\dagger D U)) = \mathrm{Tr}(V dD - [D,V] dU U^\dagger) = \\
&= \mathrm{Tr}(V dD - L dU U^\dagger) \\
\omega &= d\tau = \mathrm{Tr}(dV \wedge dD) - \mathrm{Tr}(dL \wedge dU U^\dagger) + \mathrm{Tr}(L dU U^\dagger \wedge dU U^\dagger) = \\
&= \sum_{i=1}^N dp_i \wedge dx_i - \mathrm{Tr}(dL \wedge a) + \mathrm{Tr}(La \wedge a) = \omega_{x,p} + \omega_{L,U}
\end{aligned} \tag{3.23}$$

The matrix-valued one form $a = dUU^\dagger$ is a special case of a Mauer-Cartan one form [74] with the property $da + a \wedge a = 0$. There is a separation of the (x,p) degrees of freedom in a canonical symplectic form $\omega_{x,p}$ and and the (L,U) degrees of freedom in a less obvious $\omega_{L,U}$, which is an example of the Kirillov-Kostant-Souriau form [75].

The Hamiltonian vector field corresponding to a function $f:(x,p,L,U)\to\mathbb{R}$ is the following:

$$V_f=\sum_{i=1}^{N}\frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i}-\frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i}-\mathrm{Tr}\left(\frac{\partial f}{\partial L^T}U\frac{\partial}{\partial L^T}\right)+\mathrm{Tr}\left\{\left(U\frac{\partial f}{\partial U^T}+\left[\frac{\partial f}{\partial L^T},L\right]\right)\frac{\partial}{\partial U^T}\right\},\tag{3.24}$$

and the arising Poisson brackets of two functions on the phase space:

$$\begin{aligned}\{f,g\} &= \omega(V_g,V_f)=\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i}-\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}+\\ &-\mathrm{Tr}\left\{U\left(\frac{\partial f}{\partial U^T}\frac{\partial g}{\partial L^T}-\frac{\partial g}{\partial U^T}\frac{\partial f}{\partial L^T}\right)\right\}+\mathrm{Tr}\left(L\left[\frac{\partial f}{\partial L^T},\frac{\partial g}{\partial L^T}\right]\right)\end{aligned}$$

The Poisson brackets of the relevant dynamic variables, (x,p,L) take the form

$$\{x_i,p_j\}=\delta_{ij},\quad \{L_{ij},L_{kl}\}=\delta_{il}L_{kj}-\delta_{jk}L_{il},\tag{3.25}$$

while all the other Poisson brackets vanish.

The Lie algebra spanned by the L variables

The L matrices, anti-Hermitian and traceless, belong to the $\mathfrak{su}(N)$ algebra, so it is expected that the The Poisson brackets (3.25) of L_{ij} variables will somehow reflect this fact. Indeed, let me define the real and imaginary parts of L_{ij} as:

$$L^R_{ij}=-L_{ij}+L_{ji},\qquad L^I_{ij}=i(L_{ij}+L_{ji})\tag{3.26}$$

and carefully calculate the Poisson brackets:

$$\{L^R_{ij},L^R_{kl}\} = \delta_{jk}L^R_{il}-\delta_{ik}L^R_{jl}-\delta_{jl}L^R_{ik}+\delta_{il}L^R_{jk},\tag{3.27}$$
$$\{L^R_{ij},L^I_{kl}\} = \delta_{jk}L^I_{il}-\delta_{ik}L^I_{jl}+\delta_{jl}L^I_{ik}-\delta_{il}L^I_{jk},\tag{3.28}$$
$$\{L^I_{ij},L^I_{kl}\} = -\delta_{jk}L^R_{il}-\delta_{ik}L^R_{jl}-\delta_{jl}L^R_{ik}-\delta_{il}L^R_{jk}.\tag{3.29}$$

These are exactly the commutation relations of the $\mathfrak{su}(N)$ basis elements (τ_{ij},σ_{ij}) stated in the definition 19 after a simple substitution:

$$[,]\to\{,\},\qquad \tau_{ij}\to L^R_{ij},\qquad \sigma_{ij}\to L^I_{ij}.\tag{3.30}$$

This means, that L^R and L^I variables span the off-diagonal subspace of the $\mathfrak{su}(N)$ algebra and the L^R variables alone span $\mathfrak{so}(N)$.

The orthogonal and unitary setting

It is clear, that if the initial conditions given by (X_0,Y_0) belong to the real symmetric subspace of M, the dynamics of the system is confined to this subspace as a special subset of solutions to the equations of motion (3.12)-(3.14). On the other hand the same construction on the cotangent bundle of real symmetric matrices, $T^*\mathcal{M}^R=\mathcal{M}^R\times\mathcal{M}^R=\{(X,Y):X^T=X,Y^T=Y\}$ will lead to the same equations of motion for the (x,p,L) variables. The only difference is, that since $X(t)$ and $Y(t)$ ale real and symmetric matrices, and the diagonalizing matrices are no longer unitary $U(t)$ but orthogonal $O(t)$. Moreover $L(t)$ is now real antisymmetric with the Poisson bracket equal to (3.27) up to a factor $\frac{1}{2}$, reflecting the fact that now L belongs to the $\mathfrak{so}(N)$ algebra [25, 71].

Gauge symmetry

It is a simple yet important observation that the matrix $U(t)$ which diagonalises $X(t)$ is clearly not unique. Every matrix $EU(t)$, where

$$E = diag(e^{i\phi_1}, e^{i\phi_2}, ..., e^{i\phi_N}). \tag{3.31}$$

is just as good. This results in a gauge invariance of (3.11) and consequently (3.12)-(3.14). In the orthogonal setting $\phi \in \{0, \pi\}$, which means that $E \in \mathbb{Z}_2^{\times N}$, while in the unitary setting $\phi \in [0, 2\pi)$ and $E \in U(1)^{\times N}$. This means that we can look at the trajectories $L(t) = U(t)L(0)U^\dagger(t)$ in the set of equivalence classes $[L] = \left\{L' = ELE^\dagger\right\}$, where E belongs to an appropriate set.

L and L' belong to the same equivalence class, i.e, $[L] = [L']$, if and only if

$$L'_{ij} = L_{ij}e^{i(\phi_i - \phi_j)}. \tag{3.32}$$

Writing $L_{ij} = |L_{ij}|e^{i\varphi_{ij}}$ and $L'_{ij} = |L'_{ij}|e^{i\varphi'_{ij}}$ $i < j$ we clearly get $|L_{ij}| = |L'_{ij}|$, but there must also exist such $\phi_1, \phi_2, ..., \phi_N \in [0, 2\pi)$, that for all $i < j$:

$$\varphi'_{ij} = \varphi_{ij} + \phi_i - \phi_j. \tag{3.33}$$

If such N phases exist, every triple (i, j, k) of indices must satisfy:

$$\varphi'_{ij} + \varphi'_{jk} + \varphi'_{ki} = \varphi_{ij} + \varphi_{jk} + \varphi_{ki} =: \Phi_{ijk}. \tag{3.34}$$

On the other hand if (3.34) is satisfied, $\phi_1, \phi_2, ..., \phi_N$ are well defined by (3.33). Indeed, let us define $\alpha_i = \varphi'_{i,i+1} - \varphi_{i,i+1}$, $i = 1, 2, \ldots, N-1$. Then, since L and L' are anti-Hermitian, i.e., $\varphi_{ij} = \pi - \varphi_{ji}$, $\varphi'_{ij} = \pi - \varphi'_{ji}$, we get from (3.34) $\varphi'_{j,k} - \varphi_{j,k} = \alpha_j + \ldots + \alpha_{k-1}$ for $j < k-1$. Now defining $\phi_k = \alpha_1 + \alpha_2 + \ldots + \alpha_{k-1}$ we easily obtain (3.33). Therefore (3.34) is both sufficient and necessary for two matrices to be gauge equivalent. The physical quantities, such as positions, momenta and repulsion strenghts $|L_{ij}|$ are gauge independent, as expected.

Ordinary Calogero-Moser subspace

The equations of motion for (x, p) variables defined by (3.1) can be realised with a specific choice of initial conditions (X_0, Y_0) in the big phase space. The chosen $L_0 = [X_0, Y_0]$ must remain in its gauge equivalence class as it evolves with time: $[L(t)] = [L_0]$. Since $L_{ii}(t) = 0$, $|L_{ij}|(t) = g$ and $L(t) = U(t)L_0U^\dagger(t)$, the appropriate matrix is

$$L_0 = ig\left(|e\rangle\langle e| - \mathbb{1}\right), \qquad \langle e| = (1, 1, ..., 1). \tag{3.35}$$

Indeed, $L(t) = ig\left(|f\rangle\langle f| - \mathbb{1}\right)$, where $|f\rangle = U(t)|e\rangle$. For the diagonal elements to vanish, $L_{ii}(t) = ig(|f_i|^2 - 1) = 0$, every component of $|f\rangle$ must be of the form $f_i(t) = e^{i\phi_i(t)}$, which in turn means that $|f\rangle = E|e\rangle$ thus $L(t) = E(t)L_0E^\dagger(t)$ and $[L(t)] = L_0$.

The reduction procedure

The Hamilton functions (3.17) and (3.19) and the corresponding equations of motion were obtained by changing the variables $(X, Y) \to (x, p, L, U)$ and silently eliminating the unitary degrees of freedom, as if all points (x, p, L, U) where $U \in U(N)$ were equivalent to a single point (x, p, L) in some new, reduced phase space. It is in fact guaranteed by the Marsden-Weinstein reduction theorem stated in section 2.4, that this procedure will work.

Let us take a closer look at the symplectic manifold $(M, \omega = \mathrm{Tr}(dX \wedge dY))$ with $G = U(N)$ acting by matrix conjugation. The fundamental vector field $v_M \in \mathcal{X}(M)$ corresponding to an element $v \in \mathfrak{g} = \mathfrak{u}(N)$ can be found by its action on a function $f \in C^\infty(M)$ at a point $m = (X, Y) \in M$:

$$v_M(f)(m) = \frac{d}{dt}\left(f(e^{tv}.m.e^{-tv})\right)|_{t=0} = [v, X]_{ij}\frac{\partial f}{\partial X_{ij}} + [v, Y]_{ij}\frac{\partial f}{\partial Y_{ij}}. \tag{3.36}$$

According to the definition $\omega(v_M, \ ) = d\langle\mu, v\rangle$ the moment map corresponding to this action is given by $\mu(X, Y) = [X, Y] = L$. All the points $(X, Y) \in \mu^{-1}(L)$ divide into equivalence classes given by $(D, V) = U(X, Y)U^\dagger$, where D is diagonal and these equivalence classes form a symplectic manifold.

3.1.2 Vectorial Calogero-Moser models

A different generalisation of the CM systems arises when the interaction term is described by strictly one-particle degrees of freedom, rather that by the dynamical variables L_{ij} measuring the coupling strengths between two particles [45], [46]. In this case the extended phase space is parametrized by canonical variables (x_i, p_i) of N particles on a line and vectorial degrees of freedom assigned to each particle. The vectors $|e_i) \in \mathbb{C}^d$, and their duals $(e_i| = |e_i)^\dagger$, $i = 1, 2, ..., N$ and define a symplectic structure by the following symplectic form,

$$\omega = \sum_{i=1}^{N} dp_i \wedge dx_i + i \cdot d(e| \wedge d|e) \tag{3.37}$$

The vectorial generalisation of the CM system is defined by the following Hamilton function on this phase space:

$$\mathcal{H} = \frac{1}{2}\sum_i p_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{(e_i|e_j)(e_j|e_i)}{(x_i - x_j)^2}, \tag{3.38}$$

and applying (3.6) gives us the equations of motion:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = p_i, \tag{3.39}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = \sum_{k\neq i}\frac{2(e_i|e_k)(e_k|e_i)}{(x_i - x_k)^3}, \tag{3.40}$$

$$\frac{d|e_i)}{dt} = -i \cdot \frac{\partial H}{\partial (e_i|} = -i\sum_{k\neq i}\frac{|e_k)(e_k|}{(x_i - x_k)^2}|e_i). \tag{3.41}$$

The set of N equations (3.41) have the form $|\dot{e}_i) = -iM_i|e_i)$, where M_i are Hermitian matrices, and it is useful to keep them in this form. Yet rewriting the symplectic form and the arising Poisson brackets in terms of the real and imaginary parts of $|e_i)$ vectors (rescaled by $1/\sqrt{2}$):

$$\omega = \sum_{i=1}^{N}\left(dp_i \wedge dx_i + \sum_{k=1}^{d} de^I_{i,k} \wedge de^R_{i,k}\right), \tag{3.42}$$

$$\{f, g\} = \{f, g\}_{x,p} + \sum_{i=1}^{N}\sum_{k=1}^{d}\frac{\partial f}{\partial e^R_{i,k}}\frac{\partial g}{\partial e^I_{i,k}} - \frac{\partial g}{\partial e^R_{i,k}}\frac{\partial f}{\partial e^I_{i,k}} \tag{3.43}$$

shows explicitly that there are $N \cdot d$ additional degrees of freedom due to this extension, and that the canonically conjugate variables (analogical to positions and

momenta in the additional subspace) are the real and imaginary part of each component: $\{e^R_{i,k}, e^I_{j,l}\} = \delta_{ij}\delta_{kl}$. There are two valuable conclusions from (3.41). Firstly, the scalar products $(e_i|e_i)$ are constant in time. Secondly, the dimension of the vector space spanned by $|e_i)$ vectors is effectively $d \leqslant N$ [45]. This stems from the fact that $|e_i(t)) \in span\{|e_{0,i})\}$, and as there are N vectors, at most N of which are linearly independent, hence the space spanned by the evolving vectors is at most N-dimensional (and fixed by the vectors given at t=0).

An extended phase space

A much broader set of systems may be defined in an extended phase space $\{(X, Y, \mathcal{E})\}$ where X,Y are $N \times N$ matrices, while $\mathcal{E} = (|e_1), |e_2), ..., |e_N))$ are rectangular matrices build of N d-dimensional column vectors [45]. The symplectic form

$$\omega = \mathrm{Tr}(dX \wedge dY) + i \cdot \mathrm{Tr}(d\mathcal{E}^\dagger \wedge d\mathcal{E}) \tag{3.44}$$

together with a Hamiltonian function $H(X, Y, \mathcal{E}, \mathcal{F})$ gives rise to the following equations of motion:

$$\dot{X}_{ij} = \frac{\partial H}{\partial Y_{ji}}, \quad \dot{Y}_{ij} = -\frac{\partial H}{\partial X_{ji}}, \quad \dot{\mathcal{E}}_{ij} = -i\frac{\partial H}{\partial \mathcal{E}^*_{ij}}. \tag{3.45}$$

Different assumptions about $\{(X, Y, \mathcal{E})\}$ matrices will lead to various matrix flows, which could be studied. Yet for the tools introduced in section 3.1.1 to be applicable in this case it is assumed that X, Y and $\mathcal{E}^\dagger\mathcal{E}$ are Hermitian $N \times N$ matrices. Moreover, we restrict the possible Hamilton functions $H(X, Y, \mathcal{E})$ to those with $U(N)$ symmetry, such as the functions of $\mathrm{Tr}(X^n)$,$\mathrm{Tr}(Y^n)$ or $\mathrm{Tr}((\mathcal{E}^\dagger\mathcal{E})^n)$. The solution $(X(t), Y(t), \mathcal{E}(t))$ of the equations of motion (3.45) can be acted upon by the matrix which diagonalizes $X(t)$, as it was done in the case of the (X, Y) system:

$$(D(t), V(t)) = U(t)(X(t), Y(t))U^\dagger(t) \quad E(t) = \mathcal{E}(t)U^\dagger(t). \tag{3.46}$$

The equations of motion:

$$\dot{D} = [A, D] + U\dot{X}U^\dagger, \quad \dot{V} = [A, V] + U\dot{Y}U^\dagger, \quad \dot{E} = -EA + \dot{\mathcal{E}}U^\dagger, \tag{3.47}$$

where $A = \dot{U}U^\dagger$ can be fully expressed with (D, V, E) variables for a specific solution of (3.45).

3.2 Quantum Calogero-Moser system

3.2.1 Ordinary CM system, spectrum and wavefunctions

The quantum N-body Calogero-Moser Hamiltonian considered in [4] consisted of both harmonic and inverse-square repulsion potentials between the particles:

$$\hat{H}' = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{m\omega^2}{4}\sum_{i<j}(x_i - x_j)^2 + g\sum_{i<j}\frac{1}{(x_i - x_j)^2}. \tag{3.48}$$

In this section I will consider a slightly different Hamiltonian, where the harmonic potential is external:

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{m\omega^2}{2}\sum_{i<j}x_i^2 + g\sum_{i<j}\frac{1}{(x_i - x_j)^2}. \tag{3.49}$$

and review the method used to find the spectrum and eigenstates in [4] by applying it to this case[1]. The negative values of g clearly correspond to an attractive interaction, and the scope of my research would allow not to consider them at all, but as discussed in [4] and in more detail in [62], $g \in (-\frac{\hbar^2}{4m}, 0)$ is allowed and does not affect the results presented below. A stronger attraction between particles would lead to a Hamiltonian which is unbounded from below and a 2-body collapse. The system is one-dimensional and the particles cannot overtake each other, therefore we can choose a sector of the configuration space with a fixed ordering, for example $x_1 \geqslant x_2 \geqslant .... \geqslant x_N$ and then extend the solution by $\psi(Px) = \eta_P \psi(x)$ where P denotes the permutation of positions and η_P is equal to one in case of Bose statistics, and to sign(P) in case of fermions. The solutions of the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{m\omega^2}{2} \sum_{i=1}^{N} x_i^2 + g \sum_{i<j} \frac{1}{(x_i - x_j)^2} - E \right) \psi = 0 \tag{3.50}$$

are assumed in the form:

$$\psi(R, r, z) = z^{a+\frac{1}{2}} \varphi(r) \rho(R), \tag{3.51}$$

where

$$R = \frac{1}{N} \sum_{i=1}^{N} x_i, \qquad r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2, \qquad z = \prod_{i<j} (x_i - x_j). \tag{3.52}$$

Careful calculation of $\hat{H}\psi$ leads to the separation of R and r variables:

$$\left(-\frac{d^2}{d^2\tilde{R}} + \tilde{R}^2 - \frac{2E_R}{\hbar\omega} \right) \rho(\tilde{R}) = 0, \tag{3.53}$$

$$\left(-\frac{d^2}{d^2\tilde{r}} - \frac{b}{\tilde{r}} \frac{d}{d\tilde{r}} + \tilde{r}^2 - \frac{2E_r}{\hbar\omega} \right) \varphi(\tilde{r}) = 0, \tag{3.54}$$

$$\tilde{R} = \sqrt{\frac{m\omega N}{\hbar}} R, \qquad \tilde{r} = \sqrt{\frac{m\omega}{\hbar}} r \tag{3.55}$$

$$b = N - 2 + (N-1)N(a + 1/2) \tag{3.56}$$

The crucial identity is:

$$\frac{1}{z^2} \sum_{i=1}^{N} \left(\frac{\partial z}{\partial x_i} \right)^2 = 2 \sum_{i<j} \frac{1}{(x_i - x_j)^2}. \tag{3.57}$$

which yields the term $\left(g - \frac{\hbar^2}{m}(a^2 - 1/4) \right) \sum_{i<j} \frac{1}{(x_i - x_j)^2}$ in the Schrodinger equation to disappear for

$$a = \pm \frac{1}{2} \sqrt{1 + \frac{4mg}{\hbar^2}}. \tag{3.58}$$

Of course only the positive root is acceptable, since we require the $z^{a+1/2}$ factor to vanish whenever the values of any two position variables coincide. The center of mass contribution to the solution is clearly the harmonic oscillator (or free in the case of (3.48)) eigen equation with the well known solutions:

$$\rho_n(\tilde{R}) = \frac{1}{\sqrt{2^n n!}} \left(-\frac{d}{d\tilde{R}} + \tilde{R} \right)^n e^{-\frac{\tilde{R}^2}{2}}, \qquad E_{R,n} = \hbar\omega \left(n + \frac{1}{2} \right). \tag{3.59}$$

[1] The sole difference between the two Hamiltonians is visible after separating the center of mass degree of freedom: in the former the center of mass is free and in the latter it is confined by a harmonic potential.

The solution for $\varphi(\tilde{r})$ is less straightforward to find but has a simple form as well:

$$\varphi_n(\tilde{r}) = e^{-\frac{\tilde{r}^2}{2}} L_n^\beta(\tilde{r}^2), \qquad \beta = \frac{b-1}{2}, \qquad E_{r,n} = \hbar\omega(2n+\beta+1), \tag{3.60}$$

where $L_n^\beta(x)$ are the generalised Laguerre polynomials, that is the solutions of the following ordinary differential equation:

$$xy''(x) + (\beta + 1 - x)y'(x) + ny(x) = 0, \qquad n = 0, 1, 2... \tag{3.61}$$

The complete solution to the (3.50) eigenproblem, assuming the form (3.51) is:

$$\psi_{n_R,n_r} = z^{a+\frac{1}{2}} e^{-\frac{m\omega(NR^2+r^2)}{2\hbar^2}} H_{n_R}\left(\sqrt{\frac{m\omega N}{\hbar}} R\right) L_{n_r}^\beta\left(\frac{m\omega}{\hbar} r^2\right) \tag{3.62}$$

$$E_{n_R,n_r} = \hbar\omega\left[n_R + 2n_r - 1 + \frac{N}{2} + \binom{N}{2}\left(a + \frac{1}{2}\right)\right]. \tag{3.63}$$

It has to be stressed, that the functions of (3.51) type exhaust the full set of solutions only for $N = 2$. In general, for $N \geqslant 3$ after separating the R and r variables we are left with a differential equation on the S^{N-2}, and a modified equation for $\varphi(r)$:

$$\left(-\frac{d^2}{d\tilde{r}^2} - \frac{N-2}{\tilde{r}}\frac{d}{d\tilde{r}} + \tilde{r}^2 + \frac{b_l^2}{\tilde{r}^2} - \frac{2E_r}{\hbar\omega}\right)\varphi(\tilde{r}) = 0, \tag{3.64}$$

$$\left(-\Delta_{S^{N-2}} + \frac{2mg}{\hbar} f(\bar{\theta}) - b_l^2\right)\chi_l(\bar{\theta}) = 0 \tag{3.65}$$

$$f(\bar{\theta}) = \sum_{i<j} \frac{r^2}{(x_i - x_j)^2}\Big|_{x_i \to (r,\bar{\theta})}. \tag{3.66}$$

The index l accounts for all the $N-2$ remaining quantum numbers, and the corresponding energy levers acquire an additional $\hbar\omega b_l$ term. The angular problem defined above is considered in [76].

In the case of a free system the solution in the relative variable r will be:

$$\psi(r) = \sqrt{r} J_a(kr) \tag{3.67}$$

3.2.2 Systems with spin state exchange

There is an interesting class of quantum Calogero-Moser systems, in which the spin states of the particles influence the inverse-square interaction potential [50–52]. They have a matrix structure similar to the classical one outlined in (3.1.1). Yet, as expected of spin degrees of freedom, the internal variables do not survive when the classical limit is taken. One of the characteristic traits of these systems is that in the matrix formulation the Hamiltonians have the form

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_{i,j}\left(\hat{V}^2\right)_{ij} \tag{3.68}$$

for an appropriate matrix-valued operator $\hat{V}$ instead of $\mathrm{Tr}(\hat{V}^2) = \sum_i \left(\hat{V}^2\right)_{ii}$. The diagonal terms result in a Hamiltonian acting diagonally on the internal degrees of freedom, that is an ordinary system with coupling constants possibly depending on the spin states.

Simple spin state exchange

The Hamiltonian presented in [50] is the quantum version of the Calogero-Moser system with a spin degree of freedom. The operators P_{ij} are matrices acting in spin space, exchanging the states of the i^{th} and j^{th} particle. The terms without P_{ij} should be understood as identity matrices in spin space. The x_i, p_j are standard position and momentum operators. The parameter a is a dimensionless number.

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_i p_i^2 + \frac{\hbar^2}{m}\sum_{i<j}\frac{a^2 - a\cdot P_{ij}}{(x_i - x_j)^2} \tag{3.69}$$

By analogy to the classical CM system this Hamiltonian has a corresponding pair of Lax operators V,A:

$$V_{ij} = \delta_{ij}p_i + (1-\delta_{ij})\frac{ia\hbar P_{ij}}{x_i - x_j} \tag{3.70}$$

$$A_{ij} = \frac{ia\hbar}{m}\left(-\delta_{ij}\sum_{k\neq i}\frac{P_{ik}}{(x_i - x_k)^2} + (1-\delta_{ij})\frac{P_{ij}}{(x_i - x_j)^2}\right) \tag{3.71}$$

Note that the matrix elements of these operators are themselves matrices in spin space. The properties of the V,A operators are the following:

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_{i,j}\left(V^2\right)_{ij} \tag{3.72}$$

$$\dot{V}_{ij} = \frac{1}{i\hbar}[V_{ij}, \hat{\mathcal{H}}] = [A, V]_{ij} \tag{3.73}$$

By analogy with the classical case we introduce operators $D_{ij} = \delta_{ij}x_i$ and $L_{ij} = [D, V]_{ij}$, and calculate the time dependence with $\hat{\mathcal{H}}$ and A. The results resemble the classical case:

$$L_{ij} = i\hbar\delta_{ij} + ia\hbar\,(1-\delta_{ij})\,P_{ij} \tag{3.74}$$

The time dependence of D and L:

$$\dot{D}_{ii} = \frac{1}{i\hbar}[x_i, \hat{\mathcal{H}}] = \frac{p_i}{m} = \frac{V_{ii}}{m} \tag{3.75}$$

$$[A, D]_{ii} = 0 \tag{3.76}$$

$$[A, D]_{ij} = -\frac{V_{ij}}{m} \tag{3.77}$$

$$\dot{L}_{ii} = 0 \tag{3.78}$$

$$\dot{L}_{ij} = \frac{1}{i\hbar}[L_{ij}, \hat{\mathcal{H}}] = [A, L]_{ij} \tag{3.79}$$

where (3.77) serves as a consistency check.

In conclusion, the equations for D,V and L time evolution are the following:

$$\dot{D} = [A, D] + \frac{V}{m} \tag{3.80}$$

$$\dot{V} = [A, V] \tag{3.81}$$

$$\dot{L} = [A, L] \tag{3.82}$$

The closed set of equations for D_{ii},V_{ii} and L_{ij} looks like this:

$$\dot{D}_{ii} = \frac{V_{ii}}{m} \tag{3.83}$$

$$\dot{V}_{ii} = \frac{1}{m}\sum_{j\neq i}\frac{-2L_{ij}L_{ji} + 2i\hbar L_{ij}}{(D_{ii} - D_{jj})^3} \tag{3.84}$$

$$\dot{L}_{ij} = \frac{1}{m}\sum_{k\neq i,j}[L_{ik}, L_{kj}]\left(\frac{1}{(D_{ii} - D_{kk})^2} - \frac{1}{(D_{jj} - D_{kk})^2}\right) \tag{3.85}$$

Weighted spin state exchange

The Hamiltonian presented in [51] is a generalisation of (3.69) in which interactions depend on the spin states:

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_i p_i^2 + \frac{\hbar^2}{m}\sum_{i<j}\frac{\sum_{\alpha,\beta} g_{\alpha\beta}\left(g_{\alpha\beta}X_i^{\alpha\alpha}X_j^{\beta\beta} + X_i^{\alpha\beta}X_j^{\beta\alpha}\right)}{(x_i - x_j)^2} \tag{3.86}$$

the α, β indices run over spin degrees of freedom, the $X_i^{\alpha\beta}$ matrices turn the state of the i^{th} particle from α to β. The matrices act on single particles, so the X matrix products should be understood (for $i < j$) as:

$$X_i^{\alpha\beta}X_j^{\gamma\delta} = 1^{i-1} \otimes X^{\alpha\beta} \otimes 1^{j-i-1} \otimes X^{\gamma\delta} \otimes 1^{N-j} \tag{3.87}$$

The summation over spin degrees of freedom for some general values of $g_{\alpha\beta}$ results in:

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_i p_i^2 + \frac{\hbar^2}{m}\sum_{i<j}\frac{E_{ij}^2 + E_{ij}}{(x_i - x_j)^2} \tag{3.88}$$

where E_{ij}^2 is diagonal, and $E_{ij} = \sum_{\alpha,\beta} g_{\alpha\beta}X_i^{\alpha\beta}X_j^{\beta\alpha}$ exchanges the spin states of i_{th} and j^{th} particle multiplying the state vector by a state-dependent factor.

By analogy with the (3.69) case,and looking at (3.70) and (3.71) it is possible to guess the form of the Lax pair in this more general setting:

$$V_{ij} = \delta_{ij}p_i + (1-\delta_{ij})\frac{i\hbar E_{ij}}{x_i - x_j} \tag{3.89}$$

$$A_{ij} = \frac{i\hbar}{m}\left(\delta_{ij}\sum_{k\neq i}\frac{E_{ik}}{(x_i - x_k)^2} + (1-\delta_{ij})\frac{E_{ij}}{(x_i - x_j)^2}\right) \tag{3.90}$$

The definitions $D_{ij} = \delta_{ij}x_i$ and $L_{ij} = [D, V]_{ij}$ do not change, and it turns out that:

$$\hat{\mathcal{H}} = \frac{1}{2m}\sum_{i,j}\left(V^2\right)_{ij} \tag{3.91}$$

$$\dot{V}_{ij} = \frac{1}{i\hbar}[V_{ij}, \hat{\mathcal{H}}] = [A, V]_{ij} \tag{3.92}$$

$$\dot{D}_{ij} = \frac{1}{i\hbar}[D_{ij}, \hat{\mathcal{H}}] = [A, D]_{ij} + \frac{V_{ij}}{m} \tag{3.93}$$

Whereas for the L operator:

$$L_{ij} = i\hbar\delta_{ij} + i\hbar\,(1-\delta_{ij})\,E_{ij}, \tag{3.94}$$

There is an additional commutator with a diagonal operator:

$$\dot{L}_{ij} = \frac{1}{i\hbar}[L_{ij}, \hat{\mathcal{H}}] = [A + A_D, L]_{ij} \tag{3.95}$$

which can be eliminated by the appropriate gauge transformation $L \to ELE^\dagger$.

3.3 Relation with Quantum Hall Effect

One of the features of the Calogero-Moser system which motivated my research is its link with the Quantum Hall Effect presented in [31]. The core idea is that in the strong magnetic field regime the states of the Quantum Hall system can be

projected onto the subspace of a single Landau level. The elements of this subspace admit a 1D representation, which turns out to be identical to the wavefunctions of the Calogero-Moser Hamiltonian. To see exactly how it works, we need to start from the Hamiltonian of a single electron confined to the (x, y) plane in the magnetic field $\bar{B} = B\hat{e}_z = \bar{\nabla} \times \bar{A}$:

$$\hat{H}_H = \frac{1}{2m}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2\right), \quad \hat{\pi}_i = \hat{p}_i + e\hat{A}_i. \tag{3.96}$$

In the position representation $\hat{p}_i = -i\hbar\partial_{x_i}$, and therefore $[\hat{\pi}_x, \hat{\pi}_y] = -i\hbar eB$. This relation lets us efficiently show, that:

$$\hat{H}_H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad \omega = \frac{eB}{m}, \tag{3.97}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar eB}}\left(\hat{\pi}_x - i\hat{\pi}_y\right), \quad \left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{3.98}$$

The eigenstates of this Hamiltonian can be constructed with the anihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ exactly as it is done for the quantum harmonic oscillator:

$$\hat{a}\psi_0 = 0, \quad \psi_n = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\psi_0, \quad \hat{H}_H\varphi_n = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n, \tag{3.99}$$

and in this context the quantized energy levels are called the Landau Levels. Just as we would get for the classical phase space functions with the use of the Poisson bracket, we obtain the time derivatives of the operators with the use of the commutator,i.e. the Heisenberg equations:

$$\frac{d}{dt}\begin{pmatrix}\hat{\pi}_x\\ \hat{\pi}_y\end{pmatrix} = -\frac{-1}{i\hbar}\left[\hat{H}_H, \begin{pmatrix}\hat{\pi}_x\\ \hat{\pi}_y\end{pmatrix}\right] = \begin{pmatrix}-\omega\hat{\pi}_y\\ \omega\hat{\pi}_x\end{pmatrix}. \tag{3.100}$$

The solution is of course a pair of operators rotating with the cyclotron frequency $\omega = \frac{eB}{m}$:

$$\begin{pmatrix}\hat{\pi}_x\\ \hat{\pi}_y\end{pmatrix}(t) = \begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix}\hat{\pi}_x\\ \hat{\pi}_y\end{pmatrix}(0). \tag{3.101}$$

Instead of $(\hat{x}, \hat{y})$ we will use the centre of mass position operators which commute with the Hamiltonian:

$$\begin{pmatrix}\hat{X}\\ \hat{Y}\end{pmatrix} = \begin{pmatrix}\hat{x} - \frac{\hat{\pi}_y}{m\omega}\\ \hat{y} + \frac{\hat{\pi}_x}{m\omega}\end{pmatrix}, \quad \frac{d}{dt}\begin{pmatrix}\hat{X}\\ \hat{Y}\end{pmatrix} = -\frac{-1}{i\hbar}\left[\hat{H}_H, \begin{pmatrix}\hat{X}\\ \hat{Y}\end{pmatrix}\right] = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{3.102}$$

The algebra of the four operators $(\hat{X}, \hat{Y}, \hat{\pi}_x, \hat{\pi}_y)$ is the following:

$$\left[\hat{X}, \hat{Y}\right] = il^2, \quad [\hat{\pi}_x, \hat{\pi}_y] = -i\hbar eB \tag{3.103}$$

$$\left[\hat{X}, \hat{\pi}_x\right] = \left[\hat{Y}, \hat{\pi}_y\right] = 0 \tag{3.104}$$

$$\left[\hat{X}, \hat{\pi}_y\right] = \left[\hat{Y}, \hat{\pi}_x\right] = 0 \tag{3.105}$$

where $l = \sqrt{\frac{\hbar}{eB}}$. Instead of $\hat{X}$ and $\hat{Y}$ we may define yet another pair of operators, $\hat{b}$ and $\hat{b}^\dagger$, such that:

$$\hat{b} = \frac{1}{\sqrt{2}l}\left(\hat{Y} - i\hat{X}\right), \quad \left[\hat{b}, \hat{b}^\dagger\right] = 1, \quad \left[\hat{b}, \hat{a}\right] = \left[\hat{b}, \hat{a}^\dagger\right] = 0 \tag{3.106}$$

The existence of such $\hat{b}, \hat{b}^\dagger$ operators means that the eigenvalue of $\hat{m} = \hat{b}^\dagger \hat{b}$ is a good quantum number and we may create a ladder of states numbered by m within each Landau Level:

$$\hat{a}\psi_{0,0} = \hat{b}\psi_{0,0} = 0, \tag{3.107}$$

$$\psi_{n,m} = \frac{1}{\sqrt{n!m!}} \left(\hat{a}^\dagger\right)^n \left(\hat{b}^\dagger\right)^m \psi_{0,0} \tag{3.108}$$

In a strong magnetic field, that is when the level spacing $\hbar\omega$ is much larger that $k_B T$, we may treat the system as confined to the lowest Landau Level (LLL) with the $\hat{\pi}_x$ and $\hat{\pi}_y$ frozen by the condition $\hat{a}\psi = 0$. The remaining two variables describe an effectively one-dimensional system, since in the diagonal basis of $\hat{X}$, that is for states $|s\rangle$ such that $\hat{X}|s\rangle = s|s\rangle$ the operator $\hat{Y}$ can be treated as the conjugate momentum: $\hat{Y} = -il^2\partial_s = \frac{\hat{p}_s}{eB}$.

Until now all the definitions were gauge independent, but to work out the wavefunctions in the 1D representation, a specific gauge is necessary. The choice made in [31] is the symmetric gauge, $\bar{A} = \frac{B}{2}(-y, x, 0)$, which allows for the use of very convenient complex variables $(z, z^*) = \frac{1}{l\sqrt{2}}(x + iy, x - iy)$. The operators $\hat{a}$, $\hat{b}$ and their Hermitian conjugates can be expressed in these variables[2]:

$$\hat{a} = -i\left(\partial_z + \frac{z^*}{2}\right), \quad \hat{a}^\dagger = i\left(-\partial_{z^*} + \frac{z}{2}\right) \tag{3.109}$$

$$\hat{b} = -i\left(\partial_{z^*} + \frac{z}{2}\right), \quad \hat{b}^\dagger = i\left(-\partial_z + \frac{z^*}{2}\right) \tag{3.110}$$

The LLL wavefunctions defined by $\hat{a}\psi_0 = 0$ have a general form:

$$\psi(z, z^*) = \varphi(z^*) e^{-\frac{|z|^2}{2}}. \tag{3.111}$$

To change the representation of a state $|\Psi\rangle$ from $\Psi(z, z^*) = \langle zz^*|\Psi\rangle$ to $\Psi(s) = \langle s|\Psi\rangle$, we need to calculate:

$$\Psi(s) = \langle s|\Psi\rangle = \int \langle s|zz^*\rangle\langle zz^*|\Psi\rangle dz dz^* = \int \langle s|zz^*\rangle \Psi(z, z^*) dz dz^* \tag{3.112}$$

where $\langle s|zz^*\rangle$ is the complex conjugate of normalized eigenstate of $\hat{X}$ with the eigenvalue s written in the z, z^* variables:

$$\hat{X} = \frac{x}{2} + il^2\partial_y = \frac{l}{\sqrt{2}}\left(\frac{z + z^*}{2} + \partial_{z^*} - \partial_z\right) \tag{3.113}$$

$$\Psi_s(z, z^*) = \langle zz^*|s\rangle = \frac{1}{\sqrt{l}\pi^{1/4}} e^{-\frac{s^2}{2l^2} + \frac{\sqrt{2}}{l}sz^* - \frac{(z^*)^2 + |z|^2}{2}} \tag{3.114}$$

$$\hat{X}\Psi_s(z, z*) = s\Psi_s(z, z^*). \tag{3.115}$$

The integral for $\Psi(s)$:

$$\Psi(s) = \frac{1}{\sqrt{l}\pi^{1/4}} e^{-\frac{s^2}{2l^2}} \int e^{\frac{\sqrt{2}sz}{l} - \frac{z^2}{2}} \varphi(z^*) e^{-|z|^2} dz dz^* = \tag{3.116}$$

$$= \frac{1}{\sqrt{l}\pi^{1/4}} e^{-\frac{s^2}{2l^2}} e^{-\frac{l^2}{4}\partial_s^2} \int e^{-|z|^2 + \frac{\sqrt{2}sz}{l}} \varphi(z^*) dz dz^* = \tag{3.117}$$

$$\propto e^{-\frac{s^2}{2l^2}} exp\left(-\frac{l^2}{4}\partial_s^2\right) \varphi\left(\frac{\sqrt{2}s}{l}\right), \tag{3.118}$$

[2] only in this specific, symmetric gauge, which enters into the $\hat{a}, \hat{b}$ operators through the gauge dependent $\hat{\pi}_{x,y}$

where we use the fact that $\frac{l^2}{4}\frac{\partial^2}{\partial l^2}\left(e^{\frac{\sqrt{2}sz}{l}}\right) = \frac{z^2}{2}$ and the coherent state identity:

$$\frac{1}{\pi}\int e^{-|z|^2+\alpha z}\varphi(z^*)dzdz^* = \varphi(\alpha). \tag{3.119}$$

Extending this result to many-body wavefunctions

$$\psi(z_{1,..N}, z^*_{1,...,N}) = \varphi(z^*_1, ..., z^*_N)e^{-\sum_{i=1}^N \frac{|z_i|^2}{2}} \tag{3.120}$$

and using a dimensionless parameters $\tilde{s}_i = \frac{\sqrt{2}s_i}{l}$ leads to:

$$\Psi(\tilde{s}_1, ..., \tilde{s}_N) = e^{-\frac{1}{4}\sum_{i=1}^N \tilde{s}_i^2} e^{-\frac{1}{2}\sum_{i=1}^N \frac{\partial^2}{\partial^2 \tilde{s}_i^2}} \varphi(\tilde{s}_1, ..., \tilde{s}_N) \tag{3.121}$$

In the case of the Laughlin state $\varphi(z^*_1, ..., z^*_N) = \prod_{i<j}(z^*_i - z^*_j)^m$ and the 1D representation:

$$\Psi(\tilde{s}_1, ..., \tilde{s}_N) = e^{-\frac{1}{4}\sum_{i=1}^N \tilde{s}_i^2} e^{-\frac{1}{2}\sum_{i=1}^N \frac{\partial^2}{\partial^2 \tilde{s}_i^2}} \prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m \tag{3.122}$$

In the strong magnetic field the length $l = \sqrt{\frac{\hbar}{eB}}$ is small and the long distance approximation $|\tilde{s}_i - \tilde{s}_j| >> 1$ may be used. In this approximation the derivatives:

$$\frac{\partial}{\partial \tilde{s}_k}\left(\prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m\right) = m\left(\sum_{l\neq k}\frac{1}{\tilde{s}_k - \tilde{s}_l}\right)\left(\prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m\right) << \left(\prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m\right)$$

so if we expand the exponent with the derivatives acting on $\left(\prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m\right)$, the zero order term will be dominant. This means that in the strong magnetic field:

$$\Psi(\tilde{s}_1, ..., \tilde{s}_N) \approx e^{-\frac{1}{2}\sum_{i=1}^N \tilde{s}_i^2}\left(\prod_{i<j}(\tilde{s}_i - \tilde{s}_j)^m\right) \tag{3.123}$$

which is the ground state of the Calogero-Moser Hamiltonian:

$$\hat{H}_{CM} = \hbar\omega\left(-\frac{1}{2}\sum_{i=1}^N \partial^2_{\tilde{s}_i} + \frac{1}{2}\sum_{i=1}^N \tilde{s}_i^2 + \sum_{i<j}\frac{m^2 - m}{(\tilde{s}_i - \tilde{s}_j)^2}\right). \tag{3.124}$$

Chapter 4

Classical Results

In this chapter I am presenting the results of my research on the classical generalisations of the Calogero-Moser system formulated in section (3.1). The accomplishments in this area can be summarised as follows:

1. I have proven the equivalence of flows defined by the Hamiltonian functions (3.17) and (3.38) under the condition, that all vectors $|e_i)$ are equally normalised. In this case $L_{ij}(t) \equiv i(e_i|e_j)(t)$, which means that the seemingly two-particle quantities L_{ij} are in fact defined completely by the internal states of individual particles.

2. Within the vectorial formulation I have proven that there are no initial conditions recovering the (3.2) dynamics unless $g_{ij} = g$.

3. I have proposed a model which combines the matrix and vectorial degrees of freedom as outlined in (3.1.2). A coupling between the two types of variables, after a unitary reduction, gives rise to $1/x_{ij}$ interaction potential. This is a very interesting result, as this is the only known integrable many-body model with this type of interaction.

4. I have solved the equations of motion given by a classical counterpart of a modified Hamiltonian (3.68).

5. I have examined the influence of the internal variables L_{ij} on the external degrees of freedom (x, p) and shown that a matrix model with $L_{ij}(0) = ig_{ij}$ can be used recreate the $(x, p)(t)$ trajectories defined by (3.2) with high accuracy.

6. I have studied the influence of particle collisions on the values of L_{ij} variables.

7. I have analytically found the reachable sets of the L degrees of freedom for a Calogero-Moser system of $N = 3$ particles. I have also made an attempt to generalise this result to $N > 3$.

4.1 Relation between the vectorial and matrix models

A careful look at (3.41) and (3.14) leads to an observation that $\mathcal{L}_{ij} = i \cdot (e_i|e_j)$ obeys an equation of motion which is similar to (3.14):

$$\frac{d}{dt}\mathcal{L}_{ij} = \sum_{k\neq i,j} \mathcal{L}_{ik}\mathcal{L}_{kj}\left(\frac{1}{(x_i - x_k)^2} - \frac{1}{(x_j - x_k)^2}\right) - \frac{i\mathcal{L}_{ij}}{x_{ij}^2} \cdot (c_j - c_i), \tag{4.1}$$

where $c_i = (e_i|e_i)$. It coincides with (3.14) if all the vectors are equally normalised $(e_i|e_i) = g$. Moreover, the Poisson brackets obtained with the use of $|e)$-dependent

part of the symplectic form (3.37)

$$\{\mathcal{L}_{ij},\mathcal{L}_{kl}\}=i\left(\sum_m \frac{\partial(e_i|e_j)}{|e_m)}\frac{\partial(e_k|e_l)}{(e_m|}-\frac{\partial(e_k|e_l)}{|e_m)}\frac{\partial(e_i|e_j)}{(e_m|}\right)=(\delta_{jk}\mathcal{L}_{il}-\delta_{il}\mathcal{L}_{kj})\,, \quad (4.2)$$

coincide with $\{L_{ij},L_{kl}\}=\delta_{jk}L_{il}-\delta_{il}L_{kj}$ (3.25). This means that once we find such initial $|e_{0,i})$ that recover a given anti-Hermitian L_0 matrix *via* $L_{0,ij}=i\cdot(e_{i,0}|e_{j,0})$, and $(e_{i,0}|e_{i,0})=g$, the two flows will be equivalent. Expressing the above with a rectangular matrix $\mathcal{E}=(|e_1)|e_2)...|e_N))$, we can say that any matrix $\mathcal{E}^\dagger\mathcal{E}$ with a constant diagonal defines a repulsive (3.38) system. Once the initial conditions coincide, $L_{ij}(0)=i(\mathcal{E}^\dagger\mathcal{E})_{ij}(0)$, and the diagonal elements are all equal $(\mathcal{E}^\dagger\mathcal{E})_{ii}(0)=g$, the trajectories will coincide as well: $L_{ij}(t)=i(\mathcal{E}^\dagger\mathcal{E})_{ij}(t)$. Now two questions need to be answered: 1. does every L matrix decompose into appropriately normalized $|e)$ vectors, 2. does every vectorial model translate to an L formulation (3.38)? In other words: do (3.17) and (3.38) coincide, overlap or one contains the other?

To answer this question let us decompose an L matrix into $|e_i)$ vectors. Every Hermitian, positive definite $N\times N$ matrix M can be written in terms of a Cholesky decomposition $M=\mathcal{E}^\dagger\mathcal{E}$, where $\mathcal{E}$ is upper (or lower, depending on the convention) triangular, and its column vectors $\{|\epsilon_i)\}$ span the full N-dimensional space. A positive semi-definite matrix can be decomposed likewise, only the column vectors of $\mathcal{E}$ will span a subspace of dimension $N-\mu^0$, where μ^0 is the multiplicity of the 0 eigenvalue. Negative (semi-)definite matrices will be simply $M=-\mathcal{E}^\dagger\mathcal{E}$. The L matrices are anti-Hermitian and non-definite, yet we can easily adjust such a matrix to a decomposable form

$$\pm iL+\mathbb{1}g=\mathcal{E}_g^\dagger\mathcal{E}_g, \quad (4.3)$$

with a large enough value of g, to make it positive semi-definite. This adjustment is, of course, not unique, but I am choosing the one which minimizes the dimension of $span(|\epsilon_i))$. The extreme eigenvalues $i|\lambda_+|$ and $-i|\lambda_-|$ of L give us two positive semi-definite matrices to choose from,

$$\pm iL+\mathbb{1}|\lambda_\pm|=\mathcal{E}_\pm^\dagger\mathcal{E}_\pm. \quad (4.4)$$

The one with the higher multiplicity results in the smallest possible subspace spanned by the column vectors $|\epsilon_i)$ and will be denoted as $\mathcal{E}$. The column vectors of $\mathcal{E}$ give a valid expression of the initial conditions given by L,

$$L_{ij}=i(\epsilon_i|\epsilon_j), \qquad (\epsilon_i|\epsilon_i)=|\lambda_\pm|. \quad (4.5)$$

The symmetry of the (3.38) and resulting equations of motion (3.41) imply that having found the $\{|e_i)\}$ decomposition of L, we automatically obtain a vast set of other choices:

$$|e_i)=W|\epsilon_i), \qquad W\in U(N) \quad (4.6)$$

The Cholesky decomposition allows every valid L matrix to be expressed with a set of N vectors of equal length. Yet the vectorial formulation does not restrict the norms of the $|e_i)$ vectors to be equal. This means that the role of the non-vanishing c_i-c_j term has to be checked. If it can be removed by a choice of gauge, that is if there exist $\phi_i(t), i=1,2,...,N$ such that all $i(e_i|e_j)e^{i(\phi_i-\phi_j)}$ obey (3.14), then the $E^\dagger E$ matrices with different diagonal elements are equivalent to the L formulation as well. Let us write the time derivative of $\mathcal{L}_{ij}e^{i(\phi_i-\phi_j)}$ and find the equations for

the suitable gauge:

$$\frac{d}{dt}\left(\mathcal{L}_{ij}e^{i(\phi_i-\phi_j)}\right)-\dot{\mathcal{L}}_{ij}e^{i(\phi_i-\phi_j)} = \mathcal{L}_{ij}e^{i(\phi_i-\phi_j)}i(\dot{\phi}_i-\dot{\phi}_j)= \tag{4.7}$$

$$= \frac{i}{x_{ij}^2}(c_j-c_i)\mathcal{L}_{ij}e^{i(\phi_i-\phi_j)}, \tag{4.8}$$

$$\dot{\phi}_i-\dot{\phi}_j = \frac{1}{x_{ij}^2}(c_j-c_i). \tag{4.9}$$

For a set of functions $\phi_1(t),\phi_2(t),...,\phi_N(t)$ to exist, every triple (k,l,m) of equations must sum up to 0 at every instant of time:

$$0=\dot{\phi_k}-\dot{\phi_l}+\dot{\phi_l}-\dot{\phi_m}+\dot{\phi_m}-\dot{\phi_k}=\frac{1}{x_{kl}^2}(c_l-c_k)+\frac{1}{x_{lm}^2}(c_m-c_l)+\frac{1}{x_{mk}^2}(c_k-c_m) \tag{4.10}$$

The numerator of the right hand side expanded in $x_{kl}=a, x_{lm}=b, x_{km}=a-b$ will have coefficients that vanish for any a,b only if $c_k=c_l=c_m$. Therefore, as we are free to choose any initial positions $x_i(0)$, and momenta the (4.9) conditions have no solution. Of course we can tune the initial positions so that they are met at time $t=0$, but in $t=\delta t$ it will no longer be true. The conclusion is that $E^\dagger E$ matrices with different diagonal elements do not translate into an L matrix formulation. They define a flow of $i(e_i|e_j)$ matrix elements which is not gauge equivalent to any L_{ij} evolution.

The final observation of this section is that just as the L matrices in Section 3.1.1, the vectorial variables also fall into equivalence classes due to gauge symmetry. In case of vectors this is the simplest possible $U(1)$ symmetry: $[|e_i)] = [|e_i')] \iff |e_i') = e^{i\phi}|e_i)$ and it translates automatically to gauge equivalent L matrices formed as $L_{ij}=i(e_i|e_j)$ and $L'_{ij}=i(e_i|e_j)e^{i(\phi_j-\phi_i)}=L_{ij}e^{i(\phi_j-\phi_i)}$. In other words equivalent vectors define the same one-dimensional eigenspace of a projection operator $P_i=\frac{|e_i)(e_i|}{g}$.

We can conclude that the vectorial $|e_i)$ formulation is equivalent to the $L=[X,Y]$ formulation of the Calogero-Moser system if and only if the diagonal elements are all equal: $(e_i|e_i)=g$. In this case L can be expressed *via* the Cholesky decomposition with a set of N complex vectors of equal length. If the diagonal elements are different, the differences appear in the equations of motion and are impossible to remove with a choice of gauge. This means, that the unequal values of $(e_i|e_i)$ take us beyond the L matrix formulation. This section can be summarised as folows: ***The L_{ij} variables of the generalized Calogero-Moser system (3.17) do not need to be treated as two-body, independent, dynamical interaction strengths, but may always be viewed as arising as functions of one-particle observables with internal degrees of freedom.***

This is a valuable hint for the quantization of the system.

4.2 Orbits of L classified by rank

A set of complex vectors $\mathcal{E}=(|e_1)|e_2)...|e_N))$ of the same length describes the repulsion forces in the CM system equivalently to an anti-Hermitian L matrix. The equations of motion

$$\frac{d|e_i)}{dt}=-i\sum_{k\neq i}\frac{|e_k)(e_k|}{x_{ik}^2}|e_i)=-i\cdot M_i(t)|e_i), \tag{4.11}$$

where $x_{ik}=x_i-x_k$, imply that the dimension of the subspace spanned by $|e_i(t))$ does not increase with time. It does not decrease either, due to the unitary evolution

of $L(t)$. This means that $d = rank(\mathcal{E}) = rank(L)$ is constant and the orbits of L fall into separate classes given by the value of $d = dim(span(|e_i)))$. This value can be translated into the number of independent variables needed to describe the internal (non-spatial) evolution of the CM system. The N vectors belong to an d-dimensional subspace of $\mathbb{C}^N$. Therefore we can choose a basis in $\mathbb{C}^N$ in which at most first d components of $\{|e_i)\}$ are nonzero, and treat them as elements of $\mathbb{C}^d$. Each complex vector in $\mathbb{C}^d$ is described with $2r$ real variables, yet one has to be excluded due to fixed lenght, and another one due to gauge symmetry,which leaves us with $2d-2$ real parameters per vector:

$$r = (2d - 1 - 1)N = 2(d-1)N. \tag{4.12}$$

The general solution of (4.11) is the following:

$$|e_i)(t) = U_i(t)|e_i)(0) \tag{4.13}$$

where $U_i(t) = Texp\left(-i\int_0^t M_i(\tau)d\tau\right) \in U(d)$. The unitarity of $U_i(t)$ expresses the fact that $(e_i|e_i)$ are constants of motion. These matrices are, of course, very sensitive to the initial positions $x_i(0)$ and momenta $p_i(0)$, but I am interested in the properties of $|e_i)$ variables independent of the initial conditions other that $|e_i)(0)$. In particular I am going to solve the problem of the conditions for $|L(t)| = |L(0)|$.

4.2.1 Rank $d = 1$ as the ordinary CM system

The vectors $|e_i)$ evolve within the span of the initial vectors and their length is a constant of motion. Therefore, if $|e_i)(0) = e^{i\phi_i}|e)$, which corresponds to $d = 1$, the evolving vectors will stay in this one dimensional subspace, their length is constant, and all that can change in time is the phases: $|e_i)(t) = e^{i\phi_i(t)}|e)$. In fact regardless of the phases and the choice of the vector $|e)$ this initial condition corresponds to

$$L_{ij}(t) = ig \cdot e^{i(\phi_j(t)-\phi_i(t))}. \tag{4.14}$$

In this case $|L_{ij}(t)| = |L_{ij}(0)| = g$ thus it corresponds to the ordinary CM system (3.1). The initial condition in matrix form is equivalent to (3.35). This result is in full agreement with (4.12), which indicates that for $d = 1$ we get $r = 0$ thus no room for nontrivial change of the interaction strenghts.

4.2.2 Rank $d = 2$ dynamics on the Bloch sphere

Before looking at the general features of CM systems with $d > 1$ I will explore the $d = 2$ case. For simplicity I set $g = 1$ and I parametrise the unit vectors $|\epsilon_i) \in \mathbb{C}^2$ as follows:

$$|e_i) = \begin{pmatrix} \cos\left(\frac{\theta_i}{2}\right) \\ e^{i\phi_i}\sin\left(\frac{\theta_i}{2}\right) \end{pmatrix}. \tag{4.15}$$

I map the vectors to $\mathbb{S}^2$ sphere in a standard way:

$$\bar{v}_i = (\cos\phi_i \sin\theta_i, \sin\phi_i \sin\theta_i, \cos\theta_i), \tag{4.16}$$

just like the states of a qubit are depicted on the Bloch sphere. The interparticle repulsion strengths can be expressed with Bloch vectors:

$$|L_{ij}|^2 = |(e_i|e_j)|^2 = \frac{1 + \bar{v}_i \cdot \bar{v}_j}{2}. \tag{4.17}$$

By applying (4.11) to $\bar{v}_i \in \mathbb{S}^2$ I obtain the following equations:

$$\frac{d\bar{v}_i}{dt} = \bar{\alpha}_i \times \bar{v}_i, \qquad \bar{\alpha}_i = \sum_{k\neq i} \frac{\bar{v}_k}{x_{ik}^2} \tag{4.18}$$

$$\frac{d}{dt}(\bar{v}_i \cdot \bar{v}_j) = (\bar{v}_i \times \bar{v}_j) \cdot (\bar{\alpha}_i - \bar{\alpha}_j) = (\bar{v}_i \times \bar{v}_j) \cdot \sum_{k\neq i,j} \bar{v}_k f_{ij,k}(x). \tag{4.19}$$

where $f_{ij,k}(x) = \frac{1}{x_{ik}^2} - \frac{1}{x_{jk}^2}$. This means that the motion of each $\bar{v}_i$ on the Bloch sphere is generated by an infinitesimal rotation arround the superposition of all the other vectors with position dependent weights . Moreover the time derivative of $\bar{v}_i \cdot \bar{v}_j$ has a geometrical interpretation as well: it is the projection of the sum of the weighted vectors onto the direction of $\bar{v}_i \times \bar{v}_j$.

If all the vectors are set initially in one plane, the derivative (4.19) vanishes at $t = 0$. Yet, we may expect the second derivative to be nonzero, since the vectors rotate according to (4.18) and start deviating from the initial plane. Indeed, the second derivative

$$\frac{d^2}{dt^2}(\bar{v}_i \cdot \bar{v}_j) = \frac{d(\bar{v}_i \times \bar{v}_j)}{dt} \cdot (\bar{\alpha}_i - \bar{\alpha}_j) + (\bar{v}_i \times \bar{v}_j) \cdot \frac{d(\bar{\alpha}_i - \bar{\alpha}_j)}{dt}$$

has terms of different geometrical types:

$$\begin{aligned}
\frac{d^2}{dt^2}(\bar{v}_i \cdot \bar{v}_j) &= (\bar{v}_i \times \bar{v}_j) \cdot \left(\sum_{k\neq i,j} \bar{v}_k \dot{f}_{ij,k}(x) \right) + (\bar{v}_i \times \bar{v}_j) \cdot \left(\sum_{k\neq i,j} \sum_{l\neq i,jk} \frac{\bar{v}_l \times \bar{v}_k}{x_{kl}^2} f_{ij,k}(x) \right) + \\
&- (\bar{v}_i \cdot \bar{v}_j)|\bar{\alpha}_i - \bar{\alpha}_j|^2 + (\bar{v}_i \cdot \bar{\alpha}_i)(\bar{v}_j \cdot \bar{\alpha}_i) + (\bar{v}_i \cdot \bar{\alpha}_j)(\bar{v}_j \cdot \bar{\alpha}_j) - 2(\bar{v}_i \cdot \bar{\alpha}_j)(\bar{v}_j \cdot \bar{\alpha}_i) + \\
&+ \frac{\bar{v}_i \cdot (P_j^{\perp} \bar{\alpha}_j) + \bar{v}_j \cdot (P_i^{\perp} \bar{\alpha}_i)}{x_{ij}^2} + \sum_{k\neq i,j} \left(\frac{\bar{v}_j \cdot (P_i^{\perp} \bar{v}_k)}{x_{ik}^2} - \frac{\bar{v}_i \cdot (P_j^{\perp} \bar{v}_k)}{x_{jk}^2} \right) f_{ij,k}(x),
\end{aligned}$$

where $P_a^{\perp} = \mathbb{1} - \bar{v}_a \bar{v}_a^T$. The first term is of the same type as (4.19) and will vanish if all the vectors $\bar{v}_a, a =, 1, 2, ...N$ belong to the same plane, yet in this case the other terms will depend on the positions $x = (x_1, ..., x_N)$ and will not generally result in a vanishing derivative and a constant value of $|L_{ij}| = |(e_i|e_j)|$.

4.2.3 Rank $d > 1$

The example of $d = 2$ suggests that for any rank $d > 1$ there is room for nontrivial evolution of $|L(t)|^2 = |(e_i|e_j)|^2(t)$. A heuristic argument for this is that for $(e_i|e_j)(t)$ to differ from $(e_i|e_j)(0)$ only by a phase factor, the two vectors should aquire phase factors only (as in the $d = 1$ case) or rotate in a strictly synchronised way, which is likely only for isolated positions and momenta. Let us therefore reexpress the condition $|L(t)| = |L(0)|$ in the language of vectors and look for such sets of $|e_i)(0)$, that result in stationary couplings:

$$\forall_{i\neq j} |(e_i|e_j)|^2(t) = |(e_i|e_j)|^2(0). \tag{4.20}$$

All time derivatives at $t = 0$ must vanish for a constant function, but the vanishing of first two is a necessary condition:

$$\frac{d}{dt}|(e_i|e_j)|^2(0) = i \sum_{k\neq i,j} (e_i|[P_k, P_j]|e_i) f_{ij,k}(x)(0) = 0 \tag{4.21}$$

$$\frac{d^2}{dt^2}|(e_i|e_j)|^2(0) = i \sum_{k\neq i,j} \frac{d}{dt} \left[(e_i|[P_k, P_j]|e_i) f_{ij,k}(x) \right](0) = 0 \tag{4.22}$$

The first step is to search for vectors which satisfy (4.21), regardless of the initial positions. The $\binom{N}{3}$ position dependent factors are expressed only by $N-1$ independent relative distances. This is why it cannot be automatically assume that all $(e_i|[P_k,P_j]|e_i)$ must vanish. Yet it can be shown (see Appendix A.1) that when we expand (4.21) for a fixed pair $i<j$ in terms of independent distances (for example $x_{i,j\neq i}$) over a common denominator, the obtained polynomial expression will be identically zero if and only if all the coefficients, that is $(e_i|[P_k,P_j]|e_i)$, vanish. This condition can be rewritten as:

$$\forall_{k,i,j} Im((e_i|e_k)(e_k|e_j)(e_j|e_i)) = 0 \tag{4.23}$$

There are only two types of initial conditions which satisfy this set of equalities: either all vectors project on the same one-dimensional subspace: $|e_i) = e^{i\phi_i}|e)$ (this is, of course, the $d=1$ case discussed before) or all vectors are (gauge equivalent to) real $|e_i) = |e_i^R)$. It can be shown (see App. A.2) that in case of $d \geqslant 2$ this is the only possibility to satisfy (4.23).

In the next step we find the sets of real vectors which satisfy (4.22) as well. The time derivative can be expanded as follows:

$$\frac{d^2}{dt^2}|(e_i|e_j)|^2(0) = i\sum_{k\neq i,j}\frac{d}{dt}\left[(e_i|[P_k,P_j]|e_i)\right] f_{ij,k}(x) + (e_i|[P_k,P_j]|e_i)\dot{f}_{ij,k}(x). \tag{4.24}$$

In cases when (4.21) is satisfied, the terms proportional to $(e_i|[P_k,P_j]|e_i)$ vanish and we are left with

$$\sum_{k\neq i,j}\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\frac{d}{dt}\left[(e_i|[P_k,P_j]|e_i)\right] = 0. \tag{4.25}$$

Further expansion of the derivative:

$$\begin{aligned}
&\frac{d}{dt}\left[(e_i|[P_k,P_j]|e_i)\right] = (\dot{e}_i|[P_k,P_j]|e_i) + (e_i|[P_k,P_j]|\dot{e}_i) + (e_i|\frac{d}{dt}([P_k,P_j])|e_i) = \\
&= i\left(\sum_{l\neq i}\frac{1}{x_{il}^2}(e_i|[P_l,[P_k,P_j]]|e_i) + \sum_{l\neq j}\frac{1}{x_{jl}^2}(e_i|[P_k,[P_j,P_l]]|e_i)\right) + \\
&+si\sum_{l\neq k}\frac{1}{x_{kl}^2}(e_i|[P_j,[P_l,P_k]]|e_i)
\end{aligned} \tag{4.26}$$

makes use of the fact, that $\dot{P}_k = i\sum_{l\neq k} x_{kl}^{-2}[P_k,P_l]$. The last equality in (4.26) applied to (4.25) gives us a double sum with many terms which are difficult to manage in general. Yet we require (4.25) to be true for any initial distances x_{ab}, so we can choose for example a special configuration in which two particles, i^{th} and $(i+1)^{st}$, are close to each other, while the others are much further away: $|x_{i,i+1}| << |x_{kl}|$. This lets us take only the dominant term, proportional to $|x_{i,i+1}|^{-4}$ (others are of order $|x_{i,i+1}|^{-2}|x_{kl}|^{-2}$, or even smaller $|x_{kl}|^{-2}|x_{mn}|^{-2}$) and demand that it vanishes,

$$\frac{2|(e_i|e_{i+1})|^2}{x_{i,i+1}^4}\left(|(e_i|e_j)|^2 - |(e_{i+1}|e_j)|^2\right) = 0. \tag{4.27}$$

We assume nonzero repulsion between adjacent particles, $|(e_i|e_{i+1})| \neq 0$. This leads to a condition:

$$\forall_{i,j}|(e_i|e_j)|^2 = |(e_{i+1}|e_j)|^2 \tag{4.28}$$

For $d>1$ all the vectors must be real, $|e_i) \in \mathbb{R}^d$ and in this case (4.28) simplifies to

$$\forall_{i,j}(e_i|e_j) = \pm(e_{i+1}|e_j), \tag{4.29}$$

where we can choose any pair of $N-1$ nearest neighbours, thus the scalar product must be the same, up to a $\pm$ sign, for all possible pairs:

$$\forall_{i\neq j,k\neq l}(e_i|e_j) = \pm(e_k|e_l). \tag{4.30}$$

This corresponds to a system of evenly distributed vectors, i.e:

$$\forall_{i\neq j}(e_i|e_j) = \cos(\phi_{ij}) = (-1)^{n_{ij}}g, \quad (e_i|e_i) = 1, \quad 0 < g \leqslant 1, \tag{4.31}$$

or in matrix form:

$$L_{ij} = ig(1-\delta_{ij})e^{i\pi\cdot n_{ij}}, \quad n_{ij} \in \{0,1\}. \tag{4.32}$$

Importantly, the condition arrising from the colliding particles approximation already excludes the possibility of a matrix model recovering a system (3.2) with different coupling constants.

In the case of $N = 3$ this is a rank 1 matrix equivalent to $\pm L_0$ found in Section 4.2.1. Indeed, the characteristic equation of (4.32) in this case reads:

$$\lambda^3 - 3\lambda \mp 2 = (\lambda \pm 1)^2(\lambda \mp 2) = 0, \tag{4.33}$$

where the $\pm$ sign is given by $(-1)^{n_{123}}$ ($n_{ijk} = n_{ij} + n_{jk} + n_{ki}$), but regardless of the sign there is always a double eigenvalue ∓ 1. This means that we can shift it as explained in Section 4.1 and obtain an d=1 matrix:

$$L = \pm ig\,\{2|v_1)(v_1| - |v_2)(v_2| - |v_3)(v_3|\} \longrightarrow L \mp ig\mathbf{1} = \pm 3ig|v_1)(v_1|. \tag{4.34}$$

Of course if $\forall_i|e_i) = \pm|e)$, the condition (4.30) will be satisfied for $N > 3$ as well (this is again the known $d = 1$ case). But what happens if there are repeating vectors among $|e_i)$, but some are different? It means that for some fixed i, j, k we have $|e_j) = \pm|e_i), (e_i|e_j) = \pm 1$ and $|e_k) \neq \pm|e_i), |(e_i|e_k)| < 1$. Such a set cannot satisfy (4.30), and we are left with two options: all vectors are the same up to a sign $|e_i) = \pm|e)$ (this is the case of $d = 1$) or all N vectors are non trivially different, i.e., they project on different directions.

We need to know what is the rank d in the second case and instead of looking at the characteristic equation we will take up a more geometrical approach. We have N real vectors pointing at N different, but evenly distributed directions defined by (4.31). What is the dimension of a real space in which such a construction is possible? Of course $\mathbb{R}^N$ is enough since it contains N different orthogonal vectors forming a standard basis $\{\hat{e}_i : i = 1, 2, ..N\}$. Projecting this set of vectors onto an $N-1$-dimensional subspace orthogonal to $e_+ = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\hat{e}_i$ reduces the necessary dimension to $N-1$. This is equivalent to subtracting $\mathbb{1}\lambda_{min}$ from (4.31) and making it positive semi-definite if the corresponding eigenspace is one-dimensional.

For some particular configurations of n_{ij} the eigenspace of the extreme eigenvalue can have dimension higher than one, thus the span of $|e_i)$ vectors can be reduced to $d < N-1$. The possible ranks, that is the dimensions in which $N \geqslant 4$ vectors can be packed, are presented in table 4.1. For example in $\mathbb{R}^3$ there can be up to 6 vectors - they point to the vertices of a regular icosahedron. There are 12 of them, but once a vertex is chosen,its antipodal vertex must be excluded. For $N = 8, 9, 10, 11$ vectors defined in (4.31) fit in $N-1, N-2$ or $N-3$-dimensional real space. An anomaly occurs for $N = 7$ since $\mathbb{R}^4$ accepts only up to 6 vectors.

In the last step we look at the second time derivative (4.22) in an arbitrary configuration of initial positions. Expanding (4.25) with the use of (4.26) and expressing the left hand side with a set of independent distances (see Appendix A.3) results in a condition:

$$\forall_{i\neq j\neq k\neq l}(e_i|[P_l,[P_k,P_j]]|e_i) = 0. \tag{4.35}$$

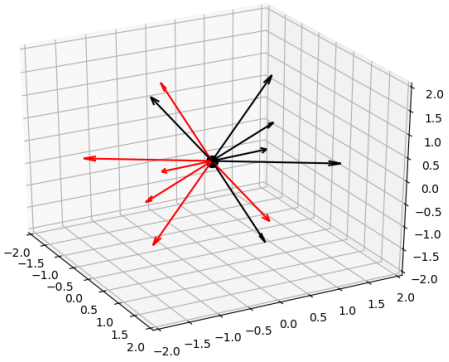


Figure 4.1: In $\mathbb{R}^3$ up to six $|e_i)$ vectors can satisfy (4.31). They point at the vertices of the regular icosahedron, and once a vertex is chosen, its antipodal must be excluded.

N	d
4	$1, 3$
5	$1, 3, 4$
6	$1, 3, 4, 5$
7	$1, 5, 6$
$8, 9, 10, 11$	$1, N-3, N-2, N-1$

Table 4.1: Possible ranks of $N \times N$ (4.32) matrices calculated numerically for different combinations of $\pm$ signs.

in case of real vectors satisfying (4.30) this is equivalent to

$$\forall_{i\neq j\neq k\neq l} n_{ij} + n_{kl} = n_{ik} + n_{jl} = n_{il} + n_{kj}, \tag{4.36}$$

which gives $\binom{N}{4}$ sets of equalities. In case of $N = 4$ there is just one, that is:

$$n_{12} + n_{34} = n_{13} + n_{24} = n_{14} + n_{23}. \tag{4.37}$$

Keeping in mind that $n_{ij} \in \{0, 1\}$ and that by addition we mean addition *modulo* 2, we can transform these equalities into

$$n_{12} + n_{23} + n_{31} = n_{12} + n_{24} + n_{42} = n_{13} + n_{34} + n_{41} = n_{23} + n_{34} + n_{42}. \tag{4.38}$$

The simple relation between n_{ij} and the phases φ_{ij} defined in Section 3.1.1, that is $\varphi_{ij} = \frac{\pi}{2} + n_{ij}\pi$, leads to the conclusion that all cyclic sums Φ_{ijk} defined by (3.33) must be the same, and the resulting L matrix is equivalent to $L_{0,ij} = ig(1 - \delta_{ij})$ found in Section 4.1. The generalisation of this fact onto $N > 4$ is simple: the $\binom{N}{4}$ relations $\Phi_{ijk} = \Phi_{ijl} = \Phi_{ikl} = \Phi_{jkl}$ imply that Φ_{abc} is the same for all choices of a, b, c, and this is the definition of matrices which are equivalent to $L_{0,ij} = ig(1-\delta_{ij})$. This proves that the matrix found in Section 4.1, that is the one which corresponds to $d = 1$, is the only one (up to gauge equivalence) which recovers the ordinary CM model. Any higher rank d allows a nontrivial unitary evolution.

We summarize the findings of this section as the following observations,

1. ***Orbits of generalized Caloger-Moser systems can be classified by rank, i.e., the dimension of the space of internal, vectorial degrees of freedom***

2. ***Rank $d = 1$ corresponds to the ordinary Calogero-Moser system. This is the only rank for which we can obtain constant values of $|L(t)|$.***

4.3 Beyond the repulsive CM model

In this section I present a model which couples the matrix and vectorial degrees of freedom. This coupling leads to a new integrable system with additional $\frac{1}{x}$ interactions between particles in the reduced phase space.

4.3.1 The CM model in the extended phase space

The special case of the Calogero-Moser Hamiltonian $H = \frac{1}{2}\mathrm{Tr}(Y^2)$ leads to a very simple solution of (3.47), for which the (D, V, E) dynamics is similar to what we know from Section 3.1.1. Let us define an anti-Hermitian matrix valued function S:

$$S = [D, V] - iE^\dagger E, \qquad \dot{S} = [A, S] \tag{4.39}$$

Initial conditions $(X_0, Y_0, \mathcal{E}_0)$ that satisfy $S_0 = [X_0, Y_0] - i\mathcal{E}_0^\dagger \mathcal{E}_0 = -ig \cdot \mathbb{1}$ define special orbits for which $S(t) = S_0$. When restricted to these orbits, trajectories of L_{ij} and $(e_i|e_j)$ coincide (this is exactly the equivalence stated in Section 4.1). Other choices of $S(0)$ will result simply in a unitary evolution $E(t) = \mathcal{E}_0 U^\dagger(t)$ with no influence of these degrees of freedom on the (X, Y) dynamics.

4.3.2 Example of interacting matrix and vectorial degrees of freedom

The matrix and vectorial degrees of freedom can be coupled through a modified Hamilton function on the extended phase space. One of the simplest examples of such a coupling is:

$$H_E = \frac{1}{2}\mathrm{Tr}((Y + \xi\mathcal{E}^\dagger\mathcal{E})^2) \tag{4.40}$$

where the free system is modified with a $\xi\mathcal{E}^\dagger\mathcal{E}$ Hermitian matrix, and ξ^{-1} has the dimension of X. The resulting equations of motion are:

$$\dot{X} = Y + \xi\mathcal{E}^\dagger\mathcal{E}, \qquad \dot{Y} = 0, \qquad \dot{\mathcal{E}} = -i\xi\mathcal{E}(Y + \xi\Phi), \qquad \dot{\Phi} = [i\xi Y, \Phi]\,, \tag{4.41}$$

where $\Phi = \mathcal{E}^\dagger\mathcal{E}$. The solutions for Y and Φ are straightforward, while $\mathcal{E}$ contain a nontrivial $u(t, \Phi) \in U(N)$ modification which cancels in the Φ evolution:

$$Y(t) = Y_0, \qquad \Phi(t) = e^{i\xi Y_0 t}\Phi_0 e^{-i\xi Y_0 t}, \qquad \mathcal{E}(t) = u(t, \Phi)\mathcal{E}_0 e^{-i\xi t Y_0}, \qquad \dot{u} = -i\xi^2\Phi u. \tag{4.42}$$

The equation for X becomes:

$$\dot{X} = Y_0 + \xi e^{i\xi t Y_0}\Phi_0 e^{-i\xi t Y_0}. \tag{4.43}$$

If $[Y_0, \Phi_0] = 0$, the solution is very simple: $X(t) = X_0 + t(Y_0 + \xi\Phi_0)$ and equivalent to the case discussed in section 3.1.1. If it is not the case, we obtain:

$$X(t) = X_0 + tY_0 + \xi\int_0^t e^{i\xi\tau Y_0}\Phi_0 e^{-i\xi\tau Y_0} d\tau, \tag{4.44}$$

which is easiest to interpret when expressed in the diagonal basis of Y_0 as:

$$(X(t))_{ij} = X_{0,ij} + t\left[\delta_{ij}(Y_{0,ii} + \xi\Phi_{0,ii}) + \xi\delta(\Delta y_{ij})\Phi_{0,ij}\right] + \frac{f(\xi\Delta y_{ij}t)}{\Delta y_{ij}}\Phi_{0,ij} \tag{4.45}$$

where $\Delta y_{ij} = Y_{0,ii} - Y_{0,jj}$ and $f(\tau) = \sin(\tau) + i(1 - \cos(\tau))$. Clearly $X(t)$ performs a spiral motion. The direction of its linear component is dictated by Y_0 (with an additional boost given by the projection of Φ_0 on this direction, if we think of the Hermitian matrices as a vector space with $(A, B) = \mathrm{Tr}(AB)$). The periodic motion takes place in the subspace perpendicular to Y_0 (again in the sense of $(A, B) = \mathrm{Tr}(AB)$ scalar product). This system turns out to have a constant of motion:

$$[X(t), Y(t)] - i\Phi(t) = [X_0, Y_0] - i\Phi_0 = S_0. \tag{4.46}$$

i.e. $[X(t), Y(t)] - i\Phi(t)$ is a constant of motion, just like for the simple Y dependent Hamilton function. The $X(t)$ trajectories, though more complicated than simple

lines obtained in 3.1.1 are still numerically accessible and it is worth looking at the dynamics of their eigenvalues. This is why in the next step I parametrize the flow with eigenvalues of $X(t)$ in the known way, with the use of (3.47). From now on the $E = (|e_1(t), |e_2(t), ..., |e_N(t))$ degrees of freedom will be tracked with entries of $\Omega(t) = U(t)\Phi(t)U^\dagger(t)$. The obtained equations are:

$$\dot{D} = [A, D] + V + \xi\Omega, \qquad \dot{V} = [A, V], \qquad \dot{\Omega} = [A + i\xi V, \Omega], \qquad \dot{L} = [A, L] - \xi[V, \Omega], \tag{4.47}$$

where $L = [D, V]$, and $A(t) = \dot{U}(t)U^\dagger(t)$ as usual. An anti-Hermitian matrix (4.39) can be introduced for a general system defined by (3.47). It may not necessarily be a Lax operator due to an additional term:

$$\dot{S} = [A, S] + U\left[\frac{d}{dt}\left([X(t), Y(t)] - i\mathcal{E}^\dagger\mathcal{E}\right)\right]U^\dagger, \tag{4.48}$$

but in the considered case this term vanishes, as stated in (4.46). This means $S = [D, V] - i\Omega$ is a Lax operator and just as for the matrix dynamics in Section 3.1.1, we have a vast set of constants of motion,

$$I_{k_1,k_2...,k_m} = \mathrm{Tr}(S^{k_1}V^{k_2}\cdots S^{k_{m-1}}V^{k_m}), \tag{4.49}$$

where $k_1, ..., k_m$ are non-negative integers. The Hamilton function (4.40) in the (D, V, Ω, L) parametrization reads:

$$H_E = \frac{1}{2}\mathrm{Tr}((V + \xi\Omega)^2). \tag{4.50}$$

In the $(x_i = D_{ii}, p_i = V_{ii}, L, \Omega)$ variables

$$H_E(x, p, L, \Omega) = \frac{1}{2}\sum_i (p_i + \xi\Omega_{ii})^2 + \sum_{i<j}\frac{|L_{ij}|^2}{x_{ij}^2} + \frac{2\xi\Re(L_{ij}\Omega^*_{ij})}{x_{ij}} + \xi^2|\Omega_{ij}|^2 \tag{4.51}$$

and it suggests that the additional degree of freedom introduces a long distance, $1/x_{ij}$ interaction potential between particles. The symplectic form (3.44) differs from (3.23) by the $i\mathrm{Tr}(d\mathcal{E}^\dagger \wedge d\mathcal{E})$ term, which results in replacing L with $S = L - i\Omega$:

$$\omega_E = \sum_{i=1}^{N} dp_i \wedge dx_i + id(e_i| \wedge d|e_i) - \mathrm{Tr}(dS \wedge a) + \mathrm{Tr}(Sa \wedge a). \tag{4.52}$$

The variables $x_i = D_{ii}$ and $p_i = V_{ii}$ are still canonically conjugate, but the mechanical momentum, let us call it π_i, contains $\Omega_{ii} = iS_{ii}$ terms as well. The equations of motion for $(x_i, \pi_i = p_i + \xi\Omega_{ii})$ contain terms with not only $1/x_{ij}^3$ and $1/x_{ij}^2$ as expected at first glance (and present in the $\dot{p}_i = -\partial_{x_i}H$ equation), but $1/x_{ij}$ as well:

$$\dot{x}_i = \pi_i \tag{4.53}$$

$$\dot{\pi}_i = \sum_{j\neq i}\frac{2|S_{ij} + i\Omega_{ij}|^2}{x_{ij}^3} + \frac{4\xi\Re(S_{ij}\Omega^*_{ij})}{x_{ij}^2} - \frac{2\xi^2\Im(S_{ij}\Omega^*_{ij})}{x_{ij}}, \tag{4.54}$$

while the equations of motion for S_{ij} and Ω_{ij} are even more involved:

$$\dot{\Omega}_{ii} = i\dot{S}_{ii} = \sum_{j\neq i} \frac{2\Re(S_{ij}\Omega^*_{ij})}{x^2_{ij}} - \frac{2\xi\Im(S_{ij}\Omega^*_{ij})}{x_{ij}} \tag{4.55}$$

$$\dot{S}_{ij} = \sum_{k\neq i,j} \left(\frac{S_{ik}+i\Omega_{ik}}{x^2_{ik}} + \frac{\xi\Omega_{ik}}{x_{ik}}\right) S_{kj} - S_{ik}\left(\frac{S_{kj}+i\Omega_{kj}}{x^2_{kj}} + \frac{\xi\Omega_{kj}}{x_{kj}}\right) + \\ +i\left(\frac{S_{ij}+i\Omega_{ij}}{x^2_{ij}} + \frac{\xi\Omega_{ij}}{x_{ij}}\right)(\Omega_{ii}-\Omega_{jj}) \tag{4.56}$$

$$\dot{\Omega}_{ij} = \sum_{k\neq i,j} \left(\frac{S_{ik}+i\Omega_{ik}}{x^2_{ik}} + \frac{i\xi S_{ik}}{x_{ik}}\right) \Omega_{kj} - \Omega_{ik}\left(\frac{S_{kj}+i\Omega_{kj}}{x^2_{kj}} + \frac{i\xi S_{kj}}{x_{kj}}\right) + \\ -\left(\frac{S_{ij}+i\Omega_{ij}}{x^2_{ij}} + \frac{i\xi S_{ij}}{x_{ij}}\right)(\Omega_{ii}-\Omega_{jj}) + i\xi\Omega_{ij}(\pi_i-\pi_j). \tag{4.57}$$

A set of initial conditions such that:

$$\forall_{i\neq j}\frac{d}{dt}\left(S_{ij}\Omega_{ji}\right) = 0, \qquad \frac{d}{dt}\left(|S_{ij}+i\Omega_{ij}|^2\right) = 0 \tag{4.58}$$

would lead to a dynamical system with a clearer physical interpretation:

$$\dot{x}_i = \pi_i, \qquad \dot{\pi}_i = \sum_{i<j}\frac{c_1}{x^3_{ij}} + \frac{c_2}{x^2_{ij}} + \frac{c_3}{x_{ij}}. \tag{4.59}$$

My next step is to explore this possibility.

Exact solution for $N = 2$

The simplest case of $N = 2$ can be solved analytically. Let us define the initial conditions *via* Pauli matrices:

$$X_0 = x_0\sigma_z \qquad Y_0 = y_0(\bar{n}_y\cdot\bar{\sigma}) \qquad \Phi_0 = \phi_0(\bar{n}_\phi\cdot\bar{\sigma}), \tag{4.60}$$

where $\bar{n}_y, \bar{n}_\phi$ are unit vectors, and y_0, ϕ_0 are positive constants. Note that in this definition all the matrices are traceless. The equations of motion for traces (which correspond to the center of mass motion) separate, and the equations for the traceless part reflect the relative motion. Naturally the $Y(t)$ matrix is constant, and:

$$\Phi(t) = \phi_0\left[(\bar{n}_y|\bar{n}_\phi)\bar{n}_y + \cos(2\xi y_0 t)(\bar{n}_\phi - (\bar{n}_y|\bar{n}_\phi)\bar{n}_y) - \sin(2\xi y_0 t)(\bar{n}_y\times\bar{n}_\phi)\right]\cdot\bar{\sigma} = \\ = \Phi_0^{||Y_0} + \cos(2\xi y_0 t)\Phi_0^{\perp Y_0} + \sin(2\xi y_0 t)\Psi^{\perp Y_0,\Phi_0} \tag{4.61}$$

$$X(t) = X_0 + t\left(Y_0 + \xi\Phi_0^{||Y_0}\right) + \frac{1}{2y_0}\left(\sin(2\xi y_0 t)\Phi^{\perp Y_0} + (1-\cos(2\xi y_0 t))\Psi^{\perp Y_0,\Phi_0}\right) \\ = \bar{d}(t)\cdot\bar{\sigma}. \tag{4.62}$$

If we treat the above matrices as vectors in $\mathbb{R}^3$, we may say that the component of $\Phi(t)$ parallel to Y_0 is constant, and the perpendicular one circulates with frequency $2\xi y_0$. The trajectory of $X(t)$ is therefore a spiral, as expected. If we interpret the eigenvalues of $X(t)$ as positions of the particles, the distance between them, $r(t) = 2|\bar{d}(t)|$) oscillates. The oscillations are vanishing slowly, which shows that there is a long distance interaction between particles. The trajectory of $X(t)$ and its eigenvalues are illustrated in Figure 4.2. Let us express the matrices in the diagonal basis of $X(t)$:

$$D(t) = \frac{r(t)}{2}\sigma_z, \qquad \Omega(t) = \bar{\Omega}(t)\cdot\bar{\sigma} \tag{4.63}$$

$$iS(t) = \bar{s}(t)\cdot\bar{\sigma} = (rv_y+\Omega_x)\sigma_x + (-rv_x+\Omega_y)\sigma_y + \Omega_z\sigma_z \tag{4.64}$$

$$V(t) = \bar{v}(t)\cdot\bar{\sigma} = \frac{(\Omega_x - s_x)\sigma_x + (S_y-\Omega_y)\sigma_y}{r} + p(t)\sigma_z \tag{4.65}$$

The Hamilton function (4.50) in this case is equal to $|\bar{v}+\xi\bar{\Omega}|^2$ and in the new parametrisation it has the form:

$$H_2(r,p,s_x,s_y,\Omega_x,\Omega_y)=(p+\xi\Omega_z)^2+\frac{|\bar{s}-\bar{\Omega}|^2}{r^2}+\frac{2\xi(\bar{s}\times\bar{\Omega})_z}{r}+\xi^2(\Omega_x^2+\Omega_y^2). \quad (4.66)$$

Adjusting the equations (4.47) equations to the vectorial form we obtain:

$$\dot{\bar{v}}=2i(\bar{a}\times\bar{v}),\qquad \dot{\bar{\Omega}}=2i(\bar{a}+i\xi\bar{v})\times\bar{\Omega},\qquad \dot{\bar{s}}=2i(\bar{a}\times\bar{s}) \quad (4.67)$$

From a geometrical point of view we have two vectors, $\bar{\Omega}$ and $\bar{s}$ rotating in $\mathbb{R}^3$ around two different axes. If some choice of $\bar{s}$ and $\bar{\Omega}$ were to recover (4.59), their difference $\bar{s}-\bar{\Omega}=i\bar{l}$ would have to have a constant length. But this would imply:

$$0=\frac{d}{dt}|\bar{s}-\bar{\Omega}|^2=-2\frac{d}{dt}(\bar{s}\cdot\bar{\Omega})=4\xi\bar{s}\cdot(\bar{v}\times\bar{\Omega}), \quad (4.68)$$

thus the three vectors, $\bar{s}$,$\bar{v}$ and $\bar{\Omega}$) stay in the same plane. We know from (4.67) that both $\bar{s}$ and $\bar{v}$ rotate around the same axis, so in their rotating frame $\bar{\Omega}$ will leave their common plane if it rotates around any other axis. For that not to happen we must require $\bar{v}\times\bar{\Omega}=0$,which in the language of matrices means that $[V(t),\Omega(t)]=0$. This vanishing commutator means that in the initial basis $[Y_0,\Phi_0]=0$ and the system is equivalent to the system without additional degrees of freedom, that is just the two particle Calogero-Moser system with $c_1>0$ and $c_2=c_3=0$ in (4.59).

Generalization to higher values of N

As stated in (4.45), the spiral motion in the matrix space is not a unique feature of the $N=2$ case. For any value of N we can express $X(t)$ in the diagonal basis of Y_0:

$$(X(t))_{ij} = X_{0,ij}+t\left(Y_0+\xi\Phi^{||Y_0}\right)_{ij}+\frac{i}{\Delta y_{ij}^2}[Y_0,\Phi_0]_{ij}(1-e^{i\xi\Delta y_{ij}t}) \quad (4.69)$$

$$(\Phi_0^{||Y_0})_{ij} = \delta_{ij}\Phi_{0,ij}+(1-\delta_{ij})\delta(\Delta y_{ij})\Phi_{0,ij},\qquad [Y_0,\Phi_0^{||Y_0}]=0 \quad (4.70)$$

where $\Delta y_{ij}=Y_{0,ii}-Y_{0,jj}$ and

$$f(\tau)=\sin(\tau)+i(1-\cos(\tau))=i(1-e^{i\tau}). \quad (4.71)$$

Therefore $X(t)-X_0$ has a linear part stemming from Y_0 and the part of Φ_0 which commutes with Y_0 and a circulating part given by the entries of $[Y_0,\Phi_0]_{ij}=\Delta y_{ij}\Phi_{0,ij}$. The frequency of oscillation of the ij^{th} entry is $\omega_{ij}=\xi\Delta y_{ij}$. Oscillations of matrix elements will be reflected in oscillating eigenvalues, and the only Φ_0 for which the oscillations vanish is the one which commutes with Y_0 and ads nothing but a boost to the (X,Y) matrix motion. Eigenvalues of $X(t)$ are interpreted as positions of particles in an $N-body$ one-dimensional system. If the positions of the 1^{st} and N^{th} particle oscillate in time, it means that the interaction potential continuously changes between attractive and repulsive. This is why we should expect that for a general N once we have $[Y_0,\Phi_0]\neq 0$, a stationary case (4.59) cannot be recovered. Nevertheless, the conditions (4.58) need to be solved with the use of (4.47):

$$\frac{d}{dt}\left(|S_{ij}+i\Omega_{ij}|^2\right)=0 \implies \sum_{ij}\frac{d}{dt}\left(|S_{ij}+i\Omega_{ij}|^2\right)=0$$

$$\frac{d}{dt}\mathrm{Tr}((S+i\Omega)^2)=0 \implies \mathrm{Tr}(S[\Omega,V])=0 \quad (4.72)$$

$$\frac{d}{dt}\left(S_{ij}\Omega_{ji}\right)=0 \implies \frac{d}{dt}\mathrm{Tr}(S\Omega)=i\xi\mathrm{Tr}(S[\Omega,V])=0. \quad (4.73)$$

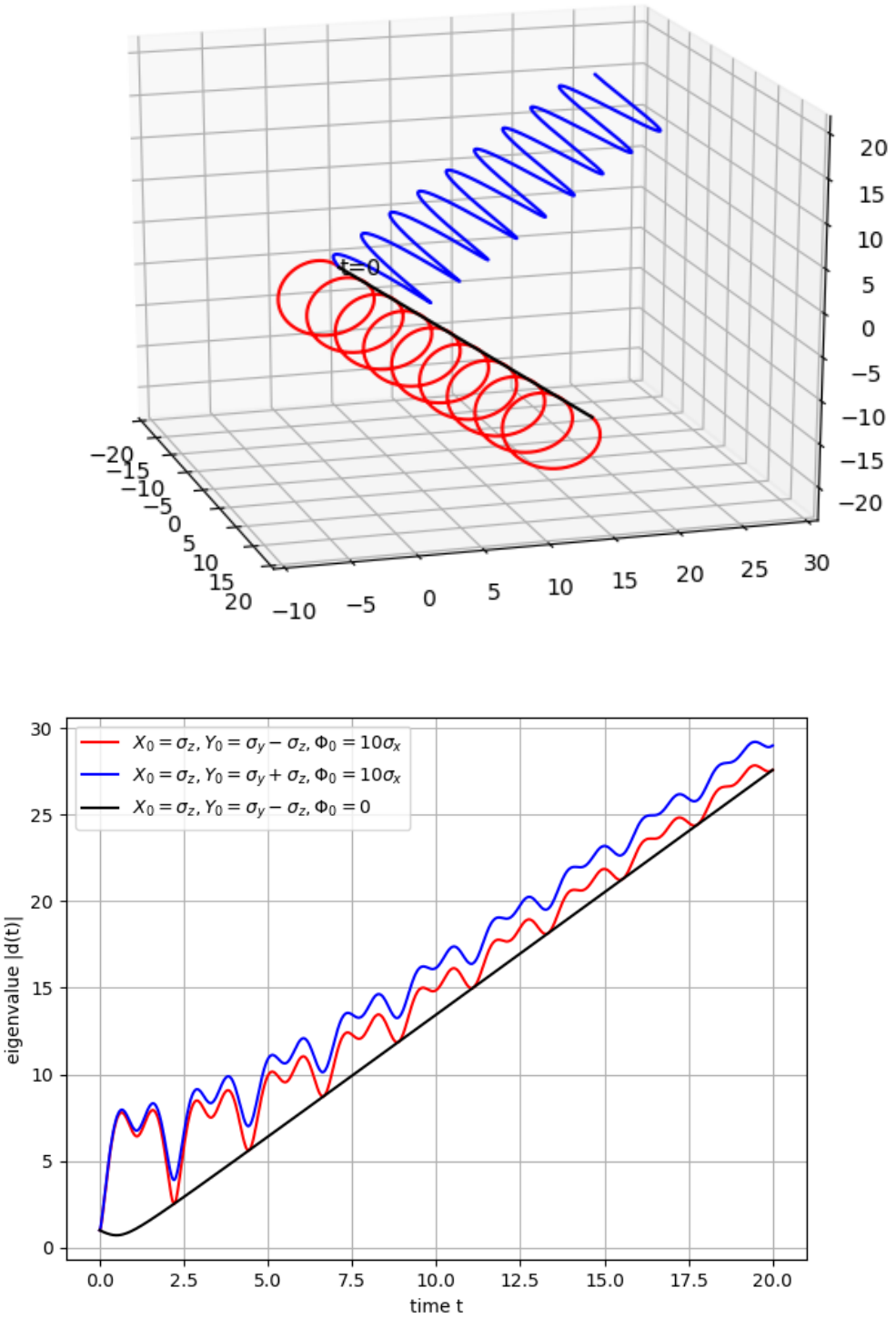


Figure 4.2: The trajectories and lengths of $\bar{d}(t)$.The black lines represent the unextended model with $\Phi_0 = 0$. The legend applies to both plots.

Both conditions in (4.47) imply a weaker condition (an N-particle counterpart of $\bar{s}\cdot(\bar{v}\times\bar{\Omega})=0$), which can be now expressed in terms of $(X,Y,\Phi)(t)$:

$$\mathrm{Tr}(S[\Omega,V]) \quad = \quad \mathrm{Tr}([D,V][\Phi,V]) = \mathrm{Tr}([X(t),Y_0][\Phi(t),Y_0]) = 0. \tag{4.74}$$

This must be true for all $t\geqslant 0$, therefore:

$$\mathrm{Tr}([X_0,Y_0][\Phi_0,Y_0]) = 0, \qquad \frac{d}{dt}\left[\mathrm{Tr}([X(t),Y_0][\Phi(t),Y_0])\right]|_{t=0} = 0. \tag{4.75}$$

The first equation implies one of two possibilities: either $[X_0,Y_0]=i\eta\Phi_0$ for some $\eta\in\mathbb{R}$ or $[\Phi_0,Y_0]=0$. The second, after applying the Leibniz rule and the equations for $(\dot{X},\dot{\Phi})(t)$ reads:

$$\mathrm{Tr}([\Phi_0,Y_0]^2) + i\xi\mathrm{Tr}([X_0,Y_0][[Y_0,\Phi_0],Y_0]) = 0. \tag{4.76}$$

It is automatically satisfied if $[\Phi_0,Y_0]=0$. If $[X_0,Y_0]=i\eta\Phi_0$, it implies:

$$(1-\xi\eta)\mathrm{Tr}([\Phi_0,Y_0]^2) = 0. \tag{4.77}$$

The chosen Φ_0 should recover (4.59) for arbitrary initial positions encoded in X_0, so η is unrestricted. Therefore $\mathrm{Tr}([\Phi_0,Y_0]^2)=0$ which corresponds to an non-extended system. The constant potential is then realised by $[X_0,Y_0]=L_0$ given by (3.35), $S_{ij}=-ig\delta_{ij}$ and $c_1=g^2$, $c_2=c_3=0$.

4.4 Classical counterparts of models with spin state exchange

The Calogero-Moser models presented in 3.1.2 have not been quantised yet, but in section 3.2.2 we have seen quantum extensions with spin degrees of freedom, where in the place of L_{ij} variables we have matrices acting on spin states. Let us recall that the above models are not a quantum analogue of $H=\frac{1}{2}\mathrm{Tr}(V^2)$ but of a sum of all terms,

$$H = \frac{1}{2}\sum_{i,j}(V^2)_{ij}, \tag{4.78}$$

which motivates us to look at the classical equations of motion for such a Hamilton function.

4.4.1 The general solution

The classical counterpart of the discussed Hamiltonians in the (X,Y) phase space:

$$H = \frac{1}{2}\sum_{i,j}(Y^2)_{ij}, \tag{4.79}$$

results in the following equations of motion:

$$\dot{X}_{ij} = Y_{ij} + \frac{1}{2}\sum_{k\neq j}Y_{ik} + \frac{1}{2}\sum_{m\neq i}Y_{mj}, \qquad \dot{Y}_{ij}=0, \tag{4.80}$$

which is again a straight line in the space of Hermitian matrices:

$$X(t) = X_0 + \frac{t}{2}[M,Y_0]_+, \qquad Y(t)=Y_0, \tag{4.81}$$

where $M_{ij} = 1$ and $[...]_+$ is the matrix anticommutator. It is much more interesting to look at the harmonic case:

$$H = \frac{1}{2}\sum_{i,j}(Y^2 + \omega^2 X^2)_{ij}, \tag{4.82}$$

where the equations of motion take the following form:

$$\dot{X} = \frac{1}{2}[M, Y]_+, \qquad \dot{Y} = -\frac{\omega^2}{2}[M, X]_+. \tag{4.83}$$

Just as in the simple harmonic oscillator case, we can solve the second order differential equation for a single matrix:

$$\ddot{X} = -\frac{\omega^2}{4}[M, [M, X]_+]_+, \tag{4.84}$$

(the same for Y). One of the possibilities is to vectorize the matrices and solve the problem using standard diagonalization, but there is a simpler way: we notice that $M = N|e_0\rangle\langle e_0|$, where $\langle e_0| = \frac{1}{\sqrt{N}}(1, 1, 1, ..., 1)$, is proportional to a one-dimensional projection. Therefore we can express the equation (4.84) in an orthonormal basis containing $|e_0\rangle$:

$$X = \begin{pmatrix} x_0 & x_{i0} \\ x_{i0}^\dagger & x_{ij} \end{pmatrix}, \quad M = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}, \quad \ddot{X} = -\frac{1}{4}[M, [M, X]_+]_+, \tag{4.85}$$

i.e., explicitly,

$$\begin{pmatrix} \ddot{x}_0 & \ddot{x}_{i0} \\ \ddot{x}_{i0}^\dagger & \ddot{x}_{ij} \end{pmatrix} = \begin{pmatrix} -(N\omega)^2 x_0 & -\frac{(N\omega)^2}{4} x_{i0} \\ -\frac{(N\omega)^2}{4} x_{i0}^\dagger & 0 \end{pmatrix}, \tag{4.86}$$

where $x_{i0} = \langle e_0|X|e_i\rangle$ is a row vector of length $N-1$, $x_{i0}^\dagger$ - a column, and $x_{ij} = \langle e_i|X|e_0\rangle$ - an $(N-1)\times(N-1)$ matrix. The same procedure applies to Y, and with the use of the first order equations:

$$\begin{pmatrix} \dot{x}_0 & \dot{x}_{i0} \\ \dot{x}_{i0}^\dagger & \dot{x}_{ij} \end{pmatrix} = \begin{pmatrix} Ny_0 & \frac{N}{2} y_{i0} \\ \frac{N}{2} y_{i0}^\dagger & 0 \end{pmatrix}, \quad \begin{pmatrix} \dot{y}_0 & \dot{y}_{i0} \\ \dot{y}_{i0}^\dagger & \dot{y}_{ij} \end{pmatrix} = \begin{pmatrix} -N\omega^2 x_0 & -\frac{N\omega^2}{2} x_{i0} \\ -\frac{N\omega^2}{2} x_{i0}^\dagger & 0 \end{pmatrix} \tag{4.87}$$

we find the solution:

$$X(t) = \begin{pmatrix} x_0 \cos(N\omega t) + \frac{y_0}{\omega}\sin(N\omega t) & x_{i0}\cos(\frac{N\omega t}{2}) + \frac{y_{i0}}{\omega}\sin(\frac{N\omega t}{2}) \\ (x_{i0}\cos(\frac{N\omega t}{2}) + \frac{y_{i0}}{\omega}\sin(\frac{N\omega t}{2}))^\dagger & x_{ij} \end{pmatrix}, \tag{4.88}$$

$$Y(t) = \begin{pmatrix} y_0 \cos(N\omega t) - \omega x_0 \sin(N\omega t) & y_{i0}\cos(\frac{N\omega t}{2}) - \omega x_{i0}\sin(\frac{N\omega t}{2}) \\ (y_{i0}\cos(\frac{N\omega t}{2}) - \omega x_{i0}\sin(\frac{N\omega t}{2}))^\dagger & y_{ij} \end{pmatrix}, \tag{4.89}$$

where the $x, y_{0,i,j}$ are the elements of the initial X_0, Y_0 matrices in the chosen basis. We see that the $|e_0\rangle\langle e_0|$ component oscillates with the frequency $N\omega$, the $|e_0\rangle\langle e_i|$ and $|e_i\rangle\langle e_0|$ components oscillate as well, but with a twice smaller frequency while the block which is orthogonal to $|e_0\rangle$ stays constant. If we take the limit $\omega \to 0$ of (4.88) and (4.89), we will obtain (4.81), that is the dynamics of the Hamiltonian (4.79). In the last step we may return to the diagonal basis of X_0, although then the results become less transparent - they are just linear combinations of constants and oscillating functions.

4.4.2 The $N = 2$ case

We shall illustrate the above considerations with explicit solutions for $N = 2$. Let us define the evolving matrices in terms of functions, together with the initial conditions:

$$X(t) = x_0(t)\mathbb{1} + \bar{x}(t)\cdot\bar{\sigma}, \qquad X_0 = x_{00}\mathbb{1} + x_{z,0}\sigma_z, \tag{4.90}$$

$$Y(t) = y_0(t)\mathbb{1} + \bar{y}(t)\cdot\bar{\sigma}, \qquad Y_0 = y_{00}\mathbb{1} + \bar{y}_0\cdot\bar{\sigma}. \tag{4.91}$$

The solution of the linear case is simple:

$$X(t) = X_0 + t(Y_0 + y_{00}\sigma_x + y_{0x}\mathbb{1}) \tag{4.92}$$

As $M = \mathbb{1} + \sigma_x$, the basis which we use in the harmonic case is the eigenbasis of σ_x: $|e_0) = (1,1)/\sqrt{2}, |e_1) = (1,-1)/\sqrt{2}$. Let us set $\omega = 1$ and write down the equations (4.84) and (4.87) in this basis:

$$\frac{d^2}{dt^2}\begin{pmatrix} x_0(t)+x_x(t) & x_z(t)+ix_y(t) \\ x_z(t)-ix_y(t) & x_0(t)-x_x(t)\end{pmatrix} = -\begin{pmatrix} 4(x_0(t)+x_x(t)) & x_z(t)+ix_y(t) \\ x_z(t)-ix_y(t) & 0\end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_0(t)+x_x(t) & x_z(t)+ix_y(t) \\ x_z(t)-ix_y(t) & x_0(t)-x_x(t)\end{pmatrix} = \begin{pmatrix} 2(y_0(t)+y_x(t)) & y_z(t)+iy_y(t) \\ y_z(t)-iy_y(t) & 0\end{pmatrix}$$

and the solutions are the following:

$$x_0(t) = \frac{1}{2}\{x_{00}(1+\cos(2t)) + (y_{00}+y_{0x})\sin(2t)\}, \tag{4.93}$$

$$y_0(t) = \frac{1}{2}\{(y_{00}+y_{0x})\cos(2t) - x_{00}\sin(2t) + y_{00} - y_{0x}\}, \tag{4.94}$$

$$\bar{x}(t) = \begin{pmatrix} x_0(t) - x_{00} \\ y_{0y}\sin(t) \\ x_{0z}\cos(t) + y_{0z}\sin(t)\end{pmatrix}, \tag{4.95}$$

$$\bar{y}(t) = \begin{pmatrix} y_0(t) - y_{00} + y_{0x} \\ y_{0y}\cos(t) \\ -x_{0z}\sin(t) + y_{0z}\cos(t)\end{pmatrix}. \tag{4.96}$$

Their linear approximations coincide with (4.92). The traces of $X(t)$ and $Y(t)$, that is $2x_0(t)$ and $\mathrm{Tr}2y_0(t)$ respectively, oscillate with a period $T = \pi$. The difference between them is a $\frac{\pi}{4}$ shift in time and a constant. This is why they easily combine to a constant of motion:

$$(\mathrm{Tr}X(t) - x_{00})^2 + (\mathrm{Tr}Y(t) - y_{00} + y_{0x})^2 = x_{00}^2 + (y_{00} - y_{0x})^2 \tag{4.97}$$

The $\bar{x}(t), \bar{y}(t)$ vectors move along periodic trajectories, which can be viewed as deformed ellipses. In particular, if the initial condition has no σ_y component, the resulting loop will be 2-dimensional.

The attractiveness of this model is revealed in higher dimensions, where out of all N^2 degrees of freedom only $N^2-(N-1)^2 = 2N-1$ evolve in time, while $(N-1)^2$ remain constant. Its major drawback on the other hand is such that the Hamilton functions constructed from the sum of all matrix elements instead of traces lacks $U(N)$ symmetry. The unitary reduction along the lines presented in Section 3.1.1 is generally impossible. Yet $U(N-1)$ transformations preserving the $|e_0\rangle$ direction are valid symmetries in this case, and the possible consequences of this symmetry can potentially be useful.

4.5 Evolution of the internal degrees of freedom

This section is focused on the interplay between the L_{ij} variables and the spatial degrees of freedom. Firstly I examine the influence of L_{ij} dynamics on the motion of particles in real space (as compared to the constant couplings g_{ij}), next I look at the dynamics of L_{ij} variables during particle collisions.

4.5.1 The influence of the internal degrees of freedom on the dynamics in physical space

The equations of motion of the ordinary N-body Calogero-Moser system (3.2) with g_{ij} coupling constants are the following:

$$\dot{x}_{i,g} = p_{i,g}, \qquad \dot{p}_{i,g} = 2\sum_{k\neq i} \frac{g_{ik}^2}{x_{ik,g}^3}, \tag{4.98}$$

while the equations of motion for (x_i, p_i, L_{ij}) variables which stem from (3.17):

$$\dot{x}_{i,L} = p_{i,L}, \qquad \dot{p}_{i,L} = \sum_{k\neq i} \frac{2|L_{ik}|^2}{x_{ik,L}^3}, \qquad \dot{L}_{ij} = \sum_{k\neq i,j} L_{ik}L_{kj}\left(\frac{1}{x_{ik,L}^2} - \frac{1}{x_{jk,L}^2}\right), \tag{4.99}$$

where $x_{ij} = x_i - x_j$, and the g and L subscripts distinguish the trajectories generated by (3.2) and (3.17). The difference of the two trajectories:

$$\begin{pmatrix} \Delta x_i(t) \\ \Delta p_i(t) \end{pmatrix} = \begin{pmatrix} x_{i,L}(t) - x_{i,g}(t) \\ p_{i,L}(t) - p_{i,g}(t) \end{pmatrix}, \tag{4.100}$$

for the two systems starting in what we may call the same state, that is

$$x_{i,g}(0) = x_{i,L}(0), \qquad p_{i,g}(0) = p_{i,L}(0), \qquad g_{ij} = |L_{ij}|(0), \tag{4.101}$$

will be influenced by the dynamics of the internal variables $L_{ij}(t)$. This influence can be detected in the Taylor expansion of $\Delta x(t)$ around $t = 0$. The chosen initial conditions yield:

$$\Delta x_i(0) = \Delta p_i(0) = 0 \tag{4.102}$$

$$\Delta \dot{p}_i(0) = \left(\sum_{k\neq i} \frac{2|L_{ik}|^2}{x_{ik,L}^3} - \frac{2g_{ik}^2}{x_{ik,g}^3}\right)(0) = 0 \tag{4.103}$$

which means:

$$\begin{aligned} \Delta x_i(t) &= \Delta x_i(0) + t\Delta p_i(0) + \sum_{n=2}^{k-1} \frac{t^n}{n!}\Delta p_i^{(n-1)}(0) + \mathcal{O}(t^k) = \\ &= \frac{t^3}{3}\sum_{k\neq i} \frac{1}{x_{ik,L}^2(0)}\left(\frac{d|L_{ik}|^2}{dt}\right)(0) + \\ &+ \frac{t^4}{12}\left(\sum_{k\neq i} \frac{1}{x_{ik,L}^2}\frac{d^2|L_{ik}|^2}{dt^2} - \frac{4\dot{x}_{ik,L}}{x_{ik,L}^3}\frac{d|L_{ik}|^2}{dt}\right)(0) + \mathcal{O}(t^5). \end{aligned} \tag{4.104}$$

Once we have expanded the expression for the positions, the result for the momenta is straightforward:

$$\Delta p_i(t) = t^2\sum_{k\neq i} \frac{1}{x_{ik,L}^2(0)}\left(\frac{d|L_{ik}|^2}{dt}\right)(0) + \tag{4.105}$$

$$+ \frac{t^3}{3}\left(\sum_{k\neq i} \frac{1}{x_{ik,L}^2}\frac{d^2|L_{ik}|^2}{dt^2} - \frac{4\dot{x}_{ik,L}}{x_{ik,L}^3}\frac{d|L_{ik}|^2}{dt}\right)(0) + \mathcal{O}(t^4). \tag{4.106}$$

The deviation of trajectories generated by (3.2) and (3.17) in the physical space appears as a 3^{rd} order effect in time, so it does not appear immediately, but should be well expected at longer time scales. On the other hand, in the system defined by (3.17) the particles interact for a short time, and then scatter like almost free particles. Therefore to see the long time influence of the additional degrees of freedom, one can look at the system enclosed in a harmonic trap (3.19). Importantly, the harmonic confinement does not affect the 3^{rd} and 4^{th} order terms of the $\Delta x_i(t)$ expansion (as well as the 2^{nd} and 3^{rd} order terms in $\Delta p_i(t)$ expansion). The reason for this is that the n^{th} order coefficient in the expansion of $\Delta x_i(t)$ will be modified by $(n-2)^{nd}$ order derivatives $\Delta x_i^{(n-2)}(0)$, and for $n = 3, 4$ these vanish.

The orthogonal vs. unitary setting

As it was mentioned in Section 3.1.1, the real and symmetric subset of initial conditions (X_0, Y_0) leads to a special subclass of trajectories confined to the subspace of real symmetric matrices (X, Y). The L matrices arising in this, as we already called it, orthogonal setting, belong to the $\mathfrak{so}(N)$ Lie algebra: $L^O \in span(\tau_{ij} : 1 \leqslant i, j \leqslant N)$. A complementary possibility corresponds to L which is purely imaginary and symmetric with zeroes on the diagonal. I will denote them as $L^I \in span(\sigma_{ij} : 1 \leqslant i, j \leqslant N)$. The two subspaces are orthogonal, and together they form an $N^2 - N$ dimensional off-diagonal subspace of the $\mathfrak{su}(N)$ algebra (spanned of course by τ_{ij} and σ_{ij} from definition 19).

The most interesting difference between the orthogonal and unitary cases is the first time derivative of absolute values of the matrix elements. It can be directly checked from (3.14) that

$$\frac{d}{dt}|L_{ij}|^2 = 2\Re\left[L_{ij}\sum_{k\neq i,j} L_{jk}L_{ki}\left(\frac{1}{x_{ik}^2} - \frac{1}{x_{jk}^2}\right)\right], \tag{4.107}$$

and therefore:

$$\forall_{L^I}\frac{d|L^I_{ij}|}{dt} = 0, \quad \forall_{L^O}\frac{d|L^O_{ij}|}{dt} = 4L^O_{ij}\sum_{k\neq i,j} L^O_{jk}L^O_{ki}\left(\frac{1}{x_{ik}^2} - \frac{1}{x_{jk}^2}\right). \tag{4.108}$$

The special initial condition L_0 (3.35) is of course of L^I type, but it turns out that all imaginary L matrices give rise to coupling which does not change at least for a short time. In the orthogonal case there is no stationary point in L space, the couplings $|L_{ij}|$ between particles change immediately.

Application of L^I matrices to g_{ij} systems

The restriction of the flow $\phi_t(x_0, p_0, L^I) = (x, p, L)(t)$ with $L^I_{ij} = ig_{ij}$ to the (x, p) phase space can be used as an approximation of the $(x_g, p_g)(t)$ flow. In case of L^I matrices the influence of L variables on the positions (4.104) is of order of t^4, since the terms with $\frac{d|L_{ik}|^2}{dt}(0)$ vanish and the first significant terms are a linear combination of $\frac{d^2|L_{ik}|^2}{dt^2}(0)$:

$$x_{i,g}(t) = x_{i,L^I}(t) - \frac{t^4}{12}\sum_{k\neq i}\frac{\frac{d^2|L^I_{ik}|^2}{dt^2}(0)}{(x_{i,L^I} - x_{k,L^I})^3(0)} + \mathcal{O}(t^5) \tag{4.109}$$

$$p_{i,g}(t) = p_{i,L^I}(t) - \frac{t^3}{3}\sum_{k\neq i}\frac{\frac{d^2|L^I_{ik}|^2}{dt^2(0)}}{(x_{i,L^I} - x_{k,L^I})^3(0)} + \mathcal{O}(t^4), \tag{4.110}$$

where:

$$\begin{pmatrix} x_{i,L^I} \\ p_{i,L^I} \end{pmatrix}(t) = \begin{pmatrix} (U(t)X(t)U^\dagger(t))_{ii} \\ (U(t)Y(t)U^\dagger(t))_{ii} \end{pmatrix} \tag{4.111}$$

$$X_{0,ij} = \delta_{ij}x_{i,g}(0) \tag{4.112}$$

$$Y_{0,ij} = \delta_{ij}p_{i,g}(0) + (1-\delta_{ij})\frac{ig_{ij}}{x_{i,g}(0) - x_{j,g}(0)} \tag{4.113}$$

Higher order terms, $\frac{d^n(x_{g,i}-x_{L,i})}{dt^n}$ with $n \geqslant 5$ could be used to improve the order of convergence even further, yet the computational cost of such an improvement should be compared with other known methods, like the symplectic Runge-Kutta algorithms [77].

I wish to conclude this subsection with an illustration of the above ideas. The first row of figure 4.3 shows the numerical solution to the equations of motion (4.98) for coupling constants $g_{12} = g_{23} = g_{34} = 10$, $g_{13} = g_{24} = 20$ and $g_{14} = 30$ together with solutions of (4.99) for $L^O_{ik} = \text{sign}(i-k)g_{ik}$ and $L^I_{ik} = i \cdot g_{ik}$. The choice of initial positions and momenta, common to all cases, is $\bar{x} = (1, 2, 3, 4)$ and $\bar{p} = (10, 5, -5, -15)$. In the second row the repulsion is an order of magnitude stronger and in the third - an order of magnitude weaker. In the moderate and strong repulsion regime the L^I matrix gives clearly a much better approximation of the motion governed by (4.98) than L^O, even without the 4^{th} order correction included in (4.109). In the weak interaction case there is little difference between the trajectories generated by the L^I and L^O models, but they both differ significantly from the trajectory of (4.98), especially in the long time scale. The possible reason for this is that when the repulsion is weak, the particles stay relatively close to each other thus the time derivatives of L matrix elements are significant and the rapidly changing repulsion strengths make the trajectories of (4.99) deviate more from the ones given by (4.98), where the repulsion is (by definition) constant.

4.5.2 The dynamics of L variables during collisions

The equations of motion

$$\dot{L}_{ij} = \sum_{k\neq i,j} L_{ik}L_{kj}\left(\frac{1}{x^2_{ik}} - \frac{1}{x^2_{kj}}\right)$$

suggest that there is perhaps not much happening with L_{ij} if the particles are far apart, but rapid changes should be expected whenever the i^{th} or j^{th} particle takes part in a collision. This is noticeable in our numerical example 4.3, where the biggest relative deviation between the matrix models and models with constant couplings is visible when the relative distances are small. My goal in this subsection is to examine the influence of collisions on the L_{ij} variables in two cases: 1. when the system is initially fully interacting, that is for all pairs i, j there is a non-zero $L_{ij}(0)$, and 2. when the system is divided in two subsystems (the L matrix has a block structure) which weakly interact through $|L_{I,I+1}| << |L_{kl}|$, where k, l entries belong to the interacting blocks.

It is convenient to parametrise the entries of the L matrix with their moduli and phases, $L_{ij} = |L_{ij}|e^{i\phi_{ij}}$, where $\varphi_{ji} = \pi - \varphi_{ij}$, so that $L_{ji} = -L^*_{ij}$. The equations of motion can than be separated into:

$$\frac{d}{dt}|L_{ij}| = \sum_{k\neq i,j} |L_{ik}||L_{kj}|\left(\frac{1}{x^2_{ik}} - \frac{1}{x^2_{kj}}\right)\cos(\Phi_{ikj}) \tag{4.114}$$

$$|L_{ij}|\frac{d\varphi_{ij}}{dt} = \sum_{k\neq i,j} |L_{ik}||L_{kj}|\left(\frac{1}{x^2_{ik}} - \frac{1}{x^2_{kj}}\right)\sin(\Phi_{ikj}), \tag{4.115}$$

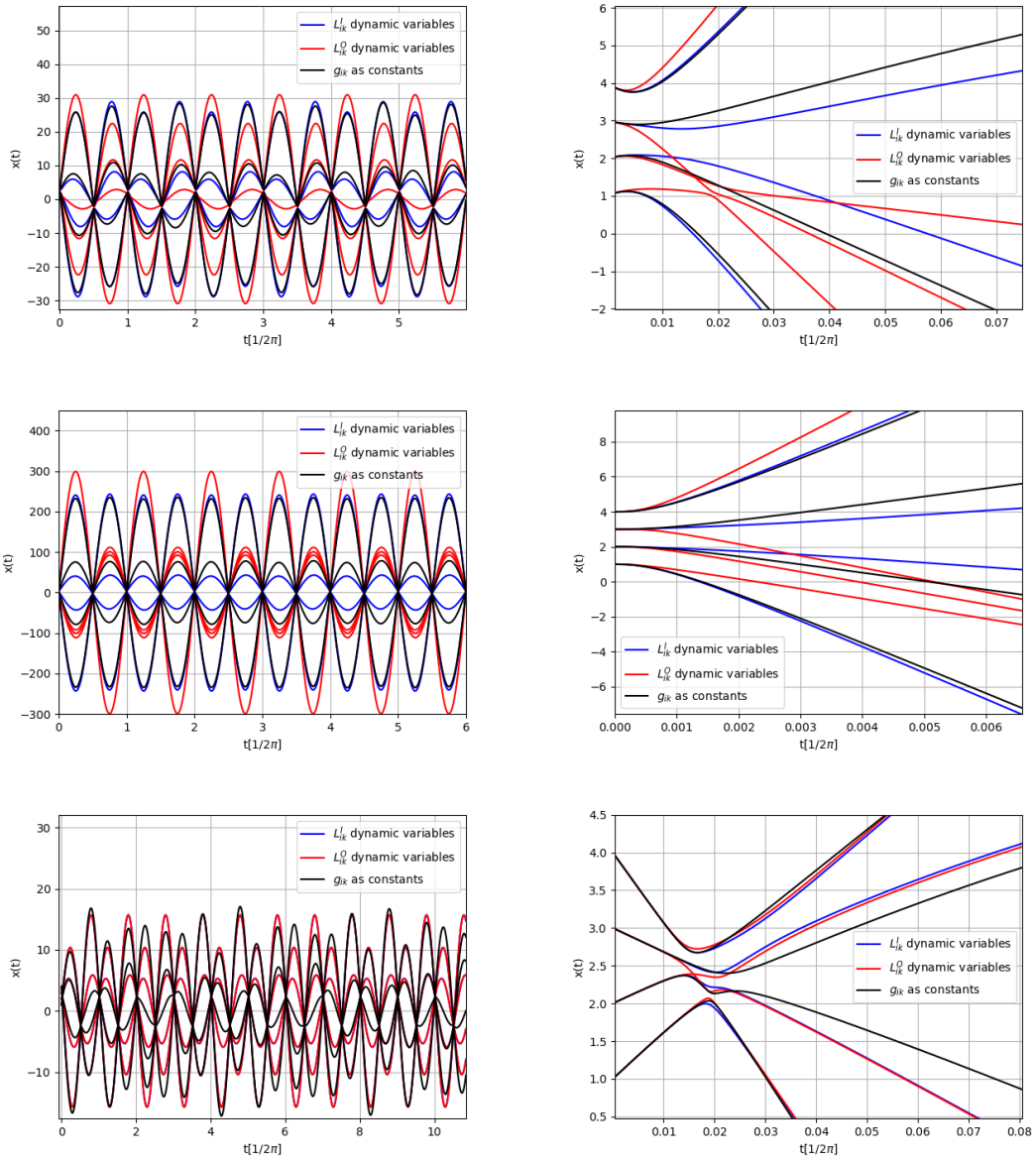


Figure 4.3: The positions of $N = 4$ particles interacting *via* constant g_{ik} values (black), in the orthogonal setting with $L^O_{ik} = \text{sign}(i-k)g_{ik}$ (red) and with purely imaginary $L^I_{ik} = i \cdot g_{ik}$ (blue). The left column shows the long time scale dynamics, the right column - the beginning of the motion. In the first row the repulsion is moderate, that is $\delta x_{ik} \cdot \delta p_{ik} \approx |L_{ik}|$, where δx_{ik} and δp_{ik} are the initial relative distance and relative momentum. In the second row $\delta x_{ik} \cdot \delta p_{ik} \approx 0.1|L_{ik}|$ (strong repulsion), and in the third $\delta x_{ik} \cdot \delta p_{ik} \approx 10|L_{ik}|$ (weak repulsion). Initial positions and momenta are the same for all cases.

where $\Phi_{ikj} = \varphi_{ik} + \varphi_{kj} + \varphi_{ji}$ coincides with (3.34) and is gauge invariant. Even permutations of indices leave it unchanged: $\Phi_{ikj} = \Phi_{kji} = \Phi_{jik}$ and odd permutations invert it around π: $\Phi_{ijk} = \Phi_{kij} = \Phi_{jki} = \pi - \Phi_{ikj}$.

Collisions in a fully interacting system

In this case we assume that all $|L_{ij}(0)| > 0$, and that two adjacent particles, for example the 1^{st} and 2^{nd} are much closer to each other than any other two particles, that is $|x_{12}| << |x_{1k}|, |x_{2k}|, |x_{kl}| \approx \lambda$ for $k, l \notin \{1, 2\}$. This means we have two typical distances, small x_{12} and large λ. The values of derivatives (4.114) and (4.115) are dominated by terms proportional to $\frac{1}{x_{12}^2}$. Let us sort the terms by order of magnitude:

$$\frac{d}{dt}|L_{12}| = \mathcal{O}\left(x_{12}\lambda^{-3}\right) \tag{4.116}$$

$$\frac{d}{dt}|L_{1k}| = \frac{|L_{12}||L_{2k}|}{x_{12}^2}\cos(\Phi_{12k}) + \mathcal{O}\left(\lambda^{-2}\right) \tag{4.117}$$

$$\frac{d}{dt}|L_{2k}| = \frac{|L_{12}||L_{1k}|}{x_{12}^2}\cos(\Phi_{21k}) + \mathcal{O}\left(\lambda^{-2}\right) \tag{4.118}$$

$$\frac{d}{dt}|L_{kl}| = \mathcal{O}\left(\lambda^{-2}\right) + \mathcal{O}\left(x_{12}\lambda^{-3}\right). \tag{4.119}$$

The time derivative of Φ_{12k} within the above approximation reads:

$$\frac{d\Phi_{12k}}{dt} = \frac{|L_{12}|}{x_{12}^2}\left(\frac{|L_{1k}|}{|L_{2k}|} - \frac{|L_{2k}|}{|L_{1k}|}\right)\sin(\Phi_{12k}) + \mathcal{O}\left(\lambda^{-2}\right) + \mathcal{O}\left(x_{12}\lambda^{-3}\right) \tag{4.120}$$

With the use of the fact that $\cos(\Phi_{21k}) = -\cos(\Phi_{12k})$, and the chain rule $\frac{d\cos(f)}{dt} = -sin(f)\frac{df}{dt}$, after discarding the small terms, we get a closed system of differential equations for $(f = |L_{1k}|, g = |L_{2k}|, \Phi = \Phi_{12k})$:

$$\dot{f} = \alpha g\cos(\Phi) \tag{4.121}$$

$$\dot{g} = -\alpha f\cos(\Phi) \tag{4.122}$$

$$\dot{\Phi} = \alpha\left(\frac{f}{g} - \frac{g}{f}\right)\sin(\Phi), \tag{4.123}$$

where $\alpha = \frac{|L_{12}|}{x_{12}^2}$ can be treated like a constant, since

$$\dot{\alpha} = \frac{1}{x_{12}^2}\left(\frac{d|L_{12}|}{dt} - \frac{2\dot{x}_{12}}{x_{12}}\right) = \mathcal{O}(x_{12}^{-1}\lambda^{-3}) - \frac{2\dot{x}_{12}}{x_{12}^3}(0) \tag{4.124}$$

and the initial relative velocity $\dot{x}_{12}(0)$ of the colliding particles can be set to as small value as needed. From (4.121) and (4.122) it is clear that $f^2 + g^2 = const.$. If we use the angular parametrisation $(f, g) = \rho(\sin(\eta(t)), \cos(\eta(t))$, we get:

$$\dot{\eta} = \alpha\cos(\Phi) \tag{4.125}$$

$$\dot{\Phi} = -2\alpha\cot(2\eta)\sin(\Phi). \tag{4.126}$$

In the special case of the orthogonal setting we have all $\phi_{ij} \in \{0, \pi\}$, and the solution of the above system in the linear approximation will be simply

$$(\eta, \Phi)(t) = (\alpha_\Phi t + \eta(0), \Phi(0)) + \mathcal{O}(t^2), \tag{4.127}$$

where $\alpha_\Phi = \alpha\cos(\Phi(0)) = \pm\alpha$. This means that depending on the value of α_Φ, the point

$$\frac{(|L_{1k}(t)|, |L_{2k}(t)|)}{\sqrt{|L_{1k}(0)|^2 + |L_{2k}(0)|^2}} \approx (\sin(\alpha_\Phi t + \eta_0), \cos(\alpha_\Phi t + \eta_0)) \tag{4.128}$$

can be anywhere on the quarter-circle. In the other extreme case, that is when $L(0)$ is purely imaginary, we have all $\Phi_{ijk} = \pm\frac{\pi}{2}$, $\dot{\eta}(0) = 0$ and in the quadratic approximation:

$$\eta(t) = \eta(0) + \alpha^2 t^2 \cot(2\eta(0)) + \mathcal{O}(t^3), \tag{4.129}$$

where $\alpha > 0$. This means, that if initially

- $0 < \eta(0) < \frac{\pi}{4}$ (this is $0 < |L_{1k}(0)| < |L_{2k}(0)|$), $\eta(t)$ will increase with time,
- $\frac{\pi}{4} < \eta(0) < \frac{\pi}{2}$ (this is $0 < |L_{2k}(0)| < |L_{1k}(0)|$), $\eta(t)$ will decrease,
- $\eta(0) = \frac{\pi}{4}$ (that is $|L_{1k}(0)| = |L_{2k}(0)|$) it will be stationary.

All in all, at least for short times, $\eta(t)$ will cover only the angles between $\eta(0)$ and $\frac{\pi}{4}$.

In the intermediate case, that is when $L(0)$ has both a real and imaginary part and $\Phi_{ijk} \notin \{0, \pm\frac{\pi}{2}, \pi\}$, we may conclude that

$$\eta(t) = \eta(0) + \alpha t \cos\Phi(0) + \alpha^2 t^2 \cot(2\eta(0)) \sin^2\Phi(0) + \mathcal{O}(t^3), \tag{4.130}$$

so this time depending on the initial values of $\eta(0)$ and $\Phi(0)$ we have two possibilities:

- if $t_0 = \frac{-\cos\Phi(0)}{\alpha \cot(2\eta(0))\sin^2\Phi(0)} < 0$ the sign of the time derivative is still dictated by the sign of $\cot(2\eta(0))$, so the angle $\eta(t)$ will stay in the segment between $\eta(0)$ and $\frac{\pi}{4}$,
- if $t_0 > 0$, the available segment will be larger. For $t \in (0, t_0/2)$ the time derivative will have an opposite sign than dictated by $\cot(2\eta(0))$, and the extreme available angle will be:

  $$\eta_M = \eta(0) - \frac{1}{4}\tan(2\eta(0))\cot^2\Phi(0) \tag{4.131}$$

  that is clearly smaller than $\eta(0)$ it $\eta(0) < \frac{\pi}{4}$ and larger if $\eta(0) > \frac{\pi}{4}$.

The main conclusion of this subsection is, that as long as the made approximations are valid, $(|L_{1k}|, |L_{2k}|)(t) = \sqrt{|L_{1k}|^2 + |L_{2k}|^2}(0)(\sin(\eta_k(t)), \cos(\eta_k(t)))$, that is the pairs of interaction strengths undergo a smooth exchange independently for every particle outside the colliding pair. The angles available for $\eta_k(t)$ depend on the initial phases $\Phi_{ijk}(0)$, which distinguish the different settings, the orthogonal, its imaginary complement and the combination of the two. In particular for the orthogonal setting the angles may vary freely depending $\alpha = \frac{|L_{12}|}{x_{12}^2}$, so the repulsion strengths $|L_{1k}|$ and $|L_{2k}|$ may achieve any values, as long as the sum of their squares stays unchanged. On the other hand for the imaginary case the angles are confined to a segment between $\eta(0)$ and $\frac{\pi}{4}$, which means that $||L_{1k}| - |L_{2k}||$ is not allowed to increase. In particular if $|L_{1k}| = |L_{2k}|$ are equal, they will stay equal, at leas up to 2^{rd} order in time.

Collisions between weakly coupled subsystems

In this subsection I am examining the influence of collisions on a different Calogero-Moser setup, that is of two subsystems one consisting of particles enumerated by $i = 1, 2, ..., I$ and the other $j = I+1, I+2, ..., I+J$. The subsystems are fully interacting and almost completely decoupled, that is initially $|L_{ij}| = 0$, apart from a weak interaction on the edge *via* a small $|L_{I,I+1}|$. I am interested in the influence of a collision between the subsystems on the values of $|L_{I,j}|$ and $|L_{i,I+1}|$ - that is the interaction of the edge particles with the bulk particles of the other subsystem,

as well as $|L_{ij}|$ with $i < I$ and $j > I+1$, that is the interaction between particles in the bulk. It is convenient to introduce the above system through its Y_0 matrix in the initial, unreduced phase space:

$$Y_0 = \begin{pmatrix} & & & 0 & ... & 0 \\ & I \times I & & 0 & ... & 0 \\ & & & \epsilon & ... & 0 \\ 0... & 0 & \epsilon^* & & & \\ 0... & 0 & 0 & J & \times & J \\ 0... & 0 & 0 & & & \end{pmatrix} \tag{4.132}$$

with a free parameter ϵ, where $|\epsilon|$ is small. We can assume that the distance between the edge particles $\delta = |x_I - x_{I+1}| > 0$ is small as well, while all the other distances are of order of $\lambda >> \delta$. The important parameter of the previous subsection was $\alpha = \frac{|L_{12}|}{x_{12}^2}$. Using the new definitions we have $|L_{I,I+1}| = \delta|\epsilon|$, which indeed is a small parameter (of second order), while $\alpha = |\epsilon|\delta^{-1}$ is in fact an arbitrary positive number. Some of the results of the previous subsection may be reused, after adjusting the indices:

$$\frac{d}{dt}|L_{I,I+1}| = \mathcal{O}\left(\delta\lambda^{-3}\right) \tag{4.133}$$

$$\frac{d}{dt}|L_{Ik}| = \alpha|L_{I+1,k}|\cos(\Phi_{I,I+1,k}) + \mathcal{O}\left(\lambda^{-2}\right) \tag{4.134}$$

$$\frac{d}{dt}|L_{I+1,k}| = -\alpha|L_{Ik}|\cos(\Phi_{I,I+1,k}) + \mathcal{O}\left(\lambda^{-2}\right) \tag{4.135}$$

$$\frac{d}{dt}|L_{kl}| = \mathcal{O}\left(\lambda^{-2}\right) + \mathcal{O}\left(\delta\lambda^{-3}\right). \tag{4.136}$$

The first conclusion is, that indeed the edge particles immediately start to interact with the other subsystem. This happens through the familiar exchange $(|L_{Ik}|, |L_{I+1,k}|)(t)$, only in this case $\eta(0) \in \{0, \pi/2\}$:

$$\begin{aligned} (|L_{Ii}|, |L_{I+1,i}|)(t) &= |L_{Ii}(0)|(\sin\eta_i(t), \cos\eta_i(t)), \quad \eta_i(0) = \frac{\pi}{2} \\ (|L_{Ij}|, |L_{I+1,j}|)(t) &= |L_{I+1,j}(0)|(\sin\eta_j(t), \cos\eta_j(t)), \quad \eta_j(0) = 0 \end{aligned}$$

for $i < I$ and $j > I+1$. The interaction between the bulk particles, that is between $i < I$ and $j > I+1$ will arise as $\mathcal{O}(t^2)$ terms but only if we allow the distances x_{iI} and $x_{j,I+1}$ to be of the order of $\delta << \lambda$.

4.6 The reachable sets of L variables

The question of finding the set of points accessible from given initial points in phase space under certain dynamics is important in many problems of control theory [78]. The considerations of the previous section, where I asked for limitations on the angles $\eta(t)$ for chosen initial conditions $(\eta(0), \Phi(0))$ were an example of such a question. In this section I will consider a more general problem. Namely I will ask what are the matrices $L(t) \in \mathfrak{g}$ that can be reached from a chosen $L \in \mathfrak{g}$ ($\mathfrak{so}(N)$ or $\mathfrak{su}(N)$ depending on the setting) assuming complete freedom of choice of initial positions and momenta. Predicting such a set of matrices, let us call it the reachable set of L, with respect to the Calogero-Moser dynamics

$$\mathcal{R}_{CM}(L) = \{L' = ULU^\dagger : U(X_0 + tY_0)U^\dagger = D(t), [X_0, Y_0] = L\} \tag{4.137}$$

is the goal of this section. Clearly as $\mathcal{R}_{CM}(L)$ is obtained by conjugating L with unitary matrices, it will be a subset of what we would get by acting with the entire

Lie group, $\mathcal{R}_G(L) = \{L' = ULU^\dagger, U \in G\}$. The motion in physical space depends on the absolute values $|L_{ij}|$, so I will also consider the projection of $\mathcal{R}_{CM}(L)$ onto a region of $\mathbb{R}^{\binom{N}{2}}$:

$$\mathcal{S}_{CM}(L) = \{(l'_{12}, l'_{13}, ..., l'_{N-1,N}) : l'_{ij} = |L'_{ij}|, L' \in \mathcal{R}_{CM}(L)\}. \tag{4.138}$$

The reachable set $\mathcal{R}_{CM}(L) \subset \mathcal{R}_G(L)$ may be computed numerically by generating trajectories $L(t) = U(t)LU^\dagger(t) \in \mathfrak{g}$ for different choices of X_0 and Y_0 with a fixed commutator $[X_0, Y_0] = L$, but my main objective is to predict it analytically.

Let me recall that if there are only $N = 2$ particles interacting, the dynamics of the L matrix is trivial. We may see it from (3.14), that with no $k \neq 1, 2$ indices to sum over automatically $\dot{L}_{12} = 0$. This is why I am starting with the smallest non-trivial case of $N = 3$. The number of variables in this case is small enough to make direct predictions and to present them visually. Next I will generalise the results to $N > 3$.

Connectivity of a Calogero-Moser system

It will be useful in the upcoming considerations to define the notion of connectivity in the context of a Calogero-Moser system. If we think of the N particles as vertices of a graph, then $|L_{ij}| \neq 0$ will represent an edge connecting the i^{th} and j^{th} vertex. I transfer the notion of paths and path lengths between two vertices directly to this context, and I call a CM system connected if there exists a path (not necessarily direct) between every two particles. On the other hand a single particle or a subset of particles that has no connection to the rest of the system I will call a subsystem disconnected. Below I am considering only connected systems, because a connection cannot arise between disconnected subsystems. The reason for this lies in the structure of matrix dynamics. Let us recall the derivation in section 3.1.1, where we had the Lax equation $\dot{L} = [A, L]$, and $A_{ij} = L_{ij}x_{ij}^{-2}$, so the A matrix has the same non-vanishing entries as L, only rescaled, in particular it inherits the block structure of L. As we already discussed in 4.5.2 disconnected subsystems correspond to block structured L matrices. In case of two subsystems we have $L = L_1 \oplus L_2$, $A = A_1 \oplus A_2$ has the same block structure, and so has $[A, L] = [A_1, L_1] \oplus [A_2, L_2]$, therefore no changes can occur outside the blocks.

4.6.1 Reachable sets for $N = 3$

In the first step I am going to predict the projected reachable sets $\mathcal{S}_{CM}(L)$ of an $N = 3$ particle Calogero-Moser system. The parametrisation I choose for the L and $L(t)$ matrices is the following:

$$L = i \begin{pmatrix} 0 & l_{12}e^{i\phi_{12}} & l_{31}e^{-i\phi_{31}} \\ l_{12}e^{-i\phi_{12}} & 0 & l_{23}e^{i\phi_{23}} \\ l_{31}e^{i\phi_{31}} & l_{23}e^{-i\phi_{23}} & 0 \end{pmatrix}, \tag{4.139}$$

$$L(t) = i \begin{pmatrix} 0 & l_{12}(t)e^{i\phi_{12}(t)} & l_{31}(t)e^{-i\phi_{31}(t)} \\ l_{12}(t)e^{-i\phi_{12}(t)} & 0 & l_{23}(t)e^{i\phi_{23}(t)} \\ l_{31}(t)e^{i\phi_{31}(t)} & l_{23}(t)e^{-i\phi_{23}(t)} & 0 \end{pmatrix}, \tag{4.140}$$

where $l_{ij} = |L_{ij}| \geqslant 0$[1] and $\Phi_{123} = \phi_{12} + \phi_{23} + \phi_{31} \in [-\pi/2, \pi/2]$ (this is a sufficient set due to gauge invariance). For $L(t) \in R_{CM}(L)$ the two matrices must first of all have the same eigenvalues. Moreover the CM evolution preserves the vanishing diagonal, that is $L_{ii}(t) = 0$. In the orthogonal setting this is true for the entire $SO(N)$ group

[1] At least two of the three values need to be positive for a connected $N = 3$ particle system

acting on the Lie algebra $\mathfrak{so}(N)$, but in the unitary setting this is not the case, and this condition imposes additional limitations on $L(t) \in R_{CM}(L)$. The characteristic equation at time t reads:

$$\lambda^3 - |\bar{l}(t)|^2\lambda + 2\cos(\Phi_{123}(t))l_{12}(t)l_{23}(t)l_{31}(t) = 0, \tag{4.141}$$

where $\bar{l}(t) = (l_{12}, l_{23}, l_{31})(t)$. The constrains for the time dependent l_{ij} and ϕ_{ij} functions stem from the initial values:

$$|\bar{l}(t)| = |\bar{l}|, \quad \cos(\Phi_{123}(t))l_{12}(t)l_{23}(t)l_{31}(t) = \cos(\Phi_{123})l_{12}l_{23}l_{31}. \tag{4.142}$$

The first equation states that $\bar{l}(t)$ is confined to a sphere of the radius given at $t = 0$ (and it is equivalent to the conservation of $\mathrm{Tr}L^2$). In the orthogonal case $\Phi_{123} = \pm\pi/2$ and the first equation is the only constraint.

The second equation says that $\bar{l}(t)$ is a point on a paraboloid $x \cdot y \cdot z = p_0$, from a family of paraboloids given by $p_0 = \cos(\Phi_{123})l_{12}l_{23}l_{31}/\cos(\Phi_{123})(t) \geqslant \cos(\Phi_{123})l_{12}l_{23}l_{31}$. The two equations limit $\bar{l}(t)$ to those paraboloids which intersect with the sphere. They cut out a spherical cap centered at the point $x = y = z = \frac{|\bar{l}|}{\sqrt{3}}$, and the edge of the cap is the circular intersection of the sphere and $x \cdot y \cdot z = \cos(\Phi_{123})l_{12}l_{23}l_{31}$. The closer $\cos(\Phi_{123})$ is to unity (and Φ_{123} - to zero), the smaller is the cap. In particular for $l_{12} = l_{23} = l_{31} = \frac{g}{\sqrt{3}}$ and $\Phi_{123} = 0$, that is for the stationary L_0 matrix (3.35), it shrinks to a point. From this point of view, L_0 is stationary because there are no other (not gauge equivalent) accessible matrices with the same eigenvalues. The figure 4.4 illustrates the above predictions compared to numerically generated trajectories projected to a sector of a 2-dimensional sphere representing the points $\bar{l}(t)$. The first image, with $\Phi_{123} = \frac{\pi}{2}$, corresponds to the orthogonal setting and shows that there is no limitations on $\bar{l}(t)$ apart from constant length. The last image, $\Phi_{123} = 0$, is the purely imaginary case, and it shows that the distance between $\bar{l}(t)$ and $\bar{l}_0 = (2, 2, 2)/\sqrt{3}$ (the stationary point in the centre of the circular region) cannot increase. The two middle illustrations, with $\Phi_{123} = \frac{\pi}{3}$ and $\frac{\pi}{6}$ show the available region growing with Φ_{123}. This in accordance with the results of the section 4.5.2: for example if we fix the value of $|l_{12}|$ the available values of $|l_{13}|$ and $|l_{23}|$ are limited to a segment of a circle in (l_{13}, l_{23}) plane determined by the initial ratio of l_{13} and l_{23} and the phase Φ_{123}. It seems plausible, that there are no further limitations on $\mathcal{S}_{CM}(L)$ for $N = 3$, but to show this I will use a different approach, which works for general $N > 3$.

4.6.2 Reachable sets for $N \geqslant 3$

The reachable set of a group action (at least its component connected to unity), whether it is $SO(N)$ or $SU(N)$ acting on its Lie algebra, can be described by:

$$\{L' = e^{tv}Le^{-tv} : t \in \mathbb{R}_+, v \in \mathfrak{g}\} \subset \mathcal{R}_G(L) \tag{4.143}$$

This can be translated to the directions in the Lie algebra in which L can be moved by the group action:

$$L' = L + t\delta L + \mathcal{O}(t^2), \quad \delta L = [v, L] \in \mathfrak{g}. \tag{4.144}$$

The choice of $v \in \mathfrak{g}$ is unrestricted. Therefore if we express δL, v and L in the basis defined in section 2.1.2, definition 18 (for the $SO(N)$ case we put all the L^I and v^I

components equal to 0):

$$\delta L = \sum_{a<b} \delta L^R_{ab}\tau_{ab} + \delta L^I_{ab}\sigma_{ab} + \sum_{a=1}^{N-1} \delta L_{aa} d_a \tag{4.145}$$

$$\delta L^R_{ab} = \sum_{k=1}^{N} v^R_{ak} L^R_{kb} - v^I_{ak} L^I_{kb} - L^R_{ak} v^R_{kb} + L^I_{ak} v^I_{kb} \tag{4.146}$$

$$\delta L^I_{ab} = \sum_{k=1}^{N} v^R_{ak} L^I_{kb} + v^I_{ak} L^R_{kb} - L^R_{ak} v^I_{kb} - L^I_{ak} v^R_{kb}, \tag{4.147}$$

we see that L can be moved in almost any direction in the Lie algebra by appropriate choice of v. The excluded directions are the ones completely disconnected from L, that is such that all L_{ak} and L_{bk} entries vanish. Clearly, the full $\mathcal{R}_G(L)$ consists of all $L' \in \mathfrak{g}$ matrices with the same eigenvalues. Yet this local expansion around L is now comparable to the local expansion of $\mathcal{R}_{CM}(L)$. The Calogero-Moser matrix dynamics imposes limitations on the directions δL. As it was discussed in section 3.1.1:

$$\delta L = \dot{L}(0) = [A(0), L], \qquad A(0) \in \mathfrak{g}, \qquad A_{ij} = \frac{L_{ij}}{x^2_{ij}}(0). \tag{4.148}$$

This means that instead of commuting L with any element $\lambda \in \mathfrak{g}$, we are now restricted to elements $A \in \mathfrak{g}$ which are in a sense similar to L, i.e. they have the same vanishing and non-vanishing components, only rescaled by positive factors x^{-2}_{ij}. Let us call $\mathfrak{g}_L \subset \mathfrak{g}$ the subspace spanned by the non-vanishing components of L. Of course $A \in \mathfrak{g}_L$, and the choice of positions $x_1 < x_2 < ... < x_N$ is completely free. Yet since the factors given by x^{-2}_{ij} are not independent, we do not have access to all elements of $\mathfrak{g}_L$. Using the results from section 4.5.2, we may say that we have access to all elements defined as

$$A_{i,i+1} = \frac{L^R_{i,i+1}\tau_{i,i+1} + L^I_{i,i+1}\sigma_{i,i+1}}{x^2_{i,i+1}} + \mathcal{O}(\lambda^{-2}), \qquad i = 1, 2, .., N-1, \qquad x_{i,i+1} << \lambda. \tag{4.149}$$

Each such element generates a rotation in the $(L^R_{ik}, L^R_{i+1,k}, L^I_{ik}, L^I_{i+1,k})$ subspaces for all $k \neq i, i+1$:

$$\frac{d}{dt}\begin{pmatrix} L^R_{ik} \\ L^I_{i,k} \end{pmatrix} = \begin{pmatrix} \alpha^R_i & -\alpha^I_i \\ \alpha^I_i & \alpha^R_i \end{pmatrix} \begin{pmatrix} L^R_{i+i,k} \\ L^I_{i+1,k} \end{pmatrix} \tag{4.150}$$

$$\frac{d}{dt}\begin{pmatrix} L^R_{i+i,k} \\ L^I_{i+1,k} \end{pmatrix} = \begin{pmatrix} -\alpha^R_i & -\alpha^I_i \\ \alpha^I_i & -\alpha^R_i \end{pmatrix} \begin{pmatrix} L^R_{i,k} \\ L^I_{i,k} \end{pmatrix}, \tag{4.151}$$

where $\alpha^{R,I}_i = \frac{L^{R,I}_{i,i+1}}{x^2_{i,i+1}}$. In the orthogonal setting, where all the imaginary entries vanish, and R superscripts may be dropped,this simplifies to:

$$\frac{d}{dt}\begin{pmatrix} L_{ik} \\ L_{i+i,k} \end{pmatrix} = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix} \begin{pmatrix} L_{i,k} \\ L_{i+1,k} \end{pmatrix} \tag{4.152}$$

and the solution:

$$\begin{pmatrix} L'_{ik} \\ L'_{i+1,k} \end{pmatrix} = \begin{pmatrix} \cos(\alpha_i t) & \sin(\alpha_i t) \\ -sin(\alpha_i t) & \cos(\alpha_i t) \end{pmatrix} \begin{pmatrix} L_{i+i,k} \\ L_{i+1,k} \end{pmatrix} = R(\eta_i) \begin{pmatrix} L_{i+i,k} \\ L_{i+1,k} \end{pmatrix}, \tag{4.153}$$

where the rotation angle $\eta_i = \alpha_i t$ is unrestricted due to the free choice of $x_{i,i+1}$. The antisymmetric L matrix can be represented by a vector $\bar{l} \in \mathbb{R}^{\binom{N}{2}}$ such that $L_{ij} =$

sign$(j-i)l_{ij}$. The effect $A_{i,i+1}$ has on L in this parametrisation can be described as:

$$\bar{l}' = [P_i(R(\eta_i) \oplus R(\eta_i) \oplus ... \oplus R(\eta_i) \otimes \mathbb{1})P_i]\bar{l}, \tag{4.154}$$

where P_i permutes the components of $\bar{l}$ so the first $2(N-2)$ components are the pairs $(l_{ik}, l_{i+1,k})$ affected by the $R(\eta_i) \in SO(2)$ rotations, and then are the other $\binom{N}{2} - 2(N-2)$ which stay unchanged. The angles $\eta_i \in [0, 2\pi]$ parametrise a loop on a $\binom{N}{2} - 1$ dimensional sphere of radius $|\bar{l}|$. The projections of this loop on the $(l_{ik}, l_{i+1,k})$ planes are full circles. In the unitary case we need to define two angles, η_i which parametrises the loop and β_i which encodes the phase of $L_{i,i+1}$:

$$\eta_i = t\sqrt{(\alpha_i^R)^2 + (\alpha_i^I)^2} = |\alpha_i| t, \tag{4.155}$$

$$(\cos\beta_i, \sin\beta_i) = \left(\frac{\alpha_i^R}{\sqrt{(\alpha_i^R)^2 + (\alpha_i^I)^2}}, \frac{\alpha_i^I}{\sqrt{(\alpha_i^R)^2 + (\alpha_i^I)^2}} \right). \tag{4.156}$$

The solution reads:

$$\begin{pmatrix} L'^R_{ik} \\ L'^I_{ik} \\ L'^R_{i+1,k} \\ L'^I_{i+1,k} \end{pmatrix} = \begin{pmatrix} \cos\eta_i & 0 & \sin\eta_i\cos\beta_i & -\sin\eta_i\sin\beta_i \\ 0 & \cos\eta_i & \sin\eta_i\sin\beta_i & \sin\eta_i\cos\beta_i \\ -\sin\eta_i\cos\beta_i & -\sin\eta_i\sin\beta_i & \cos\eta_i & 0 \\ \sin\eta_i\sin\beta_i & -\sin\eta_i\cos\beta_i & 0 & \cos\eta_i \end{pmatrix} \begin{pmatrix} L^R_{ik} \\ L^I_{ik} \\ L^R_{i+1,k} \\ L^I_{i+1,k} \end{pmatrix}.$$

This, together with the phases of the complex entries $L_{kl} = i|L_{kl}|e^{i\phi_{kl}}$ (the imaginary unit is factored out like in section 4.6.1) can be used to express the absolute values $|L'_{ik}|$ and $|L'_{i+1,k}|$:

$$\begin{aligned} |L'_{ik}|^2 &= |L_{ik}|^2\cos^2\eta_i + |L_{i+1,k}|^2\sin^2\eta_i - \sin(2\eta_i)|L_{ik}||L_{i+1,k}|\sin(\Phi_{ik,i+1}) \\ |L'_{i+1,k}|^2 &= |L_{i+1,k}|^2\cos^2\eta_i + |L_{ik}|^2\sin^2\eta_i + \sin(2\eta_i)|L_{ik}||L_{i+1,k}|\sin(\Phi_{ik,i+1}), \end{aligned}$$

so just as it was discussed in the previous subsection, for the purely imaginary initial L ($\Phi_{i,k,i+1} = 0$), the available values of $|L'_{ik}|$ and $|L'_{i+1,k}|$ are restricted to the segment between the initial $|L_{ik}|$ and $|L_{i+1,k}|$. As $\Phi_{i,k,i+1}$ grows from 0 to $\frac{\pi}{2}$ the available segment grows as in the case of $N = 3$.

We may freely use the linear combinations of the elements corresponding to disjoint pairs of particles, for example A_{12},A_{34},...,$A_{2n-1,2n}$, where $n = [N/2]$. Such a linear combination generates rotations in the subspaces of real dimension 8 (or 4 in the orthogonal case). The coordinates in these subspaces are $(L_{2i-1,2j-1}, L_{2i-1,2j}, L_{2i,2j-1}, L_{2i,2j})^{R,I}$, that is $(L_{OO}, L_{OE}, L_{EO}, L_{EE})^{R,I}_{ij}$ where $1 \leqslant i < j \leqslant n$ and we use O and E as shorthands for odd and even. The generator of the rotation in such a subspace is described by two complex parameters, $\alpha_I = \frac{L_{2i-1,2i}}{x^2_{2i-1,2i}}$ and $\alpha_J = \frac{L_{2j-1,2j}}{x^2_{2j-1,2j}}$. We may write the corresponding equation for the four component vectors L^R and L^I:

$$\frac{d}{dt}\begin{pmatrix} L^R \\ L^I \end{pmatrix} = \begin{pmatrix} A^R & A^I \\ -A^I & A^R \end{pmatrix}\begin{pmatrix} L^R \\ L^I \end{pmatrix}, \tag{4.157}$$

where

$$A^R = \begin{pmatrix} 0 & \alpha_J^R & \alpha_I^R & 0 \\ -\alpha_J^R & 0 & 0 & \alpha_I^R \\ -\alpha_I^R & 0 & 0 & \alpha_J^R \\ 0 & -\alpha_I^R & -\alpha_J^R & 0 \end{pmatrix}, \quad A^I = \begin{pmatrix} 0 & \alpha_J^I & -\alpha_I^I & 0 \\ \alpha_J^I & 0 & 0 & -\alpha_I^I \\ -\alpha_I^I & 0 & 0 & \alpha_J^I \\ 0 & -\alpha_I^I & \alpha_J^I & 0 \end{pmatrix}. \tag{4.158}$$

In the orthogonal case this simplifies to:

$$\frac{d}{dt}\begin{pmatrix} L_{OO} \\ L_{OE} \\ L_{EO} \\ L_{EE} \end{pmatrix} = \begin{pmatrix} 0 & \alpha_J & \alpha_I & 0 \\ -\alpha_J & 0 & 0 & \alpha_I \\ -\alpha_I & 0 & 0 & \alpha_J \\ 0 & -\alpha_I & -\alpha_J & 0 \end{pmatrix} \begin{pmatrix} L_{OO} \\ L_{OE} \\ L_{EO} \\ L_{EE} \end{pmatrix}. \tag{4.159}$$

and the solution is:

$$\begin{pmatrix} L'_{OO} \\ L'_{OE} \\ L'_{EO} \\ L'_{EE} \end{pmatrix} = R(\eta_{2i-1}) \otimes R(\eta_{2j-1}) \begin{pmatrix} L_{OO} \\ L_{OE} \\ L_{EO} \\ L_{EE} \end{pmatrix}. \tag{4.160}$$

In terms of vectors $\bar{l} \in \mathbb{R}^{\binom{N}{2}}$ we may write:

$$\bar{l}' = [P((R(\eta_1)\otimes R(\eta_3))\oplus(R(\eta_1)\otimes R(\eta_5))\oplus ...\oplus(R(\eta_{2n-3}\otimes R(\eta_{2n-1}))\otimes \mathbb{1})P]\bar{l}, \tag{4.161}$$

where again P is a permutation matrix which rearranges the components in the correct order and each 4×4 block $R(\eta_{2i-1})\otimes R(\eta_{2j-1})$ acts in the appropriate subspace. The $n = [N/2]$ angles $\eta_{2i-1} = \alpha_I \cdot t$ are unrestricted as previously. In the unitary case we again observe how the phases Φ_{klm}, where k, l, m are distinct triples from $\{2i-1, 2i, 2j-1, 2j\}$ affect the accessible values of $(|L_{2i-i,2j-1}|, |L_{2i-i,2j}|, |L_{2i,2j-1}|, |L_{2i,2j}|)$. The easiest way to see it is from (4.114):

$$\frac{d}{dt}\begin{pmatrix} |L_{OO}| \\ |L_{OE}| \\ |L_{EO}| \\ |L_{EE}| \end{pmatrix} = \begin{pmatrix} 0 & |\alpha_J|\sin\Phi_1 & |\alpha_I|\sin\Phi_2 & 0 \\ -|\alpha_J|\sin\Phi_1 & 0 & 0 & |\alpha_I|\sin\Phi_3 \\ -|\alpha_I|\sin\Phi_2 & 0 & 0 & |\alpha_J|\sin\Phi_4 \\ 0 & -|\alpha_I|\sin\Phi_3 & -|\alpha_J|\sin\Phi_4 & 0 \end{pmatrix} \begin{pmatrix} |L_{OO}| \\ |L_{OE}| \\ |L_{EO}| \\ |L_{EE}| \end{pmatrix}$$

where the indices from 1 to 4 in $\Phi_{1,2,3,4}$ stand for $(2i-1, 2j-1, 2j)$,$(2i-1, 2i, 2j)$,$(2i, 2j-1, 2j)$ and $(2i, 2j-1, 2j)$ respectively. The rotation generated by the above equation has again the form of $R(\xi_i)\otimes R(\xi_j)$, but the angles are modified by the phases $\Phi_{1,2,3,4}$:

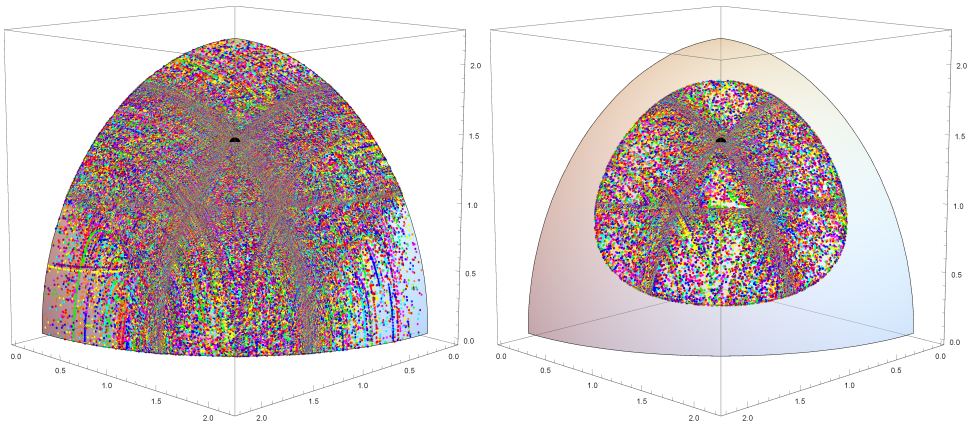
$$\xi_i = \frac{1}{2\sqrt{2}}\left(\sqrt{a+b}+\sqrt{a-b}\right) \tag{4.162}$$

$$\xi_j = \frac{1}{2\sqrt{2}}\left(\sqrt{a+b}-\sqrt{a-b}\right) \tag{4.163}$$

$$a = |\alpha_J|^2(\sin^2\Phi_1+\sin^2\Phi_4)+|\alpha_I|^2(\sin^2\Phi_2+\sin^2\Phi_3) \tag{4.164}$$

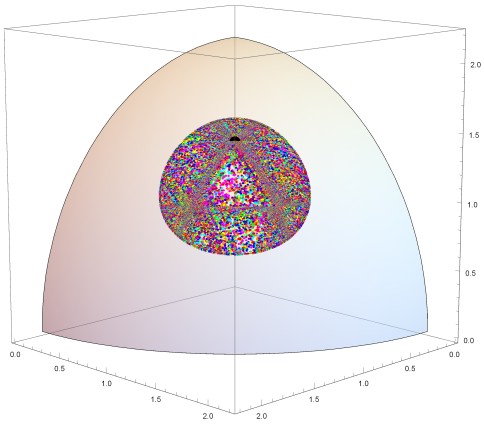
$$b = \sqrt{a^2-4(|\alpha_J|^2\sin\Phi_1\sin\Phi_4-|\alpha_I|^2\sin\Phi_2\sin\Phi_3)^2}. \tag{4.165}$$

One can check that if all $\Phi_{12,3,4} = \frac{\pi}{2}$, we obtain the result for the orthogonal setting. The question is how adding more than disjoint colliding pair components to $A \in \mathfrak{g}_L$ is further increasing the reachable set. Let us assume all the nearest neighbours are connected. In the orthogonal setting the elements A_{12},A_{34},...,$A_{2n-1,2n}$ allow us to act on the initial L matrix with all matrices $O_c = R(\eta_1)\oplus R(\eta_3)\oplus ...\oplus R(\eta_{2n-1})$ (the subscript c refers to collisions). On the other hand we know that all matrices from $SO(N)$ look like O_c in some basis (that is in the canonical form), so we have enough free parameters to perform any rotation. The only thing we need is to change the directions in $\mathfrak{g}$ preserved by the rotation generated by A from $(\tau_{12}, \tau_{34}, ..., \tau_{2n-1,2n})$ to a different set of n directions. These n directions are precisely the basis elements of a subspace commuting with A, and the choice of these directions is limited only by connectivity of L. It is very similar in the unitary case, only the phases of complex L_{ij} elements give rise to limitations on the absolute values of $|L'_{ij}|$ as we have seen in the simplest cases.

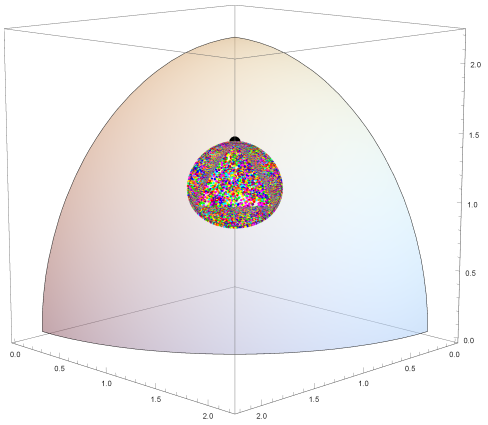


(a) $\Phi_{123} = \pi/2$

(b) $\Phi_{123} = \pi/3$



(c) $\Phi_{123} = \pi/6$



(d) $\Phi_{123} = 0$

Figure 4.4: $\mathcal{S}_{CM}(L)$ regions for the initial condition $\bar{l} = (1, 1, \sqrt{2})$ (black dots) and various values of Φ_{123}. Each image consists of 5000 trajectories generated from random initial positions and momenta. The numerical results coincide with the analytical predictions and serve as an illustration.

Chapter 5

Quantum Results

In this chapter I am presenting the results of my research on the quantum generalisations of the Calogero-Moser model. My main objective was to study a quantum version of the model defined in sections 3.1.1 and 3.1.2 and I chose two different approaches towards this task. The first one was the direct insertion of $\mathfrak{so}(N)$ or $\mathfrak{su}(N)$ matrices in the numerators of $\frac{1}{x_{ij}^2}$ terms. The second one was the canonical quantization of a free (or harmonic) system in the phase space of matrices followed by the reparametrisation with matrix eigenvalues. The main accomplishments within these two lines of thought can be summarised as follows:

1. I have studied the three-particle Hamiltonian with spin operators $\hat{s}_{x,y,z}$ in the role of quantum L operators depending on the value of total spin. I have written the Hamiltonian acting on the $2s+1$ components of the wavefunction and and diagonalized it exactly in the case of $s = 1$. I have partially generalized this result to N particles and $N \times N$ matrices $\hat{L}_{ij} = i\hbar\tau_{ij}$ (the defining representation of $\mathfrak{so}(N)$). In this case all the squares $\hat{L}_{ij}^2$ in the interacting terms are simultaneously diagonalizable. I have also made use of the result of section 4.1 and proposed a model with $\hat{L}_{ij} = \imath\hbar\hat{l}_i^\dagger \otimes \hat{l}_j$, that is the quantum analogue of the vectorial model. The commutation relations for $\hat{L}$ operators turned out to imply $\left[\hat{l}_i, \hat{l}_j^\dagger\right] = \delta_{ij}$. This means that each particle carries an internal quantum oscillator state which determines its interaction with other particles.

2. I have performed the change of variables in a quantum Hamiltonian of $N + d$ free particles from the positions X_{ij} encoded in a Hermitian (or real, symmetric) $N \times N$ matrix X to the eigenvalues $(D_1, ..., D_N)$. The value of d is $N^2 - N$ in the Hermitian and $\binom{N}{2}$ in the real symmetric case. I have obtained a quantum Calogero-Moser Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial D_i^2} - \frac{\hbar^2}{4m}\sum_{(ij)}\frac{(\hat{\lambda}_{ij})^2 + (2-\alpha)}{(D_i - D_j)^2}$$

   where (ij) means $i \neq j$ or $i < j$ in the Hermitian and the real, symmetric case respectively, and the operators $\hat{\lambda}_{ij}$ obey the commutation rules of the generators of the appropriate Lie algebra ($\mathfrak{su}(N)$ for Hermitian and $\mathfrak{so}(N)$ for real symmetric matrices), α is either 1 in the real symmetric case or 2 in the Hermitian case. I have found the reduced wavefunctions in the simplest case of $N = 2$ by projecting the plane waves defined for the initial free systems on eigenspaces of the single $\hat{L}_{12}$ operator.

5.1 Direct quantization of the generalized CM system

The classical Calogero-Moser system defined in section 3.1.1,

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{m} \sum_{i<j} \frac{|L_{ij}|^2}{(x_i - x_j)^2} \tag{5.1}$$

$$\{x_i, p_i\} = \delta_{ij}, \qquad \{L_{ij}, L_{kl}\} = \sum_{mn} f_{ij,kl}^{mn} L_{mn} \tag{5.2}$$

$$\{x_i, L_{kl}\} = \{p_j, L_{kl}\} = 0 \tag{5.3}$$

where $f_{ij,kl}^{mn}$ are the structure constants of $\mathfrak{su}(N)$ or $\mathfrak{so}(N)$ (listed in (3.27)-(3.29)), depending on the chosen setting, can be quantized canonically in the known way. The phase space variables (x_i, p_j) get promoted to operators on the Hilbert space $L^2(\mathbb{R}^N)$. The Poisson brackets translate to the commutation relations of these operators acting on functions from $L^2(\mathbb{R}^N)$:

$$x_i \to \hat{x}_i, \qquad p_j \to \hat{p}_j = -i\hbar\partial_{x_j}, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}.$$

As for the L degrees of freedom, we again may promote the functions on the phase space to Hermitian operators acting on some Hilbert space $\mathcal{V}$, with the commutation relations preserved:

$$L_{ij} \to \hat{L}_{ij}, \qquad \left[\hat{L}_{ij}, \hat{L}_{kl}\right] = i\hbar \sum_{mn} f_{ij,kl}^{mn} \hat{L}_{mn}$$

and the only decision to make is what should these operators act on. In other words we need to choose a representation of the algebra spanned by the L operators. This is the $\mathfrak{su}(N)$ or $\mathfrak{so}(N)$, and they are known to have finite-dimensional irreducible representations acting on $\mathcal{V}_n = \mathbb{C}^n$. Therefore we may define the quantized system with a Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \partial_{x_i}^2 \otimes \mathbb{1} + \frac{1}{m} \sum_{(ij)} \frac{1}{(x_i - x_j)^2} \otimes \frac{\hat{L}_{ij}\hat{L}_{ij}^\dagger + \hat{L}_{ij}^\dagger \hat{L}_{ij}}{2} \tag{5.4}$$

acting on a product $\mathcal{H} = L^2(\mathbb{R}^N) \otimes \mathbb{C}^n$. Careful examination of the commutation relations (3.27)-(3.29) shows that there is no ambiguity due to the ordering of operators. The summation over (ij) pairs means $i < j$ for the orthogonal and $i \neq j$ for the unitary setting.

5.1.1 The orthogonal setting

I will explore the properties of the above Hamiltonian in the orthogonal setting, starting with the simplest nontrivial example.

Three-body system and $so(3)$

In the case of $N = 3$ we can use the familiar spin operators and obtain

$$\hat{H}_s^{(3)} = \sum_{i=1}^{3} -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \frac{m\omega^2}{2}x_i^2 + \frac{1}{m}\left(\frac{\hat{s}_x^2}{(x_1 - x_2)^2} + \frac{\hat{s}_y^2}{(x_2 - x_3)^2} + \frac{\hat{s}_z^2}{(x_1 - x_3)^2}\right) \tag{5.5}$$

The relevant wavefunctions are $(2s+1)$-dimensional spinors and the time-independent Schrödinger equation has the following form:

$$\hat{H}_s^{(3)}\Psi_s = E\Psi_s, \qquad \Psi_s^T(x_1, x_2, x_3) = (\psi_s, \psi_{s-1}, ..., \psi_{-s})(x_1, x_2, x_3) \tag{5.6}$$

The particles carry a joint internal state, let us call it spin, and its projections onto orthogonal axes determine the strenght of interactions. The axes are completely equivalent (as long as we do not add any external field which would distinguish any of them) and the choice which axis we assign to which pair of particles is arbitrary. Here we have $(1,2) \longrightarrow x, (2,3) \longrightarrow y, (1,3) \longrightarrow z$, but permuting them is equivalent to expressing the matrix operators in a different basis (note that this will no longer be true for $N > 3$). As a mater of fact, $U\hat{s}_{x,y,z}U^\dagger, U \in U(3)$ will be just as good. An important choice to make is the eigenspace of the $\hat{s}^2$ operator, where

$$\hat{s}^2 = \hat{s}_x^2 + \hat{s}_y^2 + \hat{s}_z^2, \qquad \left[\hat{H}, \hat{s}^2\right] = 0 \tag{5.7}$$

The eigenspaces are given by half-integer numbers $s = 0, 1/2, 1, ...$ and each eigenvalue $\hbar^2 s(s+1)$ corresponds to a $(2s+1)$-dimensional matrix representation of $so(3)$. For $s = 0$ the representation is trivial and results in the ordinary CM system. For $s = \frac{1}{2}$ we have $\hat{s}_i = \frac{\hbar}{2}\sigma_i$, where $\sigma_{x,y,z}$ are Pauli matrices. They square to identity, $\sigma_i^2 = \mathbb{1}_{2\times 2}$, which means that (5.5) in this case is the ordinary system as well. The first nontrivial case is $s = 1$. Let us express the inverse-square potential part of (5.5) with $\hat{s}^2, \hat{s}_z^2$ and $\hat{s}_\pm = \hat{s}_x \pm i\hat{s}_y$:

$$\hat{H}_s^{(3)} = \left(\sum_i -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \frac{m\omega^2}{2}x_i^2\right) + \tag{5.8}$$

$$+ \frac{1}{m}\left[\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(x_2-x_3)^2}\right)\frac{\hat{s}^2 - \hat{s}_z^2}{2} + \frac{\hat{s}_z^2}{(x_1-x_3)^2}\right] + \tag{5.9}$$

$$+ \frac{1}{m}\left(\frac{1}{(x_1-x_2)^2} - \frac{1}{(x_2-x_3)^2}\right)\frac{\hat{s}_+^2 + \hat{s}_-^2}{4} \tag{5.10}$$

The (5.9) part acts diagonally on $\hat{\Psi}_s$, while (5.10) connects each ψ_m component with $\psi_{m\pm 2}$. Changing to $R = \frac{1}{3}(x_1 + x_2 + x_3), x = \frac{1}{\sqrt{2}}(x_3 - x_1), y = \frac{1}{\sqrt{6}}(x_1 + x_3 - 2x_2)$ and then expressing $x = r\sin\varphi, y = r\cos\varphi$ makes the variables in the eigenproblem (5.6) separate into $\Psi_s(R, r, \varphi) = \mathcal{R}(R)\rho(r)\Phi_s(\varphi)$, where:

$$\left(-\frac{1}{3}\frac{d^2}{dR^2} + \frac{3R^2}{l^4}\right)\mathcal{R}(R) = \frac{2m}{\hbar^2}E_R \tag{5.11}$$

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{r^2}{l^4} + \frac{b^2}{r^2}\right)\rho(r) = \frac{2m}{\hbar^2}E_r \tag{5.12}$$

$$M_s\Phi_s(\varphi) = b^2\Phi_s(\varphi) \tag{5.13}$$

and $l^2 = \frac{\hbar}{m\omega}$. The solutions for $\mathcal{R}(R)$ and $\rho(r)$ and their discrete eigenvalues are known:

$$\mathcal{R}_n(R) = \mathcal{N}_n e^{-\frac{3R^2}{2l^2}} H^{(n)}\left(\frac{\sqrt{3}R}{l}\right), \tag{5.14}$$

$$\rho_{n_r}(r) = \mathcal{N}_{n_r} r^b e^{-\frac{r^2}{2l^2}} L_{n_r}^b\left(\frac{r^2}{l^2}\right) \tag{5.15}$$

$$E_R = \hbar\omega\left(n + \frac{1}{2}\right), \qquad E_r = \hbar\omega(2n_r + b + 1), \tag{5.16}$$

$$E = E_R + E_r = \hbar\omega\left(n + 2n_r + b + \frac{3}{2}\right) \tag{5.17}$$

The variable R is of course the position of the centre of mass. The definition of r implies $2r^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2$, which means mr^2 can be

understood as the moment of inertia of the system. In particular $r = 0$ corresponds to $x_1 = x_2 = x_3$ - all particles in one place, so to say, which should not be observed in a system with singular repulsion. The factor r^b with $b > 0$ ensures that. For the Calogero-Moser system without the harmonic trap $\mathcal{R}(R)$ will be plane waves and $\rho(r)$ will be given by $\sqrt{r}J_b(kr)$, where J_b is the Bessel function and k is given by the energy in the centre of mass reference frame. The angular equation is not affected by the harmonic potential, but this is the only one which mixes the different components of Ψ_s:

$$\begin{aligned} M_s &= -\frac{d^2}{d\varphi^2} + \frac{1}{4\hbar^2}\left[\left(\frac{1}{\sin^2(\varphi+2\pi/3)} - \frac{1}{\sin^2(\varphi+4\pi/3)}\right)(\hat{s}_+^2 + \hat{s}_-^2)\right] \\ &+ \frac{1}{2\hbar^2}\left[\frac{2\hat{s}_z^2}{\sin^2\varphi} + \left(\frac{1}{\sin^2(\varphi+2\pi/3)} + \frac{1}{\sin^2(\varphi+4\pi/3)}\right)(\hat{s}^2 - \hat{s}_z^2)\right] \end{aligned}$$

Knowing the matrix elements of $\hat{s}_\pm$ in the $\hat{s}_z$ diagonal basis we can write the resulting equations (5.13):

$$\left(b^2 + \frac{d^2}{d\varphi^2} + \frac{m^2 - s(s+1)}{2} f_+(\varphi) - \frac{m^2}{\sin^2(\varphi)}\right)\psi_m = \tag{5.18}$$

$$= \frac{1}{4} f_-(\varphi)\sqrt{(s(s+1) - (m+1)^2)^2 - (m+1)^2}\,\psi_{m+2} + \tag{5.19}$$

$$+ \frac{1}{4} f_-(\varphi)\sqrt{(s(s+1) - (m-1)^2)^2 - (m-1)^2}\,\psi_{m-2} \tag{5.20}$$

$$f_\pm = \frac{1}{\sin^2(\varphi+2\pi/3)} \pm \frac{1}{\sin^2(\varphi+4\pi/3)} \tag{5.21}$$

Before we try to diagonalise this system of $2s+1$ equations, we notice that M_s acts separately on two subspaces of the internal states Hilbert space. Let $\{e_m : m = s, ..., -s\}$ be the orthonormal basis of $\hat{s}_z$ eigenstates. Then in case of integer values of s we have an $s+1$ dimensional subspace $span(e_s, e_{s-2}, ..., e_{-s})$ and a smaller, s-dimensional $span(e_{s-1}, ..., e_{-s+1})$. When s is half-integer, we have two $s + \frac{1}{2}$ dimensional subspaces $span(e_{\pm s}, ..., e_{\mp s+1})$.

Eigenvectors and eigenvalues of $M_{s=1}$

For the smallest nontrivial value of s, that is $s = 1$ the diagonalisation is very simple. The set of equations:

$$\left\{b^2 + \frac{d^2}{d\varphi^2} - \frac{1}{\sin^2\varphi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \tag{5.22}$$

$$= \left\{f_+(\varphi)\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} + f_-(\varphi)\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} \tag{5.23}$$

separates into three independent equations for ψ_0 and $\psi_\pm$:

$$\left(b^2 + \frac{d^2}{d\varphi^2} - V_0(\varphi)\right)\psi_0 = 0, \qquad \left(b^2 + \frac{d^2}{d\varphi^2} - V_\pm(\varphi)\right)\psi_\pm = 0 \tag{5.24}$$

where $V_\pm(\varphi) = V_\pm(\varphi+\pi) = \frac{1}{\sin^2\varphi} + \frac{1}{\sin^2(\pi/3\mp\varphi)}$ and $V_0(\varphi) = \frac{1}{\sin^2(\pi/3-\varphi)} + \frac{1}{\sin^2(\pi/3+\varphi)}$. Since $V_0(\varphi) = V_+(\varphi - \pi/3)$ and $V_-(\varphi) = V_+(\varphi + \pi/3)$, it is enough to find ψ_+, and

the other two are functions will be shifted accordingly: $\psi_0(\varphi) = \psi_+(\varphi - \pi/3)$ and $\psi_-(\varphi) = \psi_+(\varphi + \pi/3)$. The equations (5.24) are special instances of

$$\left[-\frac{d^2}{d\varphi^2} + \frac{g_1}{\sin^2\varphi} + \frac{g_2}{\sin^2\left(\varphi + \frac{2}{3}\pi\right)} + \frac{g_3}{\sin^2\left(\varphi + \frac{4}{3}\pi\right)}\right]\psi(\varphi) = b^2\psi(\varphi) \tag{5.25}$$

derived in [3] for the three body ordinary quantum Calogero system. It was solved analytically in two cases: $g_1 = g, g_2 = g_3 = 0$ and $g_1 = g_2 = g_3 = g$. The equation for ψ_+ corresponds to $g_1 = g_2 = 1, g_3 = 0$ and it does not admitt a solution in a closed form. Yet, since the singular points of $V_+(\varphi)$ are regular, the eigenfunctions can be found in a form of a power series up to as many terms as we wish with the Frobenius method. As it is shown in Figure 5.1, for $\varphi \in [0, \pi]$ $V_+(\varphi)$ has the form of a double, asymmetric and infinite potential well. Thus there are two orthogonal sets of solutions: $\psi_l^H(\varphi)$ localised in the higher potential region and $\psi_l^L(\varphi)$ in the lower potential region. The eigenvalues are obtained from the symmetry or antisymmetry conditions. The solutions presented in Figure 5.1(a) are ψ_+, and after a shift by $\pm\pi/3$ we have the eigenfunctions ψ_0 and ψ_- as well. The full solution of the (5.13) equation expanded in the eigenbasis of the matrix system reads:

$$\Phi^{H/L}_{s=1,l}(\varphi) = \alpha\psi^{H/L}_{+,l}(\varphi)\hat{e}_+ + \beta\psi^{H/L}_{-,l}(\varphi)\hat{e}_- + \gamma\psi^{H/L}_{0,l}(\varphi)\hat{e}_0 \tag{5.26}$$

$$\hat{e}_\pm = (1/\sqrt{2}, 0, \pm 1/\sqrt{2})^T, \quad \hat{e}_0 = (0, 1, 0)^T \tag{5.27}$$

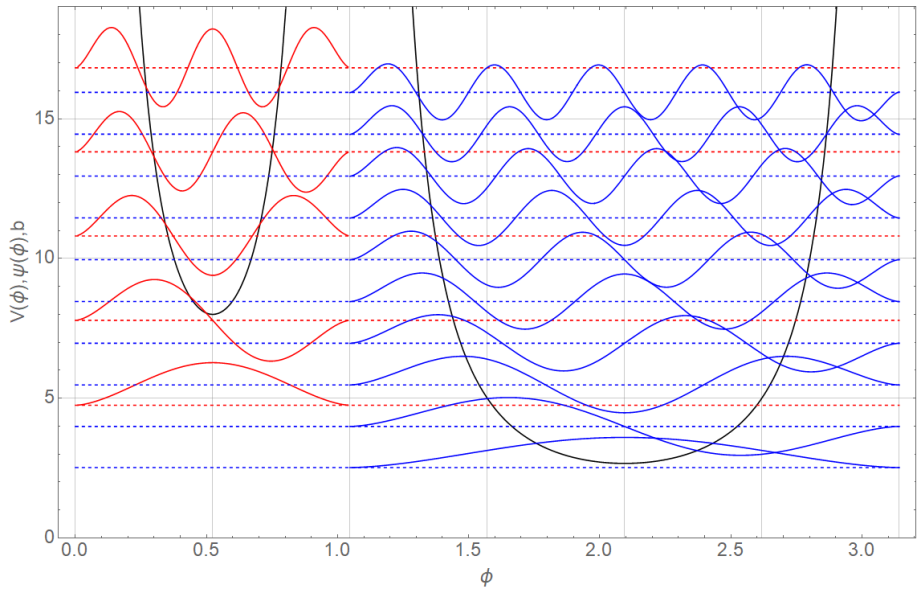
For fixed values of r and R the φ variable encodes the relative distances between particles, as shown in the bottom of Figure 5.1(b), and we can interpret the φ dependent eigenfunctions from the $x_{1,2,3}$ perspective. First of all, the ground state components: $|\psi^L_{0,l=0}|^2$ is concentrated arround $x_1 = x_3$ and maximal $|x_2 - x_{1,3}|$, while it vanishes whenever x_2 is in between x_1 and x_3. This is understandable, since $\hat{s}_z^2(0, \psi_0, 0)^T = 0$ and does not contribute to the repulsion between x_1 and x_3, while $\hat{s}^2_{x,y}(0, \psi_0, 0)^T \neq 0$. Likewise, $|\psi^L_{\pm,l=0}|^2$ present the same behaviour, but with $x_{1,2,3}$ variables cyclically permuted. The lowest energy H-type eigenfunctions, $\psi^H_{0,l=0}$ and $\psi^H_{\pm,l=0}$ are localised in regions of distinct ordering (and its reverse), maximize for $x_i - x_j = x_j - x_k$, that is when the particles are most evenly distributed, and vanish when any two positions coincide. Last but not least, we notice, that the symmetric ψ^L states (the ground state in particular) are allowed only in case of bosonic statistics.

N particles and the N-dimensional representation of $so(N)$

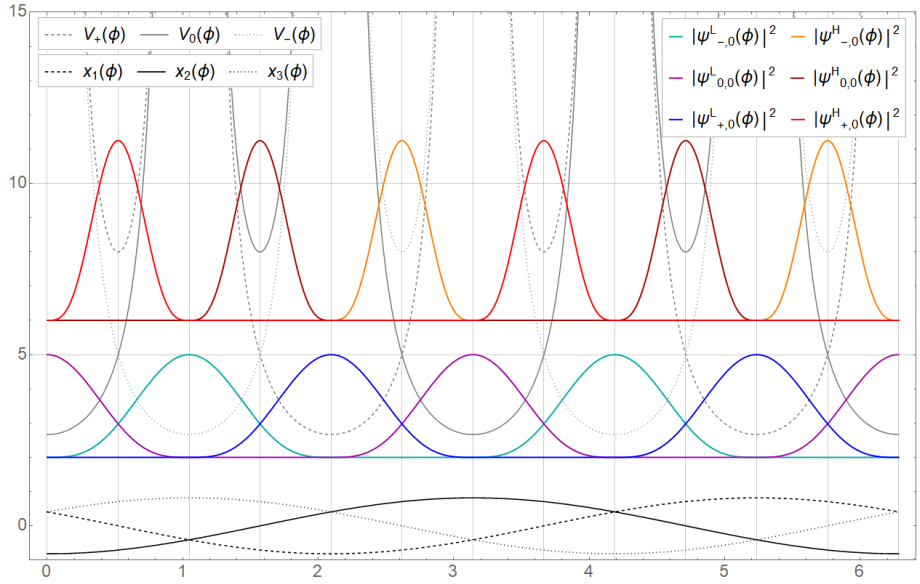
One of the key conclusions from the study of the $N = 2s + 1 = 3$ case is that the observables governing the interparticle repulsion have a common eigenbasis, that is the $\hat{e}_\pm, \hat{e}_0$ basis. In fact we would obtain the eigenvector of $\hat{M}_{s=1}$ in the form of (ψ_+, ψ_0, ψ_-),if we expressed $\hat{s}_{x,y,z}$ in this basis in the first place. The interpretation of the components in straightforward: ψ_+ contributes to the repulsion of x_1 from $x_{2,3}$ (but not between x_2 and x_3),ψ_0 - to the repulsion of x_2 with the others, but not between x_1 and x_3, and finally ψ_- - to the repulsion of x_3 with the others, but not x_1 and x_2. This can be extended to N particle systems:

$$\hat{H}^{(N)} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\partial^2_{x_i} + \frac{m\omega^2}{2}\sum_{i=1}^{N}x_i^2 + \frac{1}{m}\sum_{1\leqslant i<j\leqslant N}\frac{\hat{L}^2_{ij}}{(x_i - x_j)^2} \tag{5.28}$$

If we choose N-dimensional Hermitian matrices $(\hat{L}_{ij})_{ab} = i\hbar(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})$, that is the defining representation of $\mathfrak{so}(N)$, their squares will be $(\hat{L}^2_{ij})_{ab} = \hbar^2(\delta_{ia}\delta_{ib} + \delta_{ja}\delta_{jb})$, that is diagonal with 1 in the i^{th} and j^{th} row and 0 in the others. The set of equations



(a) Eigenfunctions $\psi^{H}_{+,l}(\varphi)$ (red) in the narrow, higher potential region and $\psi^{L}_{+,l}(\varphi)$ (blue) in the broad, lower potential region together with discrete values of b_l.



(b) Potential functions V_+, V_-, V_0 are mutually shifted and eigenfunctions $\psi^{H,L}_{-,l}(\varphi) = \psi^{H,L}_{+,l}(\varphi - \pi/3)$, $\psi^{H,L}_{0,l}(\varphi) = \psi^{H,L}_{+,l}(\varphi + \pi/3)$ are shifted accordingly. The squares of $\psi^{H,L}_{0,\pm,l=0}$ are presented together with $x_{1,2,3}(\varphi)$ values for fixed (R, r).

Figure 5.1: Eigenfunctions $\psi_+(\varphi)$ divide into two orhtogonal subsets: $\psi^{H}_{+,l}(\varphi)$ which contribute to the energy whith $b^H_l \approx 4.75 + 3l$ and $\psi^{L}_{+,l}(\varphi)$ with $b^L_l \approx 2.5 + 1.5l$. Note that the eigenvalues of (5.24) are actually b^2_l.

for the N components is the following:

$$\left(-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\partial_{x_i}^2+\frac{m\omega^2}{2}\sum_{i=1}^{N}x_i^2+\frac{\hbar^2}{m}\sum_{i\neq I}\frac{1}{(x_i-x_I)^2}\right)\psi_I(\bar{x})=E\psi_I(\bar{x}) \tag{5.29}$$

Each eigenfunction ψ_I can be interpreted as a wavefunction of a system of $N-1$ mutually noninteracticn particles, all repelled by a single particle. The only difference between them is the choice of the repulsive variable, which is x_I. We postulate the energy eigenstates in a form:

$$\hat{\Psi}(R,r,\bar{\varphi}) = \mathcal{R}(R)\rho(r)\Phi_N(\bar{\varphi}) \tag{5.30}$$
$$\Phi_N(\bar{\varphi}) = (\alpha_1\phi_1,\alpha_2\phi_2,....,\alpha_N\phi_N)^T(\bar{\varphi}) \tag{5.31}$$

where $R=(x_1+...+x_N)/N$ is the centre of mass, $Nr^2=(x_1-x_2)^2+....+(x_{N-1}-x_N)^2$ as in [4]. The angles $\bar{\varphi}=(\varphi_1,\varphi_2,...,\varphi_{N-2})$ are the spherical coordinates on S^{N-2} and encode the relations between x_i variables for fixed R and r. The eigenequations for the R,r and $\bar{\varphi}$-dependent components read:

$$\left(-\frac{1}{N}\frac{d^2}{dR^2}+\frac{NR^2}{l^4}\right)\mathcal{R}(R) = \frac{2m}{\hbar^2}E_R \tag{5.32}$$
$$\left(-\frac{d^2}{dr^2}-\frac{N-2}{r}\frac{d}{dr}+\frac{r^2}{l^4}+\frac{b^2}{r^2}\right)\rho(r) = \frac{2m}{\hbar^2}E_r \tag{5.33}$$
$$M_N\Phi_N(\bar{\varphi}) = b^2\Phi_N(\bar{\varphi}) \tag{5.34}$$

The R and r dependent eigenfunctions and eigenenergies are:

$$\mathcal{R}_n(R) = e^{-\frac{NR^2}{2l^2}}H^{(n)}\left(\frac{\sqrt{N}R}{l}\right),\quad E_R=\hbar\omega\left(n+\frac{1}{2}\right) \tag{5.35}$$
$$\rho_{n_r}(r) = r^\beta e^{-\frac{r^2}{2l^2}}L_{n_r}^\gamma\left(\frac{r^2}{l^2}\right),\quad E_r=\hbar\omega\left(2n_r+\gamma+1\right) \tag{5.36}$$
$$\beta = \sqrt{b^2+\left(\frac{N-3}{2}\right)^2}-\frac{N-3}{2},\quad \gamma=\sqrt{b^2+\left(\frac{N-3}{2}\right)^2} \tag{5.37}$$
$$E = E_R+E_r=\hbar\omega\left(n+2n_r+\sqrt{b^2+\left(\frac{N-3}{2}\right)^2}+\frac{3}{2}\right) \tag{5.38}$$

For $N=3$ the expressions for β and γ simplify to $b>0$, and the solution coincides with the results from 5.1.1. The difficult part is the eigenequation for the ψ_I components of $\hat{\Psi}_N(\bar{\varphi})$:

$$\left(-\Delta_{S^{N-2}}+f_I(\bar{\varphi})\right)\phi_I(\bar{\varphi})=b^2\phi_I(\bar{\varphi}),\quad f_I(\bar{\varphi})=\left(\sum_{j\neq I}\frac{2r^2}{(x_j-x_I)^2}\right)(\bar{\varphi}) \tag{5.39}$$

Solving the above eigenproblem as we did in section 5.1.1 for a general value of N is a hard task because of the multiple variables at play, but the general form of the solutions is predictable an has a clear interpretation. The configuration space of N particles on a line is divided into $N!$ sectors of a fixed ordering. Yet in case of a potential given by f_I the only important distinction between regions is the ordering with respect to the repulsive variable: $x_{j_1},x_{j_2},...,x_{j_k}<x_I<x_{j_{k+1}},...,x_{j_{N-1}}$. In fact, the ordering and even the choice of the x_j,$j\neq I$ variables below and above x_I does not matter, the only difference between regions is $k\in\left\{0,1,....,\left[\frac{N-1}{2}\right]\right\}$ - the number of particles on one side of x_I. In 5.1.1 we have identified two distinct

regions, where the subspaces of solutions were (ψ^L, b^L) corresponding to $k = 0$ and (ψ^H, b^H) corresponding to $k = 1$. The potential f_I is symmetric with respect to the $x_{j\neq I}$ variables and has a unique minimum in each region. It is located at $x_{j_1} = x_{j_2}, ... = x_{j_k} < x_I < x_{j_{k+1}} = ... = x_{j_{N-1}}$. The general form of the solutions will be oscillatory functions, symmetric or antisymmetric at the minimum of f_I, vanishing at the $x_I = x_{j\neq I}$ bounadries and outside the region of fixed ordering and the eigenvalues are expected to be $b_l^k = b_0^k + l\Delta_k \pm \delta_{k,l}$. The physical interpretation of the minima is straightforward: it is energetically favorable for the $N-1$ mutually noninteracting particles to be localised as far as possible from the repulsive one and depending on the partition given by k, the distances $|x_I - x_{j\neq I}|$ allowed by the fixed r will be different. In particular there is a unique bosonic ground state for $k = 0$ with all particles other than the I^{th} localised around the same point.

Solving (5.24) as an eigenproblem in l^2 Hilbert space

The most systematic and reliable approach towards (5.39) would be to expand the eigenfunction in an orthonormal basis of functions which are supported only in the selected region of the configuration space and respect its boundaries given by the singularities of f_I at $x_j = x_I$. It is still a challenge in the presence of many variables, but it certainly can be applied to the example of $N = 3$ and the $V_\pm(\varphi)$ potentials shown in Figure 5.1. For simplicity, in case of $k = 0$ (that is the lower energy subspace) we will take the V_-, since then the ψ^L functions are supported on $(0, 2\pi/3)$ and we can expand:

$$\psi^L(x) = \sum_{n=1}^{\infty} \psi_n^L \sin\left(\frac{3}{2}nx\right), \qquad \psi_n^L = \frac{3}{\pi}\int_0^{2\pi/3} \psi^L(x)\sin\left(\frac{3}{2}nx\right)dx. \tag{5.40}$$

Next, we insert the expanded ψ^L to (5.24):

$$\sum_{n=1}^{\infty} \psi_n^L \sin\left(\frac{3}{2}nx\right)\left[\left(\frac{3}{2}n\right)^2 - b^2 + \frac{1}{\sin^2 x} + \frac{1}{\sin^2(x+\pi/3)}\right] = 0 \tag{5.41}$$

and integrate both sides of this equation with another function from the basis:

$$\begin{aligned}
\int_0^{2\pi/3} \sin\left(\frac{3}{2}px\right)\left\{\sum_{n=1}^{\infty} \psi_n^L \sin\left(\frac{3}{2}nx\right)\left[\left(\frac{3}{2}n\right)^2 - b^2 + \frac{1}{\sin^2 x} + \frac{1}{\sin^2(x+\pi/3)}\right]\right\}dx &= 0 \\
\psi_p^L\left[\left(\frac{3}{2}p\right)^2 - b^2\right] + \sum_{n=1}^{\infty}\psi_n^L \int_0^{2\pi/3} \sin\left(\frac{3}{2}px\right)\sin\left(\frac{3}{2}nx\right)\left[\frac{1}{\sin^2 x} + \frac{1}{\sin^2(x+\pi/3)}\right]dx &= 0
\end{aligned}$$

and obtain an eigenproblem in l^2 Hilbert space:

$$\psi_p^L\left[\left(\frac{3}{2}p\right)^2 - b^2\right] + \frac{3}{\pi}\sum_{n=1}^{\infty}\psi_n^L I_{np} = 0 \tag{5.42}$$

Since $I_{np} = 0$ if n and p have different parity, the eigenfunctions with odd and even components form separate subspaces. The integrals I_{np} are finite and can be calculated analytically:

$$\begin{aligned}
I_{n,n} &= \frac{n}{2}\left[3\pi + \sqrt{3}\left(\Psi_0\left(\frac{n}{2}+\frac{2}{3}\right) - \Psi_0\left(\frac{n}{2}+\frac{1}{3}\right)\right)\right] && (5.43)\\
&\approx \frac{3n\pi}{2} + \frac{\sqrt{3}}{3} - |\delta_n| && (5.44)\\
I_{n,n+2m} &= I_{n+m,n+m} - I_{m,m} \approx \frac{3n\pi}{2} + |\delta_m| - |\delta_{n+m}| && (5.45)
\end{aligned}$$

where $\Psi_0(z) = \frac{d\Gamma(z)}{dz}$ is the digamma function, and $\delta_n = \frac{8\sqrt{3}}{81n^2} + O(n^{-4})$. Once we have the full I_{nm} matrix, we can solve the eigenequation numerically by using a finite cut-off at a dimension big enough so that it does not affect the result (for example dimension p+20 for the p^{th} eigenstate). The final form of the eigenequation:

$$\left[\left(\frac{3}{2}n\right)^2 + \frac{3\pi n}{2} + \frac{\sqrt{3}}{3} - \delta_n\right]\psi_n^L + \sum_{m\neq n}\left(\frac{3\pi}{2}\min(n,m) + \delta_{\frac{|m-n|}{2}} - \delta_{\frac{m+n}{2}}\right)\psi_m^L = b^2\psi_n^L \tag{5.46}$$

(where n and m are both odd or both even) justifies the intuitive prediction that asymptotically the eigenfunctions and eigenvalues will tend to those of a flat, infinite potential well: as n increases, the quadratic term will dominate the linear and constant terms. Moreover, it becomes clear why the level spacing is $\approx \frac{3}{2}$

An identical procedure for the $k = 1$ subspace leads to a similar equation:

$$((3n)^2 - b^2)\psi_n^H + \frac{6}{\pi}\sum_{m=1}^{\infty} J_{nm}\psi_m^H = 0 \tag{5.47}$$

where

$$\begin{aligned} J_{n,n} &= n\left[3\pi - \sqrt{3}\left(\Psi_0\left(n+\frac{2}{3}\right) - \Psi_0\left(n+\frac{1}{3}\right)\right)\right] = & (5.48)\\ &= 6n\pi - I_{2n,2n} & (5.49)\\ J_{n,n+2m} &= J_{n+m,n+m} - J_{m,m} & (5.50) \end{aligned}$$

The eigenvectors and eigenvalues of (5.42) and (5.47) coincide with the results obtained *via* the Frobenius method.

5.1.2 Quantum vectorial model

The classical results suggest another possible realisation of the quantum CM Hamiltonian. Let us recall the conclusion of section 4.1, that the vectorial degrees of freedom assigned to each particle turn out to be equivalent to the apparently two-particle L_{ij} variables. This motivated me to search for quantum two-particle operators $\hat{L}_{ij}$ in a form of tensor products $\hat{\lambda}_i \otimes \hat{\lambda}_j$ of single-particle operators acting on subspaces of internal states:

$$\left[\hat{x}_i, \hat{\lambda}_k\right] = \left[\hat{p}_i, \hat{\lambda}_k\right] = 0 \tag{5.51}$$

The crucial condition is the set of commutation relations:

$$\left[\hat{L}_{ij}, \hat{L}_{mn}\right] = i\hbar(\delta_{jm}\hat{L}_{in} - \delta_{in}\hat{L}_{mj}) \tag{5.52}$$

By analogy to the classical relation $L_{ij} = i(e_i|e_j)$ we postulate:

$$\hat{L}_{ij} = i\hbar\hat{l}_i^\dagger \otimes \hat{l}_j, \qquad \hat{L}_{ij}^\dagger = -i\hbar\hat{l}_i \otimes \hat{l}_j^\dagger = -\hat{L}_{ji} \tag{5.53}$$

If the $\hat{l}$ operators obey the bosonic creation and anililation commutation rules:

$$\left[\hat{l}_m, \hat{l}_n^\dagger\right] = \delta_{mn}, \tag{5.54}$$

the commutation relation (5.52) holds:

$$\begin{aligned} \left[\hat{L}_{ij}, \hat{L}_{mn}\right] &= \left[i\hbar\hat{l}_i^\dagger \otimes \hat{l}_j, i\hbar\hat{l}_m^\dagger \otimes \hat{l}_n\right] = & (5.55)\\ &= i\hbar\left(\left[\hat{l}_j, \hat{l}_m^\dagger\right] i\hbar\hat{l}_i^\dagger \otimes \hat{l}_n + i\hbar\left[\hat{l}_i^\dagger, \hat{l}_n\right] i\hbar\hat{l}_j \otimes \hat{l}_m^\dagger\right) = & (5.56)\\ &= i\hbar\left(\delta_{jm}\hat{L}_{in} - \delta_{in}\hat{L}_{mj}\right) & (5.57) \end{aligned}$$

Applying such $\hat{L}$ operators such that $-\hat{L}_{ij}\hat{L}_{ji} = \hat{L}_{ij}\hat{L}_{ij}^{\dagger}$ to the CM Hamiltoian, and making use of the $\hat{n}_k = \hat{l}_k^{\dagger}\hat{l}_k$ excitation number operator, leads to:

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_i \partial_{x_i}^2 + \frac{m\omega^2}{2}\sum_i x_i^2 + \frac{\hbar^2}{m}\sum_{i<j}\frac{(\hat{n}_i + \frac{1}{2}) \otimes (\hat{n}_j + \frac{1}{2}) - \frac{1}{4}}{(x_i - x_j)^2} \tag{5.58}$$

We notice that n_k are good quantum numbers, since

$$\left[\hat{n}_k, \hat{H}\right] = 0 \tag{5.59}$$

which is even more interesting because of the classical $L_{ii} = i(e_i|e_i)$ being constants of motion. The interaction depends on the excitation numbers in a very simple way, yet this model contains a wide class of interactions. For all $n_k = n$ we obtain an ordinary CM system with $g = n(n+1)$, in particular a noninteracting system if $n = 0$. Otherwise the system is repulsive with $g_{ij} = (n_i + 1/2)(n_j + 1/2) - 1/4$. The the classical limit of this system can be found with the use of coherent states of the internal oscillators. Taking the eigenstates of $\hat{l}$ operators $|\bar{\alpha}\rangle$ such that $\hat{l}_i|\bar{\alpha}\rangle = \alpha_i|\bar{\alpha}\rangle$ with large values of $|\alpha_i|$ we obtain the ordinary CM system with $g_{ij} \approx \frac{\hbar^2}{m}|\alpha_i|^2|\alpha_j|^2$ and no room for time evolution.

5.2 Canonical quantization in Hermitian matrix phase space

The classical generalisation of the N-particle Calogero-Moser model reviewed in section 3.1.1 made use of the configuration space of Hermitian matrices: $\mathcal{M} = \left\{X \in M_{N\times N}(\mathbb{C}) : X^{\dagger} = X\right\}$. The phase space which is generally a cotangend bundle $T^*\mathcal{M}$ here will be simply $M = \mathcal{M} \times \mathcal{M}$ consisting of pairs (X, Y) of Hermitian matrices. Each such pair of matrices can be identified with a point in $\mathbb{R}^{2N^2}$, where $1 \leqslant i < j \leqslant N$ in the following way:

$$X = \begin{pmatrix} x_{11} & \dots & \dots & \dots & \dots \\ \dots & x_{ii} & \dots & \frac{x_{ij}^R + ix_{ij}^I}{\sqrt{2}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \frac{x_{ij}^R - ix_{ij}^I}{\sqrt{2}} & \dots & x_{jj} & \dots \\ \dots & \dots & \dots & \dots & x_{NN} \end{pmatrix}, \tag{5.60}$$

$$Y = \begin{pmatrix} y_{11} & \dots & \dots & \dots & \dots \\ \dots & y_{ii} & \dots & \frac{y_{ij}^R + iy_{ij}^I}{\sqrt{2}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \frac{y_{ij}^R - iy_{ij}^I}{\sqrt{2}} & \dots & y_{jj} & \dots \\ \dots & \dots & \dots & \dots & y_{NN} \end{pmatrix} \tag{5.61}$$

The symplectic form in a $(x_{ii}, x_{ij}^R, x_{ij}^I, y_{ii}, y_{ij}^R, y_{ij}^I) \in \mathbb{R}^{2N^2}$ phase space:

$$\omega_R = \sum_{i=1}^{N} dx_{ii} \wedge dy_{ii} + \sum_{i<j} dx_{ij}^R \wedge dy_{ij}^R + dx_{ij}^I \wedge dy_{ij}^I \tag{5.62}$$

coindices with $\omega = \mathrm{Tr}(dX \wedge dY)$ for (X, Y) defined with (5.60) and (5.61) (all the imaginary terms cancel). The Poisson brackets for $i < j, k < l$:

$$\{x_{ii}, y_{jj}\} = \delta_{ij}, \qquad \left\{x_{ij}^{R,I}, y_{kl}^{R,I}\right\} = \delta_{ik}\delta_{jl} \tag{5.63}$$

$$\left\{x_{ij}^R, x_{kl}^I\right\} = 0, \qquad \left\{y_{ij}^R, y_{kl}^I\right\} = 0 \tag{5.64}$$

$$\left\{x_{ij}^R, y_{kl}^I\right\} = 0, \qquad \left\{x_{ij}^I, y_{kl}^R\right\} = 0 \tag{5.65}$$

are equivalent to $\{X_{ij}, Y_{lk}\} = \delta_{ik}\delta_{jl}$ in the matrix formulation. The variables $(\bar{x}, \bar{y}) \in \mathbb{R}^{2N^2}$ form a familiar ground for canonical quantisation. To each phase space variable we assign an operator acting on the Hilbert space $L^2(\mathbb{R}^{N^2})$:

$$x_{ii} \longrightarrow \hat{x}_{ii}, \qquad y_{ii} \longrightarrow -i\hbar\partial_{x_{ii}} \tag{5.66}$$

$$x^R_{ij} \longrightarrow \hat{x}^R_{ij}, \qquad y^R_{ij} \longrightarrow -i\hbar\partial_{x^R_{ij}} \tag{5.67}$$

$$x^I_{ij} \longrightarrow \hat{x}^I_{ij}, \qquad y^I_{ij} \longrightarrow -i\hbar\partial_{x^I_{ij}} \tag{5.68}$$

$$\{...,...\} \longrightarrow \frac{1}{i\hbar}[...,...] \tag{5.69}$$

5.2.1 Quantum reduction of a free system: preliminaries and outline

The operators $\hat{x}^{R,I}_{ij}$ and their canonical conjugates can be organised into matrices $\hat{X}$ and $\hat{Y}$ just as in (5.60),(5.60). Using such operator-valued matrices we can efficiently define free Hamiltonian together with its eigenfunctions:

$$\hat{H}_F = \frac{1}{2m}\mathrm{Tr}\left(\hat{Y}^2\right), \qquad \psi_K(X) = e^{i\mathrm{Tr}(KX)}, \tag{5.70}$$

$$\hat{H}_F\psi_K(X) = E_K\psi_K(X), \qquad E_K = \frac{\hbar^2}{2m}\mathrm{Tr}\left(K^2\right), \tag{5.71}$$

where K is a Hermitian matrix. Following the idea of classical reduction presented in section 3.1.1 I introduce new variables $(D, \bar{a}) \in \mathbb{R}^{N+d}$ where D is a diagonal matrix, $U = \exp(\bar{a}\cdot\bar{\tau})$ is unitary and $\bar{\tau} = (\tau_1, \tau_2, ..., \tau_d)$ are the anti-Hermitian basis vectors of the Lie algebra $\mathfrak{su}(N)$:

$$X = U^\dagger DU = \exp(-\bar{a}\cdot\bar{\tau})D\exp(\bar{a}\cdot\bar{\tau}). \tag{5.72}$$

The Hamiltonian and the wavefunction (5.70) can be reexpressed in these variables:

$$\psi_K(D, \bar{a}) = \exp(i\mathrm{Tr}(\exp(\bar{a}\cdot\bar{\tau})K\exp(-\bar{a}\cdot\bar{\tau})D)) = e^{i\mathrm{Tr}(K(a)D)}, \tag{5.73}$$

$$\hat{H}_{D,\bar{a}} = -\frac{\hbar^2}{2m}\sum_{i,j,k=1}^{N}\sum_{l=1}^{d}\left(\frac{\partial D_k}{\partial X_{ij}}\frac{\partial}{\partial D_k} + \frac{\partial a_l}{\partial X_{ij}}\frac{\partial}{\partial a_l}\right)^2, \tag{5.74}$$

I will prove that $H_{D,\bar{a}}$ differs only by a similarity transformation from a Calogero-Moser Hamiltonian with $(D_i - D_j)^{-2}\hat{\lambda}^2_{ij}$ interacting terms, where $\hat{\lambda}_{ij} = \sum_{k=1}^{d} l^k_{ij}\partial_{a_k}$, and the commutators are the exact counterparts of the classical Poisson brackets (3.25). The last step of the reduction procedure will consist of projecting $\psi_K(D, \bar{a})$ on the ∂_{a_l} eigenspaces in a way that will result in eigenfunctions of the CM Hamiltonian.

Before carrying out this programme I need to make two remarks. Firstly, the procedure is identical as outlined above if we start with real, symmetric matrices X and K. Then $U \in SO(N)$ and $(\tau_1, \tau_2, ..., \tau_d)$ span the $\mathfrak{so}(N)$ Lie algebra. In this case the configuration space $\mathcal{M} \equiv \mathbb{R}^{N+\binom{N}{2}}$, $N + d = N + \binom{N}{2}$, and there are as many generators and corresponding $(a_1, ..., a_d)$ variables as needed. On the other hand in the unitary case $\mathcal{M} \equiv \mathbb{R}^{N^2}$ while the dimension of $su(N)$ is equal to $N^2 - 1$, so it seems we have $N + d = N^2 + N - 1$ and $X \longrightarrow (D, a_1, ..., a_d)$ would not be a valid coordinate transformation. Fortunately we can restrict ourselves to unitary matrices with real diagonal entries due to gauge equivalence: if some matrix U diagonalises a given X, so does $U' = diag(e^{i\phi_1}, ..., e^{i\phi_N})U$, and if $U'_{ii} \in \mathbb{R}$, U' can be generated by $N^2 - N$ off-diagonal generators only. Therefore we can set $d = N^2 - N$ and stay in the off-diagonal $\mathfrak{su}(N)$ subspace.

5.2.2 The free Hamiltonian in (D, a) variables

The X_{ij} variables can be eliminated from (5.74) *via* the following expressions:

$$X_{ij}(D,\bar{a}) = (U^\dagger(\bar{a})DU(\bar{a}))_{ij}, \quad U = U(\bar{a}) = e^{\bar{a}\bar{\tau}}. \tag{5.75}$$

The derivatives $\partial_{X_{ij}} D_k$ and $\partial_{X_{ij}} a_l$ can be found as the elements of the inverted Jacobian matrix:

$$\frac{\partial(D,\bar{a})}{\partial X} = \left(\frac{\partial X}{\partial(D,\bar{a})}\right)^{-1} = \left(\frac{\partial(U^\dagger(\bar{a})DU(\bar{a}))}{\partial(D,\bar{a})}\right)^{-1}. \tag{5.76}$$

The elements of the Jacobian:

$$\frac{\partial X}{\partial(D,\bar{a})} = \left(\begin{array}{c|c} \frac{\partial X_{ii}}{\partial D_k} = |U_{ki}|^2 & \frac{\partial X_{ii}}{\partial a_l} = (U^\dagger \Omega_l U)_{ii} \\ \hline \frac{\partial X_{ij}}{\partial D_k} = U^*_{ki} U_{kj} & \frac{\partial X_{ij}}{\partial a_l} = (U^\dagger \Omega_l U)_{ij} \end{array}\right), \tag{5.77}$$

where

$$\begin{aligned} \Omega_l &= \left[D, (\partial_{a_l} U)U^\dagger\right] = \Omega_l^\dagger, && (5.78)\\ (\partial_{a_l} U)U^\dagger &= u(A)_{lk}\tau_k \in \mathfrak{g}, && (5.79)\\ u(A) &= \sum_{n=0}^{\infty} \frac{A^n}{(n+1)!}, \quad A_{ij} = a_k \cdot f^j_{ki} && (5.80) \end{aligned}$$

Now the challenging part is to invert this matrix. We can also use it to calculate the metric tensor, as shown in the appendix B.1:

$$\mathbf{g} = \left(\begin{array}{c|c} \mathbb{1}_{N\times N} & 0_{N\times d} \\ \hline 0_{d\times N} & \mathrm{Tr}(\Omega_l \Omega_k) \end{array}\right), \tag{5.81}$$

and then, due to the block structure of $\mathbf{g} = \mathbf{1}_N \oplus g$, we clearly see that the Hamiltonian splits into two parts:

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2m}\frac{1}{\sqrt{|\mathrm{det} g|}}\left(\sum_{k=1}^{N} \partial_{D_k}\sqrt{|\mathrm{det} g|}\partial_{D_k} + \sum_{k,l=1}^{d} \partial_{a_l}\sqrt{|\mathrm{det} g|}(g^{-1})_{lm}\partial_{a_m}\right) \\ &= -\frac{\hbar^2}{2m}(\Delta_D + \Delta_a) && (5.82) \end{aligned}$$

For both orthogonal and unitary matrices we prove in (B.1) that:

$$g = 2u\mathbf{D}^2 u^T, \quad \mathbf{D} = diag(D_i - D_j, (ij) \in I) \in M_{d\times d} \tag{5.83}$$

where I is the set of indices appropriate for each setting. In the orthogonal case $I = \{(ij) : 1 \leqslant i < j \leqslant N\}$ and in the unitary case $I = \{(ij) : 1 \leqslant i \neq j \leqslant N\}$. From this it automatically follows that:

$$\begin{aligned} \mathrm{det} g &= 2^d \mathcal{D}^2 |\mathrm{det} u|^2, \quad \mathcal{D} = \prod_{i<j} |D_i - D_j|^\alpha, && (5.84)\\ g^{-1} &= \frac{1}{2}(u^T)^{-1}\mathbf{D}^{-2}u^{-1} && (5.85) \end{aligned}$$

where $\alpha = \frac{2d}{N(N-1)}$, which means $\alpha = 1$ corresponds to the orthogonal and $\alpha = 2$ to the unitary case. The factorisation of g and $\det g$ into a purely D and $\bar{a}$-dependent parts allows us to simplify Δ_D (appendix B.2):

$$\Delta_D = \frac{1}{\mathcal{D}} \sum_{i=1}^{N} \frac{\partial}{\partial D_i} \left(\mathcal{D} \frac{\partial}{\partial D_i} \right) \tag{5.86}$$

$$\sqrt{\mathcal{D}} \Delta_D \frac{1}{\sqrt{\mathcal{D}}} = \sum_{i=1}^{N} \frac{\partial^2}{\partial^2 D_i} + \frac{\alpha(2-\alpha)}{2} \sum_{i<j}^{N} \frac{1}{D_{ij}^2}, \tag{5.87}$$

where (5.87) is a similarity transformation which eliminates the 1^{st} order derivatives from Δ_D (and leaves the Δ_a part unchanged). The result is a free N particle system for the unitary and an ordinary CM for the orthogonal case[1].

Before I move on to Δ_a, let me recall that the indices of the algebra degrees of freedom are ordered pairs $(ij), (pq), (rs) \in I$, where $I = \{(pq) : 1 \leqslant p < q \leqslant N\}$ in the orthogonal case and $I = \{(pq) : 1 \leqslant p \neq q \leqslant N\}$ in the unitary case. Moreover, let $\bar{a} \cdot \bar{\tau} = \sum_{i<j} a_{ij} \tau_{ij} + \sum_{i>j} a_{ij} \sigma_{ji}$ where τ_{ij} and σ_{ij} are the off-diagonal basis elements of $\mathfrak{su}(N)$ from the definition 18. In the orthogonal setting of course $a_{ij} = 0$ for $i > j$. Now we can write Δ_a explicitly:

$$\Delta_a = \frac{1}{2|\det u|} \sum_{(ij)\in I} \sum_{(pq)\in I} \sum_{(rs)\in I} \frac{\partial}{\partial a_{pq}} \left(|\det u| \frac{u^{-1}_{(ij)(pq)} u^{-1}_{(ij)(rs)}}{(D_i - D_j)^2} \frac{\partial}{\partial a_{rs}} \right) \tag{5.88}$$

$$= \frac{1}{2} \sum_{(ij)\in I} \frac{\hat{\Lambda}_{ij}}{(D_i - D_j)^2} \tag{5.89}$$

The free Hamiltonian, after similarity transformation, in the (D, a) variables:

$$\hat{H}'_F = \sqrt{\mathcal{D}} \hat{H}_F \frac{1}{\sqrt{\mathcal{D}}} = -\frac{\hbar^2}{2m} \left(\sum_{i=1}^{N} \frac{\partial^2}{\partial D_i^2} + \frac{1}{2} \sum_{(ij)\in I} \frac{\hat{\Lambda}_{ij} + (2-\alpha)}{(D_i - D_j)^2} \right), \tag{5.90}$$

has a form of a generalised Calogero-Moser Hamiltonian, where:

$$\hat{\Lambda}_{ij} = \hat{\lambda}_{ij}^2 + F_{ij} \hat{\lambda}_{ij} \tag{5.91}$$

$$\hat{\lambda}_{ij} = \sum_{(pq)\in I} u^{-1}_{(ij)(pq)} \frac{\partial}{\partial a_{pq}} \tag{5.92}$$

$$F_{ij} = \frac{1}{|\det u|} \sum_{(pq)\in I} \frac{\partial}{\partial a_{pq}} \left(|\det u| u^{-1}_{(ij)(pq)} \right) = \frac{1}{|\det u|} \sum_{(pq)\in I} \left(\frac{\partial C(u)}{\partial a_{pq}} \right)_{(pq)(ij)} \tag{5.93}$$

where $C(u)$ is the cofactor matrix of u, and the determinant $\det u$ is positive, so the absolute value is redundant. Using the definition of a divergence of a tensor field, we can rewrite (5.93) as:

$$F_{ij} = \frac{1}{\det u} \left(\nabla \cdot C(u^T) \right)_{ij}. \tag{5.94}$$

In the appendix B.3 I prove that F_{ij} vanishes identically. This means that indeed

$$\hat{\Lambda}_{ij} = \hat{\lambda}_{ij}^2 \tag{5.95}$$

[1] We notice that the interaction term is attractive in this case, but this is not the full Hamiltonian, and further $1/D_{ij}^2$ repulsive terms will appear.

and the last step is to calculate the commutation relations between the $\hat{\lambda}_{ij}$ operators. As shown in B.3, they turn out to be the following:

$$\left[\hat{\lambda}_{ij}, \hat{\lambda}_{kl}\right] = -\sum_{(mn)\in I} f^{(mn)}_{(ij)(kl)} \hat{\lambda}_{mn}, \tag{5.96}$$

which means that they form a representation of the Lie algebra $\mathfrak{so}(N)$ or $\mathfrak{su}(N)$ depending on the setting. This result is probably as elegant and as close to quantizing the generalized Calogero-Moser model as one can get. The next step is to find the eigenfunctions through appropriate integration of the plane waves (5.73).

The harmonic case

Following the same line of thought as in sections 5.2.1 and 5.2.2 for a system of N^2 or $\frac{N(N+1)}{2}$ noninteracting particles in an external harmonic potential we obtain:

$$\hat{H}_H = \mathrm{Tr}\left(\frac{1}{2m}\hat{Y}^2 + \frac{m\omega^2}{2}\hat{X}^2\right) = \hbar\omega\left(\mathrm{Tr}(\hat{a}^\dagger\hat{a}) + \frac{N+d}{2}\right), \tag{5.97}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{i}{\sqrt{m\hbar\omega}}\hat{Y} + \sqrt{\frac{m\omega}{\hbar}}\hat{X}\right), \tag{5.98}$$

$$|\psi_n\rangle = |n_{ii}, n^R_{ij}, n^I_{ij}\rangle = |n\rangle \tag{5.99}$$

$$\tag{5.100}$$

$$\hat{H}_H|\psi_n\rangle = \hbar\omega\left(\sum_{i,j} n_{ij} + \frac{N+d}{2}\right)|\psi_n\rangle, \tag{5.101}$$

where the eigenstates are of course expressed with Fock states of individual oscillators. In the position representation we will have:

$$\psi_n(X) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{(N+d)}{4}} \prod_{i,j} \frac{e^{-\frac{m\omega}{2\hbar^2}x_{ij}}}{\sqrt{2^{n_{ij}}(n_{ij})!}} H_{n_{ij}}\left(\sqrt{\frac{m\omega}{\hbar}}x_{ij}\right) \tag{5.102}$$

where $x_{ij} = X^R_{ij}$ or X^I_{ij} for $i < j$ and $i > j$ respectively, and $H_n(z)$ are the Hermite polynomials. Since $\mathrm{Tr}(X^2) = \mathrm{Tr}(D^2)$, the Hamiltonian in the (D, a) variables differs from (5.90) by a harmonic term, resulting in a generalised Calogero Hamiltonian:

$$\hat{H}'_H = \sqrt{\mathcal{D}}\hat{H}_H\frac{1}{\sqrt{\mathcal{D}}} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial D_i^2} + \frac{m\omega^2}{2}D_i^2\right) - \frac{\hbar^2}{4m}\left(\sum_{(ij)\in I}\frac{\hat{\Lambda}_{ij} + (2-\alpha)}{(D_i - D_j)^2}\right). \tag{5.103}$$

5.2.3 The Hamiltonian and the reduced wave functions for N=2

We expect the case of $N = 2$ particles to be equivalent to the ordinary CM system. With only one pair of interacting particles, there is by default only one operator $\hat{L}_{12}$ whose eigenvalue should yield a constant coupling in the interaction term $\frac{g^2}{(D_1-D_2)^2}$. This is why we will treat the $N = 2$ (orthogonal and unitary) case separately. We shall apply the results from 5.2.2 and calculate the reduced wave functions in the orthogonal and unitary case.

Example: SO(2)

This example is not entirely an original contribution, similar considerations can be found in [64]. The $SO(2)$ group has only one generator and it is known that its every

element can be expressed as:

$$U(\phi) = \exp\left(\phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{5.104}$$

Since there is only one generator, there are no nonvanishing structure constants, and therefore in the adjoint representation $A = 0$ and $u(A) = u^{-1}(A) = 1$. The single Λ operator at hand is $\hat{\Lambda}_{12} = \frac{\partial^2}{\partial\phi^2}$, and the Hamiltonian (5.90) has a simple form:

$$\hat{H}' = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial D_1^2} + \frac{\partial^2}{\partial D_2^2} + \frac{1}{2(D_1 - D_2)^2}\left(\frac{\partial^2}{\partial\phi^2} + 1\right)\right] \tag{5.105}$$

which easily translates to centre of mass and relative distance variables $R = \frac{D_1+D_2}{2}$, $r = D_1 - D_2$ and a doubled angle $\Phi = 2\phi$:

$$\hat{H}'(R, r, \Phi) = -\frac{\hbar^2}{4m}\frac{\partial^2}{\partial R^2} - \frac{\hbar^2}{m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\Phi^2} + \frac{1}{4}\right)\right]. \tag{5.106}$$

Now we need to express the eigenfinction of the unreduced, free system in these variables:

$$\begin{aligned}
\psi_K(X) &= \exp(i\mathrm{Tr}(KX)) = \exp\left[i\mathrm{Tr}\left(KU^T(\phi)DU(\phi)\right)\right] = \psi_K(D, \phi) \\
&= \exp\left[i\mathrm{Tr}\left(\begin{pmatrix} K_1 & \frac{K_{12}}{\sqrt{2}} \\ \frac{K_{12}}{\sqrt{2}} & K_2 \end{pmatrix} U(\phi)^T \begin{pmatrix} R + \frac{r}{2} & 0 \\ 0 & R - \frac{r}{2} \end{pmatrix} U(\phi)\right)\right] \\
\psi_K(R, r, \Phi) &= \exp[[i\mathrm{Tr}(KR)]\exp[i\kappa\cos(\Phi + \phi_k)r], && (5.107) \\
\kappa &= \frac{1}{2}\sqrt{(K_1 - K_2)^2 + 2K_{12}^2} = \sqrt{\mathrm{Tr}(k^2)}, && (5.108) \\
k &= K - \frac{1}{2}\mathrm{Tr}(K) && (5.109) \\
\cos\phi_k &= \frac{K_1 - K_2}{2\kappa}. && (5.110)
\end{aligned}$$

Now we make use of the fact, that $\hat{H}\psi_K = E_K\psi_K$ holds no matter in which variables we express the equation in, and also if we reverse the similarity transformation $\hat{H}' = \sqrt{r}\hat{H}\frac{1}{\sqrt{r}}$, we get:

$$\hat{H}'(R, r, \Phi)(\sqrt{r}\psi_K(R, r, \Phi)) = E_K\sqrt{r}\psi_K(R, r, \Phi). \tag{5.111}$$

The crucial step of the procedure is the integration over the Φ variable which projects $\psi_K(R, r, \Phi)$ onto an eigenspace of $\hat{\Lambda}_{12} = \partial_\Phi^2$. We perform the integration of both sides of (5.111):

$$-\frac{\hbar^2}{m}\int e^{-i\nu\Phi}\left[\frac{1}{4}\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\Phi^2} + \frac{1}{4}\right)\right](\sqrt{r}\psi_K)d\Phi = E_K\sqrt{r}\int e^{-i\nu\Phi}\psi_K d\Phi$$

where the integration cuts out a Fourier component of $\psi_K(R, r, \Phi) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\Phi}\psi_{n,K}(R, r)$. The integral $\int_0^{2\pi} e^{-i\nu\phi}\psi_K(R, r, \Phi)d\Phi = \psi_{\nu,K}(R, r)$ commutes with the R and r-dependent terms of $\hat{H}$, thus:

$$-\frac{\hbar^2}{m}\left(\frac{1}{4}\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{4r^2}\right)(\sqrt{r}\psi_{\nu,K}) - \frac{\hbar^2}{mr^2}\sqrt{r}\int_0^{2\pi} e^{-i\nu\Phi}\frac{\partial^2\psi_K}{\partial\Phi^2}d\Phi = E_K\sqrt{r}\psi_{\nu,K},$$

where the only nontrivial term can be calculated by parts as for $\nu \in \mathbb{Z}$ the boundary terms cancel:

$$\int_0^{2\pi} e^{-i\nu\Phi}\partial_\Phi^2\left(\psi_K(R, r, \Phi)\right)d\Phi = -\nu^2\int_0^{2\pi} e^{-i\nu\Phi}\psi_K(R, r, \Phi)d\Phi = -\nu^2\psi_{\nu,K}(R, r). \tag{5.112}$$

Finally, we discover that the function $\sqrt{r}\psi_{\nu,K}(R,r)$ is the eigenfunction od an ordinary 2-particle Calogero-Moser Hamiltonian with $g=\nu^2-\frac{1}{4}$:

$$-\frac{\hbar^2}{m}\left[\frac{1}{4}\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{1}{4}-\nu^2\right)\right](\sqrt{r}\psi_{\nu,K}(R,r))=E_K\sqrt{r}\psi_{\nu,K}(R,r). \quad (5.113)$$

In the final step we need to compute the integral:

$$\begin{aligned}\psi_{\nu,K}(R,r) &= e^{i\mathrm{Tr}(K)R}\sqrt{r}\int_0^{2\pi}e^{-i\nu\Phi}e^{i\kappa\cos(\Phi+\phi_K)r}d\Phi = \quad (5.114)\\ &= e^{i\nu\phi_K}e^{i\mathrm{Tr}(K)R}\int_0^{2\pi}e^{i(\kappa r\cos\Phi-\nu\phi)}d\Phi = \\ &= e^{i\nu\phi_K}e^{i\mathrm{Tr}(K)R}i^\nu\sqrt{r}J_\nu(\sqrt{\mathrm{Tr}(k^2)}r). \quad (5.115)\end{aligned}$$

As expected [3], the Calogero-Moser wave function, apart from the trivial center of mass component and some phase factors, has the form $\sqrt{r}J_\nu(\kappa r)$, where J_ν is the Bessel function. Moreover, the trace of K contributes to the energy of center of mass motion, $E_{CM}=\frac{\hbar^2\mathrm{Tr}(K)^2}{4m}$, while the traceless part $k=K-\frac{1}{2}\mathrm{Tr}(K)$ contributes to the relative motion $E_{rel}=\frac{\hbar^2}{m}\mathrm{Tr}(k^2)$, $E_{CM}+E_{rel}=E_K$.

In the harmonic case, whenever it is convenient, we will rescale all the variables in a standard way: $\tilde{x}=\sqrt{\frac{m\omega}{\hbar}}x=\frac{x}{l}$ (x stands for all X_{ij},R and r) and drop all the normalisation constants. The reduction procedure can be summarised as follows:

$$\begin{aligned}\psi_{\bar{n}}(X) &= \langle X|n\rangle = e^{-\frac{1}{2}\mathrm{Tr}(\tilde{X}^2)}H_{n_{11}}(\tilde{X}_{11})H_{n_{12}}(\tilde{X}_{12})H_{n_{22}}(\tilde{X}_{22}) \quad (5.116)\\ \langle R,r,\Phi|X\rangle &= \delta(X_{11}-R-\frac{1}{2}r\cos\Phi, X_{22}-R+\frac{1}{2}r\cos\Phi, X_{12}+\frac{\sqrt{2}}{2}r\sin\Phi)\\ \psi_{\bar{n}}(R,r,\Phi) &= \int\langle R,r,\Phi|X\rangle\langle X|n\rangle dX \quad (5.117)\\ \hat{H}'_H\sqrt{r}\psi_{\bar{n}}(R,r,\Phi) &= E_{\bar{n}}\sqrt{r}\psi_{\bar{n}}(R,r,\Phi), \quad (5.118)\\ E_{\bar{n}} &= \hbar\omega\left(n_{11}+n_{22}+n_{12}+\frac{3}{2}\right)=\hbar\omega\epsilon_{\bar{n}}. \quad (5.119)\end{aligned}$$

Again, we apply the integration over Φ:

$$\int e^{-i\nu\phi}\left[-\frac{1}{4}\frac{\partial^2}{\partial\tilde{R}^2}+\tilde{R}^2-\frac{\partial^2}{\partial\tilde{r}^2}+\frac{1}{4}\tilde{r}^2-\frac{1}{r^2}\left(\frac{\partial^2}{\partial\Phi^2}+\frac{1}{4}\right)\right](\sqrt{\tilde{r}}\psi_{\bar{n}})d\Phi=\epsilon_{\bar{n}}\int e^{-i\nu\Phi}(\sqrt{\tilde{r}}\psi_{\bar{n}})d\Phi,$$

and obtain the Calogero Hamiltonian with the eigenfunction $\sqrt{\tilde{r}}\psi_{\nu,\bar{n}}(\tilde{R},\tilde{r})=\sqrt{\tilde{r}}\int_0^{2\pi}e^{-i\nu\Phi}\psi_{\bar{n}}(\tilde{R},\tilde{r},\Phi)$:

$$\left[-\frac{1}{4}\frac{\partial^2}{\partial\tilde{R}^2}+\tilde{R}^2-\frac{\partial^2}{\partial\tilde{r}^2}+\frac{1}{4}\tilde{r}^2+\frac{1}{\tilde{r}^2}\left(\nu^2-\frac{1}{4}\right)\right](\sqrt{\tilde{r}}\psi_{\nu,\bar{n}}(\tilde{R},\tilde{r}))=\epsilon_{\bar{n}}\sqrt{\tilde{r}}\psi_{\nu,\bar{n}}(\tilde{R},\tilde{r}). \quad (5.120)$$

The last step is to compute $\psi_{\nu,\bar{n}}(\tilde{R},\tilde{r})$, which should coincide with the predictions of [3]. And for this calculation we may use the result for the free system and the overlaps between the plane waves and oscillator eigenstates of the N^2-particle system:

$$\psi_{\nu,n}(R,r)=\int\psi_{\nu,K}(R,r)\langle K|n\rangle dK \quad (5.121)$$

Example: SU(2)

In the unitary case two parameters are necessary to define a diagonalising matrix (the third parameter is redundant as stated in 5.2.1):

$$U(\theta,\phi)=\begin{pmatrix}\cos\theta & -\sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & \cos\theta\end{pmatrix}, \quad (5.122)$$

where $\theta \in [0, \pi/2]$ is a sufficient set, due to gauge symmetry. In order to find the metric tensor we need to calculate:

$$\begin{aligned}(\partial_\theta U)U^\dagger &= \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \quad (\partial_\phi U)U^\dagger = \begin{pmatrix} -i\sin^2\theta & i\sin\theta\cos\theta e^{-i\phi} \\ i\sin\theta\cos\theta e^{i\phi} & i\sin^2\theta \end{pmatrix} \\ \Omega_\theta &= r\begin{pmatrix} 0 & -e^{-i\phi} \\ -e^{i\phi} & 0 \end{pmatrix}, \quad \Omega_\phi = r\begin{pmatrix} 0 & -i\sin\theta\cos\theta e^{-i\phi} \\ i\sin\theta\cos\theta e^{i\phi} & 0 \end{pmatrix}\end{aligned}$$

where $r = D_1 - D_2$. The nontrivial block of the metric tensor, its determinant and in:verse have the following form:

$$g = \begin{pmatrix} 2r^2 & 0 \\ 0 & \frac{1}{2}r^2\sin^2(2\theta) \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \frac{1}{2r^2} & 0 \\ 0 & \frac{2}{r^2\sin^2(2\theta)} \end{pmatrix}, \quad \sqrt{|\det(g)|} = r^2\sin(2\theta). \tag{5.123}$$

The (θ, ϕ) dependent part of the Laplacian, after all the simplifications,and substitution $2\theta \longrightarrow \theta \in [0, \pi]$:

$$\begin{aligned}\Delta_{\theta,\phi} &= \frac{1}{r^2\sin(2\theta)}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin(2\theta)}{2}\frac{\partial}{\partial\theta}\right) + \frac{2}{\sin(2\theta)}\frac{\partial^2}{\partial\phi^2}\right] = \\ &= \frac{2}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = \frac{2}{r^2}\Delta_{S^2}\end{aligned} \tag{5.124}$$

turns out to be proportional to a familiar Laplace operator on the two-dimensional sphere. The eigenfunction $\psi_K(X) = \psi_K(R, r, \theta, \phi)$ in the new variables:

$$\begin{aligned}\psi_K(R, r, \theta, \phi) &= e^{\left(i\mathrm{Tr}(KU^\dagger DU)\right)} = e^{i(K_1+K_2)R}\exp\left[i\left(\kappa_1\cos\theta - \kappa_2\sin\theta\right)r\right] \\ K &= \begin{pmatrix} K_1 & \frac{K_R+iK_I}{\sqrt{2}} \\ \frac{K_R-iK_I}{\sqrt{2}} & K_2 \end{pmatrix} \qquad (5.125) \\ \kappa_1 &= \frac{K_1 - K_2}{2} \qquad (5.126) \\ \kappa_2 &= \frac{K_R\cos\phi - K_I\sin\phi}{\sqrt{2}} = |K_{12}|\cos(\phi + \phi_k) \qquad (5.127) \\ \cos\phi_k &= \frac{K_R}{\sqrt{K_R^2 + K_I^2}} \qquad (5.128)\end{aligned}$$

decomposes into spherical harmonics:

$$\begin{aligned}\psi_K(R, r, \theta, \phi) &= \sum_{l,m} Y_{l,m}(\theta, \phi)\psi_{l,m,K}(R, r) \qquad (5.129) \\ \psi_{l,m,K}(R, r) &= \int_{S^2} Y^*_{l,m}(\theta, \phi)\psi_K(R, r, \theta, \phi)\sin\theta d\theta d\phi. \qquad (5.130)\end{aligned}$$

We use this decomposition in a similar way as in the orthogonal case, only this time $\sqrt{\mathcal{D}} = r$:

$$\hat{H}\psi_K = E_K\psi_K, \quad \hat{H}'(r\psi_K) = E_K(r\psi_K), \quad \int Y^*_{l,m}\hat{H}'(r\psi_K) = E_K\int Y^*_{l,m}(r\psi_K), \tag{5.131}$$

and since we can integrate by parts: $\int Y^*_{l,m}(\Delta_{S^2}\psi_K) = \int(\Delta_{S^2}Y^*_{l,m})\psi_K = -l(l+1)\int Y^*_{l,m}\psi_K$, we obtain a Calgero-Moser Hamiltonian with a coupling constant $l(l+1)$:

$$\left[-\frac{\hbar^2}{4m}\frac{\partial^2}{\partial R^2} + \frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right)\right]r\psi_{l,m,K} = E_K r\psi_{l,m,K}. \tag{5.132}$$

As previously, the final step is to calculate the reduced function:

$$\begin{aligned}
\psi_{l,m,K}(R,r) &= \int\int Y^*_{l,m}(\theta,\phi)\psi_K(R,r,\theta,\phi)\sin\theta d\theta d\phi = \\
&= e^{i\mathrm{Tr}(K)R}\int\int P_l^m(\cos\theta)e^{-im\phi}\exp\left[i\left(\kappa_1\cos\theta-\kappa_2\sin\theta\right)r\right]\sin\theta d\theta d\phi = \\
&= e^{i\mathrm{Tr}(K)R}\int P_l^m(\cos\theta)e^{i\kappa_1\cos\theta r}\left(\int e^{-im\phi-i|K_{12}|\sin\theta\cos(\phi+\phi_k)r}d\phi\right)\sin\theta d\theta = \\
&= e^{i\mathrm{Tr}(K)R+im\phi_k}\int P_l^m(\cos\theta)e^{i\kappa_1\cos\theta}J_m(|K_{12}|r\sin\theta)\sin\theta d\theta
\end{aligned}$$

The relative contains the Bessel function which arises from the integration over ϕ. It looks complicated, but an analytical solution exists [79], and has the form:

$$I_l^m(K,r)\propto P_l^m\left(\frac{2\kappa_1}{\kappa}\right)j_l(\kappa r), \tag{5.133}$$

where $\kappa=\sqrt{\mathrm{Tr}(k^2)}$ and k is the traceless part of K, just like in the orthogonal case, and $j_l(x)$ is the spherical Bessel function defined as:

$$j_l(x)=\sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x). \tag{5.134}$$

This means that the inverse square root from j_l and the r factor from the similarity transformation simplify, and as it should be expected from the two-particle Calogero-Moser system with $g=\frac{\hbar^2}{4m}l(l+1)$, the relative wavefunction is $\sqrt{r}J_{l+\frac{1}{2}}(r)$.

To summarize, for $N=2$ the reduction of a free system results in ordinary Calogero-Moser systems with quantized values of coupling constants. The orthogonal setting recovers the case of $g=\frac{\hbar^2}{4m}\left(l^2-\frac{1}{4}\right)$ and the relative wavefunctions in the form of $\psi_l(r)=\sqrt{r}J_l(\kappa r)$, and the unitary one recovers $g=\frac{\hbar^2}{4m}l\left(l+1\right)$ and $\psi_{l+\frac{1}{2}}(r)=\sqrt{r}J_{l+\frac{1}{2}}(\kappa r)$.

5.2.4 The Hamiltonian and reduced wavefunctions for $N\geqslant 3$

For the simplest case of $N=2$ it was possible to compute the metric tensor and the Laplace operator directly. For $N\geqslant 3$ we have the general Hamiltonian:

$$\hat{H}'_F = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial D_i^2}+\frac{1}{4m}\sum_{(ij)\in I}\frac{\hat{L}_{ij}^2+\hbar^2(\alpha-2)}{(D_i-D_j)^2} \tag{5.135}$$

$$\hat{L}_{ij} = -i\hbar\hat{\lambda}_{ij},\qquad \left[\hat{L}_{ij},\hat{L}_{kl}\right]=\sum_{(mn)}i\hbar f^{(mn)}_{(ij)(kl)}\hat{L}_{mn} \tag{5.136}$$

$$\hat{L}_{ij}^\dagger = (-i\hbar\hat{\lambda}_{ij})^\dagger = i(-\hbar\hat{\lambda}_{ij})=\hat{L}_{ij} \tag{5.137}$$

Having more than one interacting pair of particles and more than a single $\hat{L}$ operator makes it difficult to produce eigenfunctions of the above Hamiltonian from plane waves defined in the X variables. The reason for this is that $\hat{L}_{ij}^2$ operators generally do not commute thus are impossible to diagonalize simultaneously. The solution is to seek for representations of the algebra of $\hat{L}$ operators in which their squares commute, such as the defining representation studied in the direct approach presented in 5.1.1. The wave-function, which had the form:

$$\sqrt{\mathcal{D}}\psi_K(D,a) = \sqrt{\prod_{i<j}(D_i-D_j)^\alpha}\cdot e^{i\mathrm{Tr}(K(a)D)}, \tag{5.138}$$

$$\hat{H}'_F(\sqrt{\mathcal{D}}\psi_K(D,a)) = E_K(\sqrt{\mathcal{D}}\psi_K(D,a)) \tag{5.139}$$

prior to the reduction, now will be a spinor of the appropriate dimension. Still, the resulting probability distribution,

$$|\sqrt{\mathcal{D}}\psi_K(D,a)|^2 = \prod_{i<j}(D_i - D_j)^\alpha,$$

where $\alpha = 1, 2$ in the orthogonal and unitary setting respectively, resembles the predictions for level repulsion in random matrix theory, and indicates that in the symplectic setting there would be $\alpha = 4$ contributing to stronger repulsion.

Example: SO(3)

As a final example I will show the orthogonal 3×3 case. The matrices used in this case can be parametrised as follows:

$$U(\bar{a}\bar{\tau}) = \exp(\bar{a}\bar{\tau}) = \exp\begin{pmatrix} 0 & -a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ -a_{13} & a_{23} & 0 \end{pmatrix} = \mathbb{1} + \sin a(\bar{n}\bar{\tau}) + (1-\cos a)(\bar{n}\bar{\tau})^2, \tag{5.140}$$

where $a \in (-\pi, \pi)$, $\bar{n} \in S^2$, $\bar{a} = (a_{12}, a_{13}, a_{23}) = a\bar{n}$ and the a_{ij} notation emphasises the link between the $(ij)^{th}$ off-diagonal matrix elements in the big phase space and the interaction between the i^{th} and j^{th} particle in the reduced phase space. In the case of $SO(3)$ the $N = 3$ dimensional defining representation and the $\binom{N}{2} = 3$ dimensional adjoint representation coincide, thus $A = \bar{a}\bar{f} = \bar{a}\bar{\tau}$ and:

$$u(\bar{a}\cdot\bar{\tau}) = \mathbb{1} + \frac{1-\cos a}{a}(\bar{n}\bar{\tau}) + \frac{a-\sin a}{a}(\bar{n}\bar{\tau}) \tag{5.141}$$

$$u^{-1}(\bar{a}\cdot\bar{\tau}) = \mathbb{1} - \frac{a}{2}(\bar{n}\bar{\tau}) + \left(1 - \frac{a}{2}\operatorname{ctg}\left(\frac{a}{2}\right)\right)(\bar{n}\bar{\tau})^2 \tag{5.142}$$

$$\det u = \frac{2(1-\cos a)}{a^2}. \tag{5.143}$$

With the above results the operators $\hat{\lambda}_{ij}$ can be directly calculated:

$$\hat{\lambda}_{ij} = f(a)\bar{\nabla}_{ij} + \frac{1}{2}\left(\bar{a}\times\bar{\nabla}\right)_{ij} + (1-f(a)))\, n_{ij}(\bar{n}\cdot\bar{\nabla}) \tag{5.144}$$

$$F_{ij} = \frac{1}{|\det u|}\sum_{(kl)}\left(\frac{\partial C(\omega)}{\partial a_{kl}}\right)_{(ij)(kl)} = 0 \tag{5.145}$$

$$\hat{\Lambda}_{ij} = \hat{\lambda}_{ij}^2, \tag{5.146}$$

where $f(a) = \frac{a}{2}\operatorname{ctg}\left(\frac{a}{2}\right)$. The commutation relations are known for the general $N \times N$ case, but in this smallest nontrivial example it can be checked that indeed:

$$\left[\hat{\lambda}_{ij}, \hat{\lambda}_{kl}\right] = -\epsilon_{(ij)(kl)(mn)}\hat{\lambda}_{mn} \tag{5.147}$$

which means that they span an algebra which is isomorphic to $\mathfrak{so}(3)$, and there are representations in which all $\hat{\Lambda}_{ij} = \hat{\lambda}_{ij}^2$ commute with each other, namely:

- the 2×2 Pauli matrices (up to a proportionality constant) which lead to the ordinary CM system,
- the 3×3 generators of the defining representation, which lead to the case similar to the one solved in 5.1.1, only shifted due to the $\alpha(\alpha-2) = -1$ terms. This shift, as its contribution is attractive, may lead to interesting effects which need to be studied.

Chapter 6

Conclusion and outlook

The key motivations of this thesis were:

- the potential application of the known relationships of the classical Calogero-Moser system with random matrices and chaotic dynamics to the quantum setting
- the possible influence of additional degrees of freedom on the relationship of the quantum Calogero-Moser system with the Quantum Hall Effect, in a form of possible topological effects and further applications in condensed matter physics.

The necessary step towards these goals was to find a reliable quantization scheme for the generalized Calogero-Moser system. This in turn required a closer look at the dynamics of the classical degrees of freedom. What may have looked like taking two steps backwards, in fact turned out to be fruitful for the study of this many-body interacting system. The results I have obtained may be summarized in a broader perspective, together with an outlook on further further research:

- I have shown the equivalence between the classical $\mathfrak{su}(N)$ matrix variables of the generalized Calogero-Moser model and vectorial degrees of freedom carried by each particle. The equivalence is given by the relation $L_{ij}(t) = i(e_i|e_j)(t)$, which means that the L_{ij} variables are functions of one-particle functions on phase space. This property transfered directly onto the quantum operators $\hat{L}_{ij} = i\hbar\hat{l}_i^\dagger \otimes \hat{l}_j$ results in a system which has a different classical limit, namely a system with coupling constants g_{ij} in stead of dynamic variables $L_{ij}(t)$. This fact leads to the conclusion, that the quantum operators $\hat{L}_{ij}$ which will be still dynamical variables in the classical limit, must not factorize into one-particle operators.
- The trajectories of $L(t)$ matrices fall into separate classes given by the dimension of the space spanned by the initial vectors $d = span\{|e_i(0))\}$. I have proven that the matrices which give rise to stationary coupling constants belong to the class of $d = 1$, which implies all the coupling constants to coincide: $g_{ij} = g$. The models with distinct coupling constants do not admit corresponding matrix models. In the light of the derivation of integrable many-body systems with interactions by Olshanetsky and Perelomov [12] it can be concluded that such systems are not integrable.
- Nevertheless, the dynamics of a Calogero-Moser system with distinct coupling constants g_{ij} can be approximated by a matrix model with the same initial positions and momenta. The optimal matrix $L \in \mathfrak{su}(N)$ for this approximation is purely imaginary: $L_{ij} = ig_{ij}$. The study of reachable sets of the L matrices

is compatible with this result. The purely imaginary matrices are in fact the most restricted in terms of the reachable values of $|L_{ij}|$.

- I have combined the matrix and vector degrees of freedom in an augmented phase space as hinted by Gibbons [45]. A Hamilton function which mimics the one of a charged particle in a magnetic field gave rise to an integrable dynamical system with spiral trajectories in the matrix space. The unitary reduction along the lines presented in section 3.1.1 led to an unexpected result: the reduced Hamilton function contained $\frac{1}{x_{ij}}$ interactions, so far unseen in completely integrable systems. The dynamical variables in the numerators of these "Coulombic" terms do not have stationary points, unless they vanish.

- The canonical quantization of the classically reduced degrees of freedom (x, p, L) and the quantum reduction of a canonically quantized free (or harmonic) system almost coincide. The only difference is the attractive term $-\frac{\hbar^2}{4m}\sum_{i<j}\frac{1}{(x_i-x_j)^2}$, which arises for the latter scheme in the real symmetric case. In the case of $N = 2$ this additional term results in two distinct sets of wavefunctions for $l = 0, 1, 2...$: $\psi_l(r) = \sqrt{r}J_l(\kappa r)$ in the orthogonal and $\psi_{l+\frac{1}{2}}(r) = \sqrt{r}J_{l+\frac{1}{2}}(\kappa r)$ in the unitary case. Without this modification, the orthogonal case would yield a different set of functions, $\sqrt{r}J_{a_l}(r)$, where $a_l = \sqrt{l^2 + \frac{1}{4}}$. This off-course makes no difference for large values of l (as expected in the classical limit). Yet if we consider the system in a harmonic trap, the energy (3.63) contains a term

$$E_a = \hbar\omega\binom{N}{2}\left(a + \frac{1}{2}\right).$$

  This means that the ground state is significantly affected, as the $a = 0$ case is excluded in favour of $a = \frac{1}{2}$ recovered in both orthogonal and unitary setting. This suggests, that the attractive term is crucial at the quantum level and the reduction of a canonically quantized matrix system (considered in section 5.2) is expected to have a richer set of solutions than the canonically quantized reduced system (section 5.1) for a general value of N.

- The Hamiltonian which I diagonalized in the case of $N = 3$ and the $2s + 1 = N = 3$ dimensional representation of $\mathfrak{so}(3)$ can be thought of as a toy model and needs to be corrected with the attractive term discussed above. First and foremost it has to be checked if such a Hamiltonian is bounded from below. The defining representation of $\mathfrak{su}(3)$ needs to be considered as well. The expected result is a rescaled version of the solutions obtained in 5.1.1. The next step will be to generalize the results to $N > 3$.

- The common trait of the quantum models reviewed in section 3.2.1 and the ones I have considered in sections 5.1 and 5.2 is that two variables, namely the centre of mass $R = \frac{1}{N}\sum x_i$ and $r^2 = \frac{1}{N}\sum x_{ij}^2$, separate easily. The difficulty lies in the remaining angular differential equation on $\mathbb{S}^{N-2}$. The case of $N = 3$ reduces to an ordinary differential equation on a circle and is manageable through the Fourier Transform, as shown in section 5.1.1. The challenge of generalizing this result to $N > 3$ lies in finding an orthonormal basis of functions on $\mathbb{S}^{N-2}$, which will satisfy the boundary conditions imposed by the repulsive potential.

- The attractive term in the quantum orthogonal setting, and its expected influence on the eigenstates as opposed to the purely repulsive unitary setting (as well as the ordinary CM system) shows the correspondence with the classical

dynamics. In both cases the $L \in \mathfrak{so}(N)$ degrees of freedom have more impact than the $L \in \mathfrak{su}(N)$ on the spatial evolution of the system.

- The prefactors $\prod_{i<j}(D_i - D_j)^\alpha$ in the probability distribution of positions in the reduced free system, with $\alpha = 1, 2$ in the orthogonal and unitary setting respectively, are not really surprising. Just as in the case of level repulsion predicted by random matrix theory, we are considering a probability distribution of eigenvalues of a matrix, only its interpretation is different. Still, this suggest that the procedure should be repeated for a symplectic case, and the exponent is expected to be $\alpha = 4$ and a positive repulsive term should appear in the Hamiltonian.

To summarize, the study of the dynamics of classical degrees of freedom of the generalized CM systems has not only provided a sufficient foundation for the quantum considerations but also resulted in unexpected additional result in the form of an integrable system with $\frac{1}{x_{ij}}$ interactions. One of the possible extensions of this result would be to quantize the matrix and vector degrees of freedom in the augmented phase space.

Moreover, as the classical L_{ij} degrees of freedom govern the energy level repulsion for quantum systems defined by $H(\lambda) = H_0 + \lambda V$, the properties of the purely imaginary L matrices (the stationary $|L_{ij}|$ and minimal reachable sets) are expected to have an impact on the statistics of level repulsion for large values of N. The possible next step is to study the statistics of energy levels for an ensemble of purely imaginary V matrices. The downside is that this subset of Hermitian matrices does not form a universality class, but the resulting probability distribution could be interpreted as a conditional one (i.e. the random matrices V are drawn from a unitary ensemble and the imposed condition is that the matrix is purely imaginary). The expectation is that the level repulsion will be stronger than in the case of the full unitary ensemble, meaning $P(s) \propto s^{2+\delta}$, where $\delta > 0$, and the level spacing $s \to 0$.

The study of quantization schemes, on the other hand, with the reliable derivation of a quantum generalized CM Hamiltonian from a canonically quantized free (or harmonic) system, opens a plethora of exciting paths for further research. Apart from the direct continuation outlined above, there is the problem of quantum integrability, and the diagonalization (if possible) of the system obtained in section 5.2 for other representations of the underlying Lie algebra. Moreover, the finite-dimensional component of the Hilbert space, on which the $\hat{L}_{ij}$ operators act may be treated as a synthetic second dimension [80], which yields the system fit for the study of topological effects, such as edge states.

Appendix A

Details of the proofs in 4.2

A.1 Necessary conditions for (4.21)

We assume that a rational function of $N-1$ independent relative positions $x_i = x_{i1}, x_{i2}, .., x_{i,i-1}, x_{i,i+1}, ..x_{ij}, ..., x_{iN}$:

$$f(x_i) = \sum_{k\neq i,j} \left(\frac{1}{x_{ik}^2} - \frac{1}{(x_{ik}-x_{ij})^2} \right) a_{ijk} \tag{A.1}$$

is identically equal to zero. This means that for any $k \neq i, j$:

$$\partial_{x_{ik}} f = \left(-\frac{2}{x_{ik}^3} + \frac{2}{(x_{ik}-x_{ij})^3} \right) a_{ijk} = 2a_{ijk} \left[\frac{x_{ik}^3 - (x_{ik}-x_{ij})^3}{x_{ik}^3 (x_{ik}-x_{ij})^3} \right] = 0 \tag{A.2}$$

$$\partial_{x_{ik}} f = 0 \iff a_{ijk} = 0 \vee x_{ik} = x_{ik} - x_{ij} \tag{A.3}$$

It is forbidden for x_{ij} to vanish, which means $a_{ijk} = 0$, and this applies to all $k \neq i, j$. In case of (4.21) $a_{ijk} = (\epsilon_i|[P_j, P_k]|\epsilon_i)$, therefore it means that (4.23) is the necessary condition for (4.21) to vanish.

A.2 Necessary conditions for (4.23)

The imaginary part of $(\epsilon_i|\epsilon_j)(\epsilon_j|\epsilon_k)(\epsilon_k|\epsilon_i)$ will vanish if all the vectors point in the same direction: if $|\epsilon_i) = e^{i\phi_i}|e)$ then $\mathrm{Im}((\epsilon_i|\epsilon_j)(\epsilon_j|\epsilon_k)(\epsilon_k|\epsilon_i)) = \mathrm{Im}|e|^6 = 0$. If all the vectors point in one of two directions, and for the chosen triple i, j, k we have for example $e^{-i\phi_i}|\epsilon_i) = e^{-i\phi_j}|\epsilon_j) = |e)$ and $e^{-i\phi_k}|\epsilon_k) = |f)$, $\mathrm{Im}((\epsilon_i|\epsilon_j)(\epsilon_j|\epsilon_k)(\epsilon_k|\epsilon_i)) = \mathrm{Im}(|e|^2|(e|f)|^2) = 0$. Three distinct directions $|\epsilon_{1,2,3})$ can be expressed as follows:

$$\begin{aligned} (\epsilon_1| &= (1,0,0,...), \\ (\epsilon_2| &= (\bar{e}_2^1, \bar{e}_2^2, 0, 0, 0...), \\ (\epsilon_3| &= (\bar{e}_3^1, \bar{e}_3^2, \bar{e}_3^3, 0, 0...), \end{aligned}$$

and $e_{2,3}^1$ can be made real by a correct choice of gauge. The basis independent value of (4.23) reads:

$$\mathrm{Im}(e_2^1(e_2^1 e_3^1 + \bar{e}_2^2 e_3^2)e_3^1) = 0 \tag{A.4}$$

which is equivalent to $\mathrm{Im}(\bar{e}_2^2 e_3^2) = 0$, meaning that e_2^2 and e_3^2 have a common phase factor. Applying the same procedure to all tripples of vectors, but expressing every-

thing in the initial basis, we find out that

$$\begin{aligned}
(\epsilon_1| &= (1,0,0,...),\\
(\epsilon_2| &= (e_2^1, e^{i\phi_2}e_2^2, 0, 0, 0...),\\
(\epsilon_3| &= (e_3^1, e^{i\phi_2}e_3^2, e^{i\phi_3}e_3^3, 0, 0...)\\
(\epsilon_4| &= (e_4^1, e^{i\phi_2}e_4^2, e^{i\phi_3}e_4^3, e^{i\phi_4}e_4^4, 0...),\\
&..\\
(\epsilon_N| &= (e_N^1, e^{i\phi_2}e_N^2, e^{i\phi_3}e_N^3, ..., e^{i\phi_N}e_N^N),
\end{aligned}$$

where all the $e_k^l \in \mathbb{R}$, and the resulting $L_{ij} = i(\epsilon_i|\epsilon_j)$ matrix is equal to the one given by purely real vectors without the phase factors.

A.3 Necessary conditions for (4.25)

Let us find the necessary condition for:

$$\sum_{k\neq i,j}\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)(\epsilon_i|\sum_{l\neq i}\frac{[P_l,[P_k,P_j]]}{x_{il}^2}+\sum_{l\neq j}\frac{[P_k,[P_j,P_l]]}{x_{jl}^2}+\sum_{l\neq k}\frac{[P_j,[P_l,P_k]]}{x_{kl}^2}|\epsilon_i)=0.$$

provided that the vectors are real $|\epsilon_i) = |e_i) \in R^r$, and as a concequence $(e_a|e_b) = (e_b|e_a)$. First of all we shall introduce convenient shorthands:

$$\begin{aligned}
[abc] &= -[acb] = (e_i|[P_a,[P_b,P_c]]|e_i) =\\
&= 2(e_i|e_a)(e_b|e_c)((e_a|e_b)(e_c|e_i)-(e_a|e_c)(e_b|e_i)), &\text{(A.5)}\\
0 &= [abc]+[bca]+[cab] &\text{(A.6)}\\
(e_a|e_b) &= (ab) = (ba). &\text{(A.7)}
\end{aligned}$$

The sums over l exclude just one index each, for example i, but allow j,k, and therefore we have:

$$\begin{aligned}
0 &= \sum_{k\neq i,j}\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\left(\sum_{l\neq i,j,k}a_{ijkl}+b_{ijk}\right) &\text{(A.8)}\\
a_{ijkl} &= \frac{[lkj]}{x_{il}^2}+\frac{[kjl]}{x_{jl}^2}+\frac{[jlk]}{x_{kl}^2} &\text{(A.9)}\\
b_{ijk} &= \frac{[jkj]+[kji]}{x_{ij}^2}+\frac{[kkj]+[jik]}{x_{ik}^2}+\frac{[kjk]+[jjk]}{x_{jk}^2} &\text{(A.10)}
\end{aligned}$$

We use the righthand side of (A.5) to express the $[abc]$ commutators in the (A.10) part of the equation:

$$b_{ijk} = 2\left(\frac{(ij)^2((jk)^2-(ik)^2)}{x_{ij}^2}+\frac{(ik)^2((ij)^2-(jk)^2)}{x_{ik}^2}+\frac{(jk)^2((ik)^2-(ij)^2}{x_{jk}^2}\right) \tag{A.11}$$

The terms which are proportional to $x_{ik}^{-4}, k = i+1$ are dominant in the colliding pair approximation. Here it is visible how the (4.27) condition arises from the equation for general $\bar{x} = (x_{i,1}, x_{i,2}, ..., x_{i,N})$. The (A.10) part will vanish for $(ab) = (e_a|e_b) = g(-1)^{n_{ab}}$. Now we have to check the conditions for the (A.9) part in this case (using the Jacobi identity $[jkl] = -[lkj] - [kjl]$):

$$\begin{aligned}
(e_a|e_b) &= g(-1)^{n_{ab}}, \quad \Phi_{iabc} = (-1)^{n_{ia}+n_{ab}+n_{bc}+n_{ci}}, &\text{(A.12)}\\
[abc] &= 2g^4(-1)^{n_{ia}+n_{bc}}((-1)^{n_{ab}+n_{ic}}-(-1)^{n_{ac}+n_{ib}}) = 2g^4(\Phi_{iabc}-\Phi_{iacb}), &\text{(A.13)}\\
0 &= 2\sum_{k\neq l\neq i,j}\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\left(\left(\frac{1}{x_{il}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{ilkj}-\Phi_{iljk})+\left(\frac{1}{x_{jl}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{ikjl}-\Phi_{iklj})\right).
\end{aligned}$$

In the next step we modifying the sum so that we have all k, l dependent terms grouped together:

$$\begin{aligned}
0 &= \sum_{k<l\neq i,j} \left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\left(\left(\frac{1}{x_{il}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{ilkj}-\Phi_{iljk})+\left(\frac{1}{x_{jl}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{ikjl}-\Phi_{iklj})\right)+ \\
&+ \left(\frac{1}{x_{il}^2}-\frac{1}{x_{jl}^2}\right)\left(\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{iklj}-\Phi_{ikjl})+\left(\frac{1}{x_{jk}^2}-\frac{1}{x_{kl}^2}\right)(\Phi_{iljk}-\Phi_{ilkj})\right)= \\
&= \sum_{k<l\neq i,j} A_{ijkl}\left[\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\left(\frac{1}{x_{il}^2}-\frac{1}{x_{kl}^2}\right)-\left(\frac{1}{x_{il}^2}-\frac{1}{x_{jl}^2}\right)\left(\frac{1}{x_{jk}^2}-\frac{1}{x_{kl}^2}\right)\right]+ \\
&+ B_{ijkl}\left[\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{jk}^2}\right)\left(\frac{1}{x_{jl}^2}-\frac{1}{x_{kl}^2}\right)-\left(\frac{1}{x_{il}^2}-\frac{1}{x_{jl}^2}\right)\left(\frac{1}{x_{ik}^2}-\frac{1}{x_{kl}^2}\right)\right],
\end{aligned}$$

where $A_{ijkl}=\Phi_{ilkj}-\Phi_{iljk}$ and $B_{ijkl}=\Phi_{ikjl}-\Phi_{iklj}$. If we parmetrise the expression with $N-1$ independent distances $x_{i1}, x_{i2}, ...x_{i,i-1}, x_{i,i+1}, ..., x_{ij}, ..x_{iN}$, we can write:

$$0=\sum_{k<l} A_{ijkl}F(x_{ij},x_{ik},x_{il})+B_{ijkl}G(x_{ij},x_{ik},x_{il}), \tag{A.14}$$

where

$$\begin{aligned}
F(x,y,z) &= f(y,y-x)f(z,y-z)-f(z,z-x)f(y-x,z-y), &\text{(A.15)}\\
G(x,y,z) &= f(y,y-x)f(z-x,y-z)-f(z,z-x)f(y,y-z), &\text{(A.16)}\\
f(a,b) &= \frac{1}{a^2}-\frac{1}{b^2}. &\text{(A.17)}
\end{aligned}$$

In case of $i=1, j=2$ and $N=4$ this sum has only one term corresponding to $k=3, l=4$ and it gives us the condition

$$AF(x_{12},x_{13},x_{14})+BG(x_{12},x_{13},x_{14})=0 \tag{A.18}$$

for all $(x_{12},x_{13},x_{14})\in\mathbb{R}^3$ such that $0<x_{12}<x_{13}<x_{14}$, where $A=A_{1234}, B=B_{1234}$. Choosing any two linearly independent configurations, for example $(1,2,3)$ and $(1,2,4)$ results in a system of equations satisfied solely by $A=B=0$. This means that $(e_1|[P_4,[P_3,P_2]]|e_1)=(e_1|[P_3,[P_4,P_2]]|e_1)=0$, and $n_{13}+n_{24}=(n_{14}+n_{23}) \bmod 2$. By choosing another (i,j) pair we obtain the equality of $n_{12}+n_{34}=n_{13}+n_{24}=(n_{14}+n_{23}) \bmod 2$. To extend this result to $N>4$ without dealing with multiple terms,we notice that for four fixed values of i,j,m,n only one term in (A.14) depends on both x_{im} and x_{in}. We can therefore keep the distances x_{ij}, x_{im} and x_{in} finite, and take all the others to infinity. This leaves us with

$$A_{ijmn}F(x_{ij},x_{im},x_{in})+B_{ijmn}G(x_{ij},x_{im},x_{in})=0 \tag{A.19}$$

and the same argument as for $N=4$ leads to the conclusion that $A_{ijmn}=B_{ijmn}=0$ for any quadruple of indices. This means (4.36) is satisfied.

Appendix B

Details of proofs in 5.2

B.1 Derivation of the formulae (5.77)-(5.83)

Let us start with the entries of the Jacobian matrix (5.77):

$$\begin{aligned}
\frac{\partial X_{ij}}{D_k} &= \frac{\partial}{\partial D_k}\left(U^\dagger D U\right)_{ij} = \frac{\partial}{\partial D_k}\left(\sum_{m=1}^{N} U^\dagger_{im} D_m U_{mj}\right) = U^*_{ki} U_{kj} \\
\frac{\partial X_{ij}}{a_l} &= \frac{\partial}{\partial a_l}\left(U^\dagger(\bar{a}) D U(\bar{a})\right)_{ij} = \left(\partial_{a_l} U^\dagger D U + U^\dagger D \partial_{a_l} U\right)_{ij} = \left(U^\dagger D \partial_{a_l} U - U^\dagger (\partial_{a_l} U) U^\dagger D U\right)_{ij} = \\
&= \left[U^\dagger \left(D(\partial_{a_l} U)U^\dagger - (\partial_{a_l} U)U^\dagger D\right) U\right]_{ij} = \left(U^\dagger \left[D, (\partial_{a_l} U)U^\dagger\right] U\right)_{ij} = \left(U^\dagger \Omega_l U\right)_{ij}
\end{aligned}$$

Next, we derive the expression for $(\partial_{a_l} U)U^\dagger$ stated in (5.78):

$$\begin{aligned}
(\partial_{a_l} U)U^\dagger &= -U\partial_{a_l} U^\dagger = -e^{\bar{a}\cdot\bar{\tau}}\partial_{a_l} e^{-\bar{a}\cdot\bar{\tau}} = -e^{\bar{a}\cdot\bar{\tau}} e^{-\bar{a}\cdot\bar{\tau}} \left(\frac{\mathbb{1} - e^{-(-ad_{\bar{a}\cdot\bar{\tau}})}}{-ad_{\bar{a}\cdot\bar{\tau}}}\right) \partial_{a_l}(-\bar{a}\cdot\bar{\tau}) = \\
&= \left(\frac{e^{ad_{\bar{a}\cdot\bar{\tau}}} - \mathbb{1}}{ad_{\bar{a}\cdot\bar{\tau}}}\right)\tau_l = \sum_{n=0}^{\infty} \frac{(ad_{\bar{a}\cdot\bar{\tau}})^n}{(n+1)!}\tau_l = \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \underbrace{[\bar{a}\cdot\bar{\tau}, [\bar{a}\cdot\bar{\tau}, ..., [\bar{a}\cdot\bar{\tau}, \tau_l]\,...]]}_{n} = \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} a_{k_n} f^{m_n}_{k_n, m_{n-1}} .... a_{k_2} f^{m_2}_{k_2, m_1} a_{k_1} f^{m_1}_{k_1, l} \tau_{m_n} = \\
&= \sum_{n=0}^{\infty} \frac{(a_k f_k)^n_{lm}}{(n+1)!}\tau_m = \sum_{n=0}^{\infty} \frac{(A^n)_{lm}}{(n+1)!}\tau_m = u(A)_{lm}\tau_m \in \mathfrak{g} \\
(\Omega_l)_{ij} &= u(A)_{lm}\left[D, \tau_m\right]_{ij} = u(A)_{lm}(D_i - D_j)(\tau_m)_{ij},
\end{aligned}$$

where we make use of a general formula found for instance in [56] for a smooth matrix valued function $X(t)$:

$$\frac{de^{X(t)}}{dt} = e^{X(t)}\left[\frac{\mathbb{1} - e^{-ad_{X(t)}}}{ad_{X(t)}}\left(\frac{dX(t)}{dt}\right)\right]. \tag{B.1}$$

Importantly, since we use anti-Hermitian generators, the structure constants $f^k_{ij} \in \mathbb{R}$, thus $u(A)$ is a real matrix in both unitary and orthogonal case. Now we calculate

the entries of the metric tensor (5.81):

$$g_{D_k,D_m} = \sum_{i,j} \frac{\partial X_{ij}}{D_k}\frac{\partial X_{ji}}{D_m} = \sum_{i,j} U^*_{ki}U_{kj}U^*_{mj}U_{mi} = \delta_{km} \tag{B.2}$$

$$g_{D_k,a_l} = \sum_{i,j} \frac{\partial X_{ij}}{D_k}\frac{\partial X_{ji}}{a_l} = \sum_{ij} U^*_{ki}U_{kj}\left(U^\dagger \Omega_l U\right)_{ji} = (\Omega_l)_{kk} = 0 \tag{B.3}$$

$$\begin{aligned} g_{a_l,a_m} &= \sum_{i,j} \frac{\partial X_{ij}}{a_l}\frac{\partial X_{ji}}{a_m} = \sum_{i,j}\left(U^\dagger\Omega_l U\right)_{ij}\left(U^\dagger\Omega_m U\right)_{ji} = \\ &= \mathrm{Tr}(\Omega_l\Omega_m) \end{aligned} \tag{B.4}$$

We notice that the nontrivial block $g_{lm} = Tr(\Omega_l\Omega_m)$ of the metric tensor is in fact a Gram matrix of $\Omega_{1,2,...,d}$ treated as vectors in the vector space of Hermitian matrices. This means that the determinant can be expressed with the exterior product of the vectors:

$$\det g = ||\Omega_1 \wedge \Omega_2 \wedge ... \wedge \Omega_d||^2, \tag{B.5}$$

but to make use of this structure in the proofs of (5.83)-(5.85), we need to switch to ordered pair indices, as in the main text, $(ij), (pq), (rs) \in I$, where $I = \{(pq) : 1 \leqslant p < q \leqslant N\}$ in the orthogonal case and $I = \{(pq) : 1 \leqslant p \neq q \leqslant N\}$ in the unitary case, and express the Ω vectors in an orthogonal basis:

$$\hat{e}_{ij} = \begin{cases} |i\rangle\langle j| + |j\rangle\langle i| = -i\tau_{ji}, & i < j \\ i\left(|i\rangle\langle j| + |j\rangle\langle i|\right) = \sigma_{ji}, & i > j \end{cases}, \qquad Tr(\hat{e}_{ij}\hat{e}_{kl}) = 2\delta_{ik}\delta_{jl}. \tag{B.6}$$

The expressions for Ω_{ij} and $g_{(ij)(kl)}$ are as follows:

$$\begin{aligned} \Omega_{ij} &= \sum_{(pq)\in I} u_{(ij)(pq)}\,[D, \tau_{pq}] = \sum_{(pq)\in I} u_{(ij)(pq)}(D_p - D_q)\hat{e}_{pq} \\ g_{(ij)(kl)} &= Tr(\Omega_{ij}\Omega_{kl}) = \sum_{(pq)\in I}\sum_{(rs)\in I} u_{(ij)(pq)}(D_p - D_q)u_{(kl)(rs)}(D_r - D_s)Tr(\hat{e}_{pq}\hat{e}_{rs}) = \\ &= 2\sum_{(pq)\in I}\sum_{(rs)\in I} u_{(ij)(pq)}(D_p - D_q)u_{(kl)(rs)}(D_r - D_s)\delta_{pr}\delta_{qs} = \\ &= 2\sum_{(pq)\in I} u_{(ij)(pq)}u_{(kl)(pq)}(D_p - D_q)^2 = \sum_{(pq)\in I} u_{(ij)(pq)}(D_p - D_q)^2(u^T)_{(pq)(kl)} = \\ &= 2(u\mathbf{D}^2u^T)_{(ij)(kl)}. \end{aligned}$$

The factorised form of g given by (5.83) leads automatically to (5.84),(5.85) and the expressions for Δ_a and $\hat{\Lambda}_{ij}$.

B.2 Derivation of (5.87)

The action of the similarity transformation (5.87) can be demonstrated with the use of a $\mathcal{C}^2$ class test function $f = f(D_1, D_2, ..., D_N)$, and the fact that:

$$\frac{\partial \mathcal{D}}{\partial D_i} = \frac{\partial}{\partial D_i}\left(\prod_{k<l}(D_k - D_l)^\alpha\right) = \alpha\mathcal{D}\left(\sum_{k\neq i}\frac{1}{D_i - D_k}\right). \tag{B.7}$$

The calculations are as follows:

$$\begin{aligned}
\sqrt{\mathcal{D}}\Delta_D\left(\frac{f}{\sqrt{\mathcal{D}}}\right) &= \frac{1}{\sqrt{\mathcal{D}}}\sum_{i=1}^{N}\frac{\partial}{\partial D_i}\left[\mathcal{D}\frac{\partial}{\partial D_i}\left(\frac{f}{\sqrt{\mathcal{D}}}\right)\right] = \sum_{i=1}^{N}\left\{\frac{\partial^2 f}{\partial D_i^2} + a_i(D)\left(\frac{\partial f}{\partial D_i}\right)\right\} + b(D)f \\
\sqrt{\mathcal{D}}a_i(D) &= \frac{\partial\sqrt{\mathcal{D}}}{\partial D_i} + \mathcal{D}\frac{\partial}{\partial D_i}\left(\frac{1}{\sqrt{\mathcal{D}}}\right) = \frac{1}{2\sqrt{\mathcal{D}}}\frac{\partial\mathcal{D}}{\partial D_i} - \mathcal{D}\frac{1}{2\sqrt{\mathcal{D}}^3}\frac{\partial\mathcal{D}}{\partial D_i} = 0 \\
b(D) &= \sum_{i=1}^{N}\frac{1}{\sqrt{\mathcal{D}}}\frac{\partial}{\partial D_i}\left[\mathcal{D}\frac{\partial}{\partial D_i}\left(\frac{1}{\sqrt{\mathcal{D}}}\right)\right] = \sum_{i=1}^{N}\frac{1}{\sqrt{\mathcal{D}}}\frac{\partial}{\partial D_i}\left[-\mathcal{D}\left(\frac{1}{2\sqrt{\mathcal{D}}^3}\frac{\partial\mathcal{D}}{\partial D_i}\right)\right] = \\
&= -\frac{\alpha}{2\sqrt{\mathcal{D}}}\sum_{i=1}^{N}\frac{\partial}{\partial D_i}\left(\sqrt{\mathcal{D}}\sum_{k\neq i}\frac{1}{D_i - D_k}\right) = \\
&= -\frac{\alpha}{2\sqrt{\mathcal{D}}}\sum_{i=1}^{N}\left[\frac{1}{2\sqrt{\mathcal{D}}}\frac{\partial\mathcal{D}}{\partial D_i}\sum_{k\neq i}\frac{1}{D_i - D_k} - \sqrt{\mathcal{D}}\sum_{k\neq i}\frac{1}{(D_i - D_k)^2}\right] = \\
&= \frac{\alpha}{2}\sum_{1\leqslant i\neq j\leqslant N}\frac{1}{(D_i - D_j)^2} - \frac{\alpha^2}{4}\left[\sum_{1\leqslant i\neq j\leqslant N}\frac{1}{(D_i - D_j)}\right]^2 = \\
&= \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)\sum_{1\leqslant i\neq j\leqslant N}\frac{1}{(D_i - D_j)^2} - \frac{\alpha^2}{4}\sum_{i<j<k}\left[\frac{D_k - D_j + D_i - D_k + D_j - D_i}{(D_i - D_j)(D_j - D_k)(D_k - D_i)}\right] \\
&= \frac{\alpha\,(2-\alpha)}{2}\sum_{1\leqslant i\leqslant j\leqslant N}\frac{1}{(D_i - D_j)^2} = \frac{1}{2}\sum_{(ij)}\frac{2-\alpha}{(D_i - D_j)^2}
\end{aligned}$$

B.3 Derivation of (5.95) and (5.96)

Let M be a $d\times d$ matrix of real functions: $M_{ij}:\mathbb{R}^d\to\mathbb{R}$. At every point $(x_1,..,x_d)\in\mathbb{R}^d$ where M is invertible, we may say that:

$$M^{-1} = \frac{C(M)^T}{\det M}$$

where $C(M)$ is the matrix of cofactors of M. This means that:

$$\begin{aligned}
\det M\delta_{ij} &= \sum_{l=1}^{d} M_{il}C(M)_{jl} \\
\partial_i(\det M) &= \sum_{j=1}^{d}\partial_j(\det M)\delta_{ij} = \sum_{j,l=1}^{d}\partial_j(M_{il}C(M)_{jl}) \\
&= \sum_{j,l=1}^{d}\partial_j(M_{il})C(M)_{jl} + M_{il}\partial_j(C(M)_{jl}).
\end{aligned}$$

On the other hand:

$$\partial_i(\det M) = \sum_{j,l=1}^{d}\frac{\partial\det M}{\partial M_{jl}}\frac{\partial M_{jl}}{\partial x_i} = \sum_{j,l=1}^{d}C(M)_{jl}\partial_i(M_{jl}), \tag{B.8}$$

which means that

$$\sum_{l=1}^{d}M_{il}\sum_{j=1}^{d}\partial_j(C(M)_{jl}) = \sum_{j,l=1}^{d}C(M)_{jl}(\partial_i M_{jl} - \partial_j M_{il}). \tag{B.9}$$

If M^T happens to be a Jacobian matrix of some map $m : \mathbb{R}^d \to \mathbb{R}^d$, the right-hand side of the above equation vanishes identically:

$$\partial_i M_{jl} - \partial_j M_{il} = \partial^2_{ij} m_l - \partial^2_{ji} m_l = 0. \tag{B.10}$$

For the left-hand side to vanish, the sum $F_l = \sum_{j=1}^{d} \partial_j (C(M)_{jl})$ must vanish identically as well. This is so, because wherever $\det M \neq 0$, $M_i = (M_{i1}, M_{i2}, ..., M_{id})$ can be treated as d linearly independent vectors and for the scalar product $(M_i, F) = \sum_l M_{il} F_l$ to vanish for all of them $F = 0$ identically. This proof can be found for example in [81].

Let us now translate the equation (B.9) to the language of the u matrix and a_{ij} variables. The indices change in the following way: $j \to pq$, $l \to ij$, $i \to ab$:

$$(\det u) \sum_{(ij)\in I} u_{(ab)(ij)} F_{ij} = \sum_{(ij),(pq)\in I} C(u)_{(pq)(ij)} \left(\frac{\partial u_{(pq)(ij)}}{\partial a_{ab}} - \frac{\partial u_{(ab)(ij)}}{\partial a_{pq}} \right). \tag{B.11}$$

The matrix elements of u can be derived from the definition:

$$\frac{\partial U}{\partial a_{pq}} U^\dagger = \sum_{(ij)\in I} u_{(pq)(ij)} \tau_{ij} \implies u_{(pq)(ij)} = -\frac{1}{2} \mathrm{Tr} \left[\frac{\partial U}{\partial a_{pq}} U^\dagger \tau_{ij} \right] \tag{B.12}$$

and its derivatives have the following form:

$$\begin{aligned}
\frac{\partial u_{(rs)(mn)}}{\partial a_{pq}} &= -\frac{1}{2} \mathrm{Tr} \left[\left(\frac{\partial^2 U}{\partial a_{pq} \partial a_{rs}} \right) U^\dagger \tau_{mn} \right] - \frac{1}{2} \mathrm{Tr} \left(\frac{\partial U}{\partial a_{rs}} \frac{\partial U^\dagger}{\partial a_{pq}} \tau_{mn} \right) = \\
&= -\frac{1}{2} \mathrm{Tr} \left[\left(\frac{\partial^2 U}{\partial a_{pq} \partial a_{rs}} \right) U^\dagger \tau_{mn} \right] - \frac{1}{2} \mathrm{Tr} \left(\frac{\partial U}{\partial a_{rs}} U^\dagger U \frac{\partial U^\dagger}{\partial a_{pq}} \tau_{mn} \right) = \\
&= -\frac{1}{2} \mathrm{Tr} \left[\left(\frac{\partial^2 U}{\partial a_{pq} \partial a_{rs}} \right) U^\dagger \tau_{mn} \right] + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial U}{\partial a_{rs}} U^\dagger \frac{\partial U}{\partial a_{pq}} U^\dagger \tau_{mn} \right) = \\
&= -\frac{1}{2} \mathrm{Tr} \left[\left(\frac{\partial^2 U}{\partial a_{pq} \partial a_{rs}} \right) U^\dagger \tau_{mn} \right] + \qquad (B.13) \\
&\quad + \frac{1}{2} \sum_{(ab),(cd)\in I} u_{(rs)(ab)} u_{(pq)(cd)} \mathrm{Tr}(\tau_{ab} \tau_{cd} \tau_{mn}) \qquad (B.14)
\end{aligned}$$

where we again use the definition of u and the fact that $(\partial U) U^\dagger = -U \partial U^\dagger$. The symmetric parts cancel in the difference of derivatives:

$$\begin{aligned}
\frac{\partial u_{(pq)(ij)}}{\partial a_{ab}} - \frac{\partial u_{(ab)(ij)}}{\partial a_{pq}} &= \sum_{(mn),(rs)\in I} \frac{1}{2} u_{(pq)(mn)} u_{(ab)(rs)} \mathrm{Tr}([\tau_{mn}, \tau_{rs}] \tau_{ij}) = \\
&= -\sum_{(mn),(rs)\in I} u_{(pq)(mn)} u_{(ab)(rs)} f^{(ij)}_{(mn)(rs)} = \\
&= (u f_{ij} u^T)_{(ab)(pq)},
\end{aligned}$$

This difference does not vanish identically, which means u^T is not a Jacobian matrix of any map, nevertheless we may apply it to the right-hand side of (B.11):

$$\begin{aligned}
\sum_{(ij),(pq)\in I} (u f_{ij} u^T)_{(ab)(pq)} C(u)_{(pq)(ij)} &= \sum_{(ij),(pq)\in I} (u f_{ij} u^T)_{(ab)(pq)} (u^T)^{-1}_{(pq)(ij)} \det u = \\
&= \det u \sum_{(ij)\in I} (u f_{ij})_{(ab)(ij)} = 0.
\end{aligned}$$

It turns out to be 0 due to the fact that the structure constants are antisymmetric in every pair of indices and $f^{(ij)}_{(ij)(kl)} = 0$. Using the same arguments as for the general

matrix M I conclude that for the left-hand side of (B.11) to vanish, the functions F_{ij} must vanish as well.

The commutator of $\hat{\lambda}$ operators can be calculated in the following way:

$$\begin{aligned}
\left[\hat{\lambda}_{ij}, \hat{\lambda}_{kl}\right] &= \sum_{(pq),(rs)\in I} \left[u^{-1}_{(ij)(pq)}\partial_{pq}, u^{-1}_{(kl)(rs)}\partial_{rs}\right] = \sum_{(rs)\in I} \mu^{(rs)}_{(ij)(kl)}\partial_{rs} = \\
&= \sum_{(mn)} \nu^{(mn)}_{(ij)(kl)}\hat{\lambda}_{mn} \\
\mu^{(rs)}_{(ij)(kl)} &= \sum_{(pq)\in I} u^{-1}_{(ij)(pq)}(\partial_{pq}u^{-1}_{(kl)(rs)}) - u^{-1}_{(kl)(pq)}(\partial_{pq}u^{-1}_{(ij)(rs)}) \\
\sum_{(rs)} \mu^{(rs)}_{(ij)(kl)}\partial_{rs} &= \sum_{(rs)(mn),(uv)} \mu^{(rs)}_{(ij)(kl)}u_{(rs)(mn)}u^{-1}_{(mn)(uv)}\partial_{uv} = \sum_{(mn)} \nu^{(mn)}_{(ij)(kl)}\hat{\lambda}_{mn} \\
\nu^{(mn)}_{(ij)(kl)} &= \sum_{(rs)\in I} \mu^{(rs)}_{(ij)(kl)}u_{(rs)(mn)}
\end{aligned}$$

where ∂_{pq} stands for $\frac{\partial}{\partial a_{pq}}$ and for the final calculation of ν we use the formulae (B.13) and (B.14):

$$\begin{aligned}
\nu^{(mn)}_{(ij)(kl)} &= \sum_{(rs)(pq)} u^{-1}_{(ij)(pq)}(\partial_{pq}u^{-1}_{(kl)(rs)}) - u^{-1}_{(kl)(pq)}(\partial_{pq}u^{-1}_{(ij)(rs)})u_{(rs)(mn)} = \\
&= \sum_{(rs)(pq)} u^{-1}_{(ij)(pq)}\left[\partial_{pq}(u^{-1}_{(kl)(rs)}u_{(rs)(mn)}) - u^{-1}_{(kl)(rs)}\partial_{pq}u_{(rs)(mn)}\right] + \\
&- \sum_{(rs)(pq)} u^{-1}_{(kl)(pq)}\left[\partial_{pq}(u^{-1}_{(ij)(rs)}u_{(rs)(mn)}) - u^{-1}_{(ij)(rs)}\partial_{pq}u_{(rs)(mn)}\right] = \\
&= \sum_{(pq),(rs)} \left(u^{-1}_{(kl)(pq)}u^{-1}_{(ij)(rs)} - u^{-1}_{(ij)(pq)}u^{-1}_{(kl)(rs)}\right)\partial_{pq}u_{(rs)(mn)} = \\
&= \frac{1}{2}\sum_{(pq),(rs)} \left(u^{-1}_{(ij)(pq)}u^{-1}_{(kl)(rs)} - u^{-1}_{(kl)(pq)}u^{-1}_{(ij)(rs)}\right)\mathrm{Tr}\left[(\partial^2_{pq,rs}U)U^\dagger\tau_{mn}\right] + \\
&+ \frac{1}{2}\sum_{(pq)(rs)}\sum_{(ab)(cd)} u^{-1}_{(kl)(pq)}u^{-1}_{(ij)(rs)}u_{(rs)(ab)}u_{(pq)(cd)}\mathrm{Tr}(\tau_{ab}\tau_{cd}\tau_{mn}) + \\
&- \frac{1}{2}\sum_{(pq)(rs)}\sum_{(ab)(cd)} u^{-1}_{(ij)(pq)}u^{-1}_{(kl)(rs)}u_{(rs)(ab)}u_{(pq)(cd)}\mathrm{Tr}(\tau_{ab}\tau_{cd}\tau_{mn}) = \\
&= \frac{1}{2}\sum_{(ab)(cd)} (\delta_{(ij)(ab)}\delta_{(kl)(cd)} - \delta_{(kl)(ab)}\delta_{(ij)(cd)})\mathrm{Tr}(\tau_{ab}\tau_{cd}\tau_{mn}) = \\
&= \frac{1}{2}\mathrm{Tr}([\tau_{ij},\tau_{kl}]\tau_{mn}) = -f^{(mn)}_{(ij)(kl)}
\end{aligned}$$

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