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Theoretical and numerical analysis of novel COVID-19 via fractional order mathematical model

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ABSTRACT

In the work, author's presents a very significant and important issues related to the health of mankind's. Which is extremely important to realize the complex dynamic of inflected disease. With the help of Caputo fractional derivative, We capture the epidemiological system for the transmission of Novel Coronavirus-19 Infectious Disease (nCOVID-19). We constructed the model in four compartments susceptible, exposed, infected and recovered. We obtained the conditions for existence and Ulam's type stability for proposed system by using the tools of non-linear analysis. The author's thoroughly discussed the local and global asymptotical stabilities of underling model upon the disease free, endemic equilibrium and reproductive number. We used the techniques of Laplace Adomian decomposition method for the approximate solution of consider system. Furthermore, author's interpret the dynamics of proposed system graphically via Mathematica, from which we observed that disease can be either controlled to a large extent or eliminate, if transmission rate is reduced and increase the rate of treatment.

Introduction

Human society, equipped with modern technology is presently faced with a terrible virus known as the nCOVID-19. It always affected the developing countries rather than the developed ones. This virus was detected in peoples for first time in the city of Wuhan, China [1]. Although, the origin of the virus is still a mystery, certain theories has already been formulated in this regard. According to some of these theories, the virus was to animated form animals to humans, when certain cases of patients affected with same virus were reported from local fish market in Wuhan [2]. Soon after it was found that the virus could transmit from one person to another [3]. According to the statistics provided by World Health Organization (WHO) (15th September, 2020), till the composition of this article the virus had already affected 210 countries and traceries around the globe. The conformed cases till now crosses the figures of 39,388,169 out of this alarming 1,105,895 deaths, were reported to WHO [4]. However, the number may exceed the given figure. This virus badly effected USA, Italy and Spain, where the death ratio has been higher than other place. This testifying to the severity of the nCOVID-19. The common symptoms of the infection include respiratory issues, high fever and sever coughing. which shows severity of nCOVID-19. Besides, other symptoms such as gastroenteritis and neurological sickness of contradictory strictness have also been noted [5].

The major cause of infection are the droplets from the mouth or nose of the affected person at the time of speaking or sneezing. The people around the effected person are liable to contract the disease. As preventive majors, almost every major country of the world adopted the

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lock-down policy to ensure the safety of their citizens. In such situations doctors and paramedics staff have committed themselves to provide health services to the effected peoples. Researchers of related fields thinks that root cause of nCovid-2019, was first resulted due the bats, which are identical to that of SARS (Severe Acute Respiratory Syndromes), which took birth in China and the rest of the world in 2003 [6,7]. After that some experts compared the current virus with MERS and SARS to identify family of the virus to which it belongs so as to handle the current virus with the help of the studies done to deal with SARS and MERS in the past. Lu argued that the current nCovid-2019 relates to Beta-corona virus genus, like SARS-Cov and MERS-Cov [8]. For further study related to the corona virus, we recommend [9–13].

The accurate mathematical tools that describes either the real world phenomena, to investigate the dynamical behaviors of biological disease or to specifically deal with an engineering problem, a powerful mathematical techniques which manufactures more consistent results is recognized as mathematical modeling. In this connection, various mathematical techniques are used to study the communication and has been developed an enhanced plan for the avoidance of humans from these noxious infectious diseases, (see [14–17]). One of the remarkable challenges in the society for humans that has been observed is proper implementation and understanding of the controlling strategies against the transmission of spreading diseases. In this regards, up-to large extent the concerned techniques play a vital role to eliminate, prevent against the deadly diseases from the community, (see [18,19]). The researchers present mathematical perspectives of nCovid-19 infection model. The proposed epidemic model and many more biological models are actually expressed in system of mathematical concepts with natural order derivatives. Recently, it was observed that mathematical model with fractional order derivatives are mostly used to describe the universal laws, due to its great degree of reliability and accuracy, the researchers are focusing on fractional order differential equations (FODES). The concept of FODES was successfully implemented in various fields such as control theory, patter organization, image and signal processing, and finance and economics, see [20–22]. Keeping these applications in mind, the researchers took keen interest in fractional differential equations as compared to natural order. Natural order derivatives are local in nature and cannot interpret between two integer values. In contrast to the natural derivatives, the proposed derivatives are non-local, possess memory effects and hereditary properties, which make the proposed derivatives more superior to natural order derivatives. To overcome the aforementioned difficulties of natural order derivatives, the researchers are interested to utilize the fractional order derivatives. Another application of the proposed derivatives is that the situation where the state of problem depends on previous stages rather than current stage of the problem, the researchers used fractional derivatives, because it possesses the memory effects. In existing literature, varieties of fractional orders differential operators are present. The applications of these differential operators are available in different articles, (see [23,24]). Keeping these applications in view, the researchers studied different aspects of the real situated problems such as qualitative analysis, optimization, and approximate solution of different models, (see[25–27]). One of the important types of fractional operator is known as Caputo fractional differential operator. Due to its applications and geometrical interpretation, the concerned operator was vastly used by the researchers in different articles, (see [28–31]).

In modern era, the number of experimental facts demonstrate that natural dynamics follow fractional calculus. The concerned field is indeed fastest growing area of research, due to its wide range of applications in numerous fields of engineering and science, (see [32–41]). Furthermore, one of the biggest challenge facing by researchers is the evident that trade with the dynamical system having memory effects. Since the concerned filed has direct linkage with a such system with memory effect. Therefore, researchers introduced the novel techniques for modeling of these phenomena via FEDs to handles such a problems, (see [42–45]). The field of FDEs attendant the attention of researchers, because it posses some extraordinary properties as compared to traditional class of DEs, such as greater degree of freedom, heredity and globally in nature. Due to such remarkable behaviors of concerned class of DEs, the researchers are intersected to investigate different aspects of concerned theory, like existence and stability analysis, numerical approximation etc. They obtained the desired results, by using the various techniques of analysis, fixed point theory and numerical analysis, we refer [46–48] for detail study. Another significant aspect of proposed theory is stability analysis and numerical approximation. In some dynamical or biological problems, we have too much complicated situation to obtain exact analytic solutions. In such circumstance, stability analysis and numerical approximation play vital role to obtained the desired solutions for FDEs. Although, there are various types of stabilities in the existence literature, (see [49–52]). To best of our knowledged one the most probably and consistence type of stability is Ullam-Hyers (UH) stability, which was further modified to Ullam-Hyer's Rassias (UHR) and Generalized Ullam-Hyer's (GUH) stability, [52–56].

One of the important aspect of FDEs has approximate solutions. In order to obtained the numerical solutions for FDEs, there are verities of numerical techniques for solutions of FDEs, (see [57–60]). Probably, the most efficient and reliable technique is Laplace Transform (LT) coupled with Adomian Decomposition Method (ADM) as so called LADM. LADM is a powerful technique used to solve the initial value problems (IVPs), for FDEs as well as FPDEs. With the help of Laplace transform, first we convert the proposed FDEs into algebraic equations, which are known as subsidiary equations. The solution of subsidiary equation is obtained via usual algebraic techniques and the subsidiary solution is converted back to get the desired solution. In the concerned technique, we directly obtain the particular solution of FDEs, with out finding the general solution of proposed problem. While in case of non–homogeneous FDEs, we can find the required result with out first finding solution of the corresponding homogeneous FDEs. Beside this, Laplace transform is crucial in the study of control system, and hence in the study of HVAC (heating, ventilation and air conditioning) control system. It is used to solve the problems related to communication and network analysis and many more.

In the is work, authors report the comparative results of the evolution of COVID-19 outbreak in Pakistan. In addition, we will provide practical analysis for the time and magnitude of epidemic peak i.e, maximum number of infected individuals, also measure the effect of drastic containment measure based on simple quantitative model. We collect data form WHO (World Health Organization). According to WHO report at 16th of July, the total number of confirmed cases in Pakistan are about 2,57,914, while 2145 new cases were recorded in last 24 h and 5,426 people lost their lives. This virus is controlled up-to some extent, but if people care then they could be safe. In Pakistan large number of people recovered but more and more peoples are infected too. In the case of Iran, 2,64,561 cases are confirmed, while 13,410 deaths were counted in Iran. In Iran people get recovered in large number as the people became aware. 968.876 are the total confirmed cases in India while 28,498 new cases were counted. 24,915 people lost lives also 553 new lost were recorded in last 24 h. In case of India more people died and new cases were recorded that are higher than that of Pakistan and Iran. This means that in India peoples are infected in high rate than that of Pakistan and Iran. This information are collected from WHO at 10:00 CEST, July 16th, 2020 [63]. In the report, dated July 16, Pakistan has estimated 13 corona virus associated deaths per million of its inhabitant. Pakistan suffer the second-highest death rate from the novel corona virus in comparison to its neighboring countries, the highest mortality figures were recorded from Iran, which is about 108 deaths per million of its inhabitants, while India ranked on third position. If infected rate is high, some people also get recovered and unfortunately some die. Using the most simple model for the epidemic outbreak, the population is divided into four different classes, which are S (susceptible), E (exposed), I (infected) and R (recovered) scheme SEIR. Let initial population for susceptible population is S_0. The SEIR epidemic model in sense of fractional derivative is described in Eq. (1) along with initial conditions.

Furthermore, in this regard we have tried to inspect the capture nCOVID-19 infection system for existence and numerical computation. The concerned epidemic model and many other biological model are also expressed in Mathematical language with natural order derivatives. To describe the universal law, mathematical model with fractional derivative are mostly used by researchers due to reliability and accuracy. In this paper, the author's used the tools of fixed point theory to developed analytical techniques for the existence, uniqueness of the solution and stability analysis under Caputo fractional order derivative and numerical solution are obtain via LADM for underlying nCOVID-19 model given by

$$\begin{cases} {}_0^C D_t^\eta S(t) = a - \beta S(t)I(t) - (\mu + d_1)S(t), \\ {}_0^C D_t^\eta E(t) = \beta S(t)I(t) - (\mu + d_2 + \alpha + \gamma_1)E(t), \\ {}_0^C D_t^\eta I(t) = \alpha E(t) - (\mu + \gamma_2 + d_3)I(t), \\ {}_0^C D_t^\eta R(t) = \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t), \end{cases} \tag{1}$$

subject to initial condition,

$$S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ R(0) = R_0. \tag{2}$$

The proposed model is especially consider for the ongoing pandemic in the whole world started from China and reached to almost every part on the globe. This scheme has better chances to at-least the gross future of the full time course of outbreak, although the mentioned model is little crude. Parameters which are used in the model are captured in the following table Table 1,2.

The author's presents the existence results corresponding to model (1), via tools of fixed point theory. We utilize Schaefer's and Banach theorems in order to obtain the desired results for existence theory of proposed system. We discussed the positive equilibrium points i,e disease free, endemic equilibrium points and reproductive number for proposed system. With help of reproductive number, we developed the conditions for local asymptotically stability for both disease free and endemic equilibrium. We provide the numerical approximation of system (1) for arbitrary order via LADM. We use the real data of Pakistan to established the results for approximate solution of nCOVID-19 outbreak. For justification of the desired results obtained via LADM, we use Mathemaica and assigned different values to the concerned parameters involved and supplement conditions.

Preliminaries

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Definition 0.1. *[42,43,61]* "Caputo fractional order derivative of a function ϕ on the inter val $[0,\infty)$, is defined as

$${}^C D_{0+}^\eta \phi(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-s)^{n-\eta-1} \phi^n(s) ds,$$

where $n = [\eta] + 1$ and $[\eta]$ represent the integral part of η."

Table 1
Parameters used in model (1).

Parameters	Description
$S(t)$	Susceptible Human Individuals
$E(t)$	Exposed Human Individuals
$I(t)$	Infectious Human Individuals
$R(t)$	Recovered Human Individuals
a	Birth rate
β	Transmission rate from Susceptible to Exposed Human Individuals
α	Transmission rate from Susceptible to Infected Human Individuals
γ_1	Recovery rate of Exposed Human Individuals
γ_2	Recovery rate of Infected Human Individuals
μ	Natural Death Rate
d_1	Death rate of Susceptible Human Individuals due to infection
d_2	Death rate of Exposed Human Individuals due to infection
d_3	Death rate of Infected Human Individuals due to infection

Table 2
Description of parameters used in model (1).

Parameters	Pakistan (Descriptions)
S_0	(8065518) Susceptible Human Individuals
E_0	(200000) Exposed Human Individuals
I_0	(28234) Infectious Human Individuals
R_0	(0) Recovered Human Individuals
a	(0.0018) Birth rate
β	(0.03) Transmission rate from Susceptible to Exposed Human Individuals
α	(0.25) Transmission rate from Susceptible to Infected Human Individuals
γ_1	(0.28) Recovery rate of Exposed Human Individuals
γ_2	(0.12) Recovery rate of Infected Human Individuals
μ	(0.01) Natural Death Rate
d_1	(0.26) Death rate of Susceptible Human Individuals due to infection
d_2	(0.21) Death rate of Exposed Human Individuals due to infection
d_3	(0.35) Death rate of Infected Human Individuals due to infection

Lemma 0.1. *For fractional differential equations, the following result holds*

$$I^\eta\left[{}^C D_{0+}^\eta \phi(t)\right] = \phi(t) + a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1},$$

for arbitrary $a_i \in R$, i = 0,1,2,3,…,n-1, where $n = [\eta] + 1$ and $[\eta]$ represent the integral part of η.

Lemma 0.2. *[42,43,61] "Let $g(\theta) \in C([0,T])$, then the solution of fractional differential equation*

$$\begin{cases} {}^C D_0^\eta \psi(\theta) = g(\theta),\ \theta \in [0,T],\ n-1 < \theta < n, \\ \psi(0) = g_0, \end{cases}$$

is given by"

$$\psi(\theta) = \sum_{i=0}^{n-1} N_i \theta^i + \frac{1}{\Gamma(\eta)} \int_0^\theta (\theta - s)^{\eta-1} g(s) ds.$$

For $N_i \in R$, i = 0, 1, 2, 3,…, n-1.

Theorem 0.2. *[61] "Let X be a Banach space and $\mathfrak{T} : X \to X$ is compact and continuous, if the set*

$$E = \{\psi \in X : \psi = m\mathfrak{T}\psi,\ m \in (0,1)\},$$

is bounded, then $\mathfrak{T}$ has a unique fixed point".

Definition 0.3. Laplace transform of Caputo derivative is given by

$$\mathfrak{L}\left[{}^C D_{0+}^\eta \psi(t)\right] = s^\eta \psi(s) - \sum_{m=0}^{n-1} s^{\eta-m-1} \psi^m(0),\ n-1 < \eta < n,\ n \in N.$$

where $n = [\eta] + 1$ and $[\eta]$ represent the integral part of η.

Qualitative analysis

In this section, of research work the author's is provide the qualitative theory for proposed system (1), via the tools of fixed point theory. The concerned theory for existence ensure that capture model for any dynamics of biological phenomena is well posed. For this purpose the researchers used different tools to investigate these systems. Banach contraction theorem is one of the tool to investigated the fixed point for proposed problem. This theorem provides information about the existence and uniqueness of the solution for the considered model.

For the considered model, we construct the function define by

$$\begin{cases} \Theta_1(t,S,E,I,R) = a - \beta S(t)I(t) - (\mu + d_1)S(t), \\ \Theta_2(t,S,E,I,R) = \beta S(t)I(t) - (\mu + d_2 + \alpha + \gamma_1)E(t), \\ \Theta_3(t,S,E,I,R) = \alpha E(t) - (\mu + \gamma_2 + d_3 I(t), \\ \Theta_4(t,S,E,I,R) = \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t), \end{cases} \quad (3)$$

Assume that $\mathbb{B}C[0,T]$ be Banach space, with norm defined by

$$||\psi(t)|| = Sup_{\mathcal{T}\in[0,T]}[|S(t)| + |E(t)| + |I(t)| + |R(t)|],$$

where

$$\psi\left(t\right) = \begin{cases} S(t), \\ E(t), \\ I(t), \\ R(t), \end{cases} \quad \psi_0\left(t\right) = \begin{cases} S_0, \\ E_0, \\ I_0, \\ R_0, \end{cases} \quad \Phi\left(t,\psi\left(t\right)\right) = \begin{cases} \Theta_1(t,S,E,I,R), \\ \Theta_2(t,S,E,I,R), \\ \Theta_3(t,S,E,I,R), \\ \Theta_4(t,S,E,I,R) \end{cases}. \quad (4)$$

With the help of (4), the given system (1) can be expressed in the form of

$$\begin{aligned} &{}^C D^\eta \psi(t) = \Phi(t,\psi(t)), \quad t \in [0,T], \\ &\psi(0) = \psi_0, \end{aligned} \quad (5)$$

Eq. (5), in the light of (0.1), can be expressed as

$$\psi\left(t\right) = \psi_0 + \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right)ds, \quad t \in \left[0,T\right]. \quad (6)$$

Furthermore, we assume that assumptions given below holds for further consequences of proposed problem.(P1) There exists constants K_*, M_* such that

$$|\Phi(t,\psi(t))| \leqslant K_*|\psi|^q + M_*.$$

(P2) There exists constant $L_* > 0$, such that for each $\psi, \overline{\psi}$

$$|\Phi(t,\psi) - \Phi(t,\overline{\psi})| \leqslant L_*||\psi - \overline{\psi}||.$$

Further we define operator $\mathfrak{T} : \mathbb{B} \to \mathbb{B}$ as

$$\mathfrak{T}\psi\left(t\right) = \psi_0 + \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right)ds. \quad (7)$$

Theorem 0.4. *Under the assumption (P_1) and (P_2), the problem (5), has at least one fixed point. Which means that our considered problem has at leat one solution.*

Proof. We will use Schaefer's fixed point theorem, for this we proceed in the following steps.

Step I: First of all, we have to show that $\mathfrak{T}$ is continuous. We consider that Ψ_j is continuous for j = 1,2,3,4. Thus $\Phi(s,\psi(s))$ is continuous. Let ψ_n. $\psi \in X$ such that $\psi_n \to \psi$, we must have $\mathfrak{T}\psi_n \to \mathfrak{T}\psi$.

For his consider

$$\begin{aligned} &||\mathfrak{T}\psi_n - \mathfrak{T}\psi|| = max_{t\in[0,T]}|\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Phi_n\left(s,\psi_n\left(s\right)\right) \\ &ds - \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right)ds| \leqslant max_{t\in[0,T]}\int_0^t |\frac{(t-s)^{\eta-1}}{\Gamma(\eta)}||\Phi_n\left(s,\psi_n\left(s\right)\right) \\ &-\Phi\left(s,\psi\left(s\right)\right)|ds, \leqslant \frac{T^t}{\Gamma(\eta+1)}||\Phi_n - \Phi|| \to 0 as n \to \infty. \end{aligned} \quad (8)$$

Since Φ is continuous, therefore $\mathfrak{T}\psi_n \to \mathfrak{T}\psi$, as the result $\mathfrak{T}$ is continuous.

Step II: To show that the map $\mathfrak{T}$ is bounded. For any $\psi \in X$, the map $\mathfrak{T}$ satisfies the following growth condition:

$$\begin{aligned} ||\mathfrak{T}\psi|| &= max_{t\in[0,T]}|\psi_0 + \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right)ds|, \\ &\leqslant |\psi_0| + max_{t\in[0,T]}\frac{1}{\Gamma(\eta)}\int_0^t |(t-s)^{\eta-1}||\Phi\left(s,\psi\left(s\right)\right)|ds, \\ &\leqslant |\psi_0| + \frac{T^\eta}{\Gamma(\eta+1)}\left[K_*||\psi||^q + M_*\right]. \end{aligned} \quad (9)$$

Consider $\mathbb{S}$, a bounded subset of X, To show that $\mathfrak{T}(\mathbb{S})$ is bounded. for any $\psi \in \mathbb{S}$, as $\mathbb{S}$ is bounded so there exists $K_q \geqslant 0$, such that

$$||\psi|| \leqslant K_q, \quad \forall \ \psi \in \mathbb{S}. \quad (10)$$

Hence for any $\psi \in \mathbb{S}$ using the above growth condition, we have

$$||\mathfrak{T}\psi|| \leqslant |\psi_0| + \frac{T^\eta}{\Gamma(\eta+1)}\left[K_*||\psi^q|| + M_*\right] \leqslant |\psi_0| + \frac{T^\eta}{\Gamma(\eta+1)}\left[K_*K_q + M_*\right]. \quad (11)$$

Hence, $\mathfrak{T}(\mathbb{S})$ bounded.

Step III: For equi-continuity, let $t_1, t_2 \in [0,T]$, such that $t_1 \geqslant t_2$ then

$$\begin{aligned} &|\mathfrak{T}\psi\left(t_1\right) - \mathfrak{T}\psi\left(t_2\right)| = |\frac{1}{\Gamma(\eta)}\int_0^t (t_1-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right) \\ &ds - \frac{1}{\Gamma(\eta)}\int_0^t (t_2-s)^{\eta-1}\Phi\left(s,\psi\left(s\right)\right)ds|, \leqslant |\frac{1}{\Gamma(\eta)}\int_0^t (t_1-s)^{\eta-1} \\ &-\frac{1}{\Gamma(\eta)}\int_0^t (t_2-s)^{\eta-1}||\Phi\left(s,\psi\left(s\right)\right)|ds, \leqslant \frac{T^\eta}{\Gamma(\eta+1)}\left[K_*||\psi||^q + M_*\right]\left[t_1 - t_2\right]. \end{aligned} \quad (12)$$

Hence with the help of Arzela$'$-Ascoli theorem, $\mathfrak{T}(\mathbb{S})$ is relatively compact.

Step IV: Finally, we have to show that the set

$$\mathbf{E} = \{\psi \in X : \psi = m\mathfrak{T}\psi, \ m \in (0,1)\}, \quad (13)$$

is bounded. for this let $\psi \in \mathbf{E}$, then for each $t \in [0,T]$, we have

$$||\psi|| = m||\mathfrak{T}\psi|| \leqslant m\left[\left|\psi_0\right| + \frac{T^\eta}{\Gamma(\eta+1)}K_*\left|\psi\right|^q + M_*\right]. \quad (14)$$

Hence, this shows that the set **E** is bounded. using Schaefer's fixed point theorem. $\mathfrak{T}$ has at least one fixed point, and hence our consider problem (5) has at least one solution.

Remark 1. If the assumption (P_1) is formulated for q = 1, then the conclusion of theorem (0.4) remain hold, if $\frac{T^\eta K_*}{\Gamma(\eta+1)} < 1$.

Theorem 0.5. *The problem (5) has unique solution, if* $\frac{T^\eta K_*}{\Gamma(\eta+1)} < 1$.

Proof. Using Banach contraction theorem, let $\psi, \overline{\psi} \in X$, then

$$\begin{aligned} &||\mathfrak{T}\psi - \mathfrak{T}\overline{\psi}\ || \leqslant \max_{t\in[0,T]}\frac{1}{\eta}\int_0^t |(t-s)^{\eta-1}||\Phi\left(s,\psi\left(s\right)\right) - \Phi\left(s,\overline{\psi}\left(s\right)\right)|ds, \\ &\leqslant \frac{T^\eta L_\Phi}{\Gamma(\eta+1)}||\psi - \overline{\psi}||. \end{aligned} \quad (15)$$

Hence $\mathfrak{T}$ has a unique fixed point, and consequently, the considered problem (5) has a unique solution.

Stability results

One of the important aspect of FDEs is that of stability analysis. There are different types and form of stability, among these types, one of the important type of stability is UH stability. Ulam introduced this stability in 1940 and further studied by Hyers. This stability was generalized by Rassias to more generalized form known as UHR stability. The authors adopted the above mentioned stability in this work. The following are some results of the aforesaid stability from [62].

Let $\mathscr{H} : X \to X$ be an operator satisfying

$$\mathscr{H}(\psi) = \psi, \quad \psi \in X. \tag{16}$$

Definition 0.6. The Eq. (16) is UH stable, if for $\in_* > 0$ and let $\psi \in \mathscr{H}$ be any solution of the inequality given by

$$||\psi - \mathscr{H}\psi|| \leqslant \in_*, \; for \; t \in [0, T], \tag{17}$$

there exist a unique solution $\overline{\psi}$ of the Eq. (16) with constant $C_q > 0$ satisfying

$$||\overline{\psi} - \psi|| \leqslant C_q \in_*, \; for \;\; t \in \left[0, T\right]. \tag{18}$$

Definition 0.7. If $\overline{\delta}$ be the unique solution and δ any solution of Eq. (16), such that

$$||\overline{\psi} - \psi|| \leqslant \theta(\in_*), \tag{19}$$

where $\theta \in C(R, R)$ and $\theta(0) = 0$, then Eq. (16) is GUHS.

Remark 2. If there exists $\xi_*(t) \in C(J = [0, T], R)$, then $\overline{\psi} \in X$satisfies (17), if

(i) $|\xi_*(t)| \leqslant \in_*, \;\; \forall\, t \in [0, T]$,
(ii) $\mathscr{H}\overline{\psi}(t) = \overline{\psi} + \xi_*(t), \; \forall \; t \in [0, T]$.

Consider the following perturbed equation of the perturbed problem (5) as

$$\begin{cases} {}^{C}D^{\eta}_{+0}\psi(t) = \Phi(t, \psi(t)) + \xi_*(t), \\ \psi(0) = \psi_0. \end{cases} \tag{20}$$

Lemma 0.3. *The result mentioned below holds for Eq. (20),*

$$|\psi\left(t\right) - \mathfrak{T}\psi\left(t\right)| \leqslant a\in_*, \;\; where \;\; a = \frac{T^{\eta}}{\Gamma(\eta + 1)}. \tag{21}$$

Proof. This is simple result of Remark 2 and Lemma 0.2.

Theorem 0.8. *Under lemma (0.3), the solution of the concerned problem (5) is Ulam-Hyers stable and also generalized Ulam-Hyers stable, if* $\frac{T^{\eta}L_{\varphi}}{\Gamma(\eta+1)} < 1$.

Proof. Let $\psi \in X$ be any solution and $\overline{\psi} \in X$ be the unique solution of Eq. (5), then

$$|\psi(t) - \overline{\psi}\,(t)| = |\psi(t) - \mathfrak{T}\overline{\psi}(t)|, \leqslant |\psi(t) - \mathfrak{T}\psi(t)| + |\mathfrak{T}\psi(t) \\ + \mathfrak{T}\overline{\psi}(t)|, \leqslant a\in_* + \frac{T^{\eta}L_{\phi}}{\Gamma(\eta + 1)}|\psi\left(t\right) - \overline{\psi}\left(t\right)|, \leqslant \frac{a\in_*}{1 - \dfrac{T^{\eta}L_{\theta}}{\Gamma(\eta + 1)}}. \tag{22}$$

Which shows that problem (5) is Ulam-Hyers stable, also it is generalized Ulam-Hyers stable, by defining

$$Y\left(\in_*\right) = \frac{a\in_*}{1 - \frac{T^{\eta}L_{\theta}}{\Gamma(\eta+1)}}. \tag{23}$$

Such that $Y(0) = 0$.

Definition 0.9. Eq. (16) is Ulam-Hyers-Rassias stable for $g \in C([0, T], R)$, if for $\in > 0$ and let $\psi \in X$ be any solution of the inequality given by

$$||\psi - H\psi|| \leqslant g(t)\in_*, \tag{24}$$

there exists a unique solution $\overline{\psi}$ of (16) with $K_q > 0$, such

$$||\overline{\psi} - \psi|| \leqslant K_q g\left(t\right)\in_*, \;\; \forall\, t \in \left[0, T\right]. \tag{25}$$

Definition 0.10. For $g \in C([0, T], R)$, if there exists $K_{q,g}$ and for $\in_* > 0$, consider that ψ be any solution of (24) and $\overline{\psi}$ be any solution of (16), such that

$$||\overline{\psi} - \psi|| \leqslant K_{q,g}g\left(t\right), \;\; \forall \;\; t \in \left[0, T\right], \tag{26}$$

then Eq. (16) is generalized Ulam-Hyers-Rassias stable.

Remark 3. If there exists $\xi_*(t) \in C([0, T], R)$, then $\overline{\psi} \in X$ satisfies (17), if

(i) $|\xi_*(t)| \leqslant \in_* g(t), \;\; \forall\, t \in [0, T]$,
(ii) $\mathscr{H}\overline{\psi}(t) = \overline{\psi} + \xi_*(t), \; \forall \; t \in [0, T]$.

Lemma 0.4. *The following result holds for Eq. (20),*

$$|\psi\left(t\right) - \mathfrak{T}\psi\left(t\right)| \leqslant ag\left(t\right)\in_*, \;\; where \;\; a = \frac{T^{\eta}}{\Gamma(\eta + 1)}. \tag{27}$$

Proof. By simplicity of proof, we omit it.

Theorem 0.11. *Using Lemma 0.4, the solution of the concerned problem (5) is UHR stable and GUHR stable, also if* $\frac{T^{\eta}L_{\phi}}{\Gamma\eta+1} < 1$.

Proof. Let $\psi \in X$ be any solution and $\overline{\psi} \in X$ be the unique solution of Eq. (5), then

$$|\psi(t) - \overline{\psi}\,(t)| = |\psi(t) - \mathfrak{T}\overline{\psi}(t)|, \leqslant |\psi(t) - \mathfrak{T}\psi(t)| + |\mathfrak{T}\psi(t) \\ + \mathfrak{T}\overline{\psi}(t)|, \leqslant a.g\left(t\right)\in_* + \frac{T^{\eta}L_{\phi}}{\Gamma(\eta + 1)}|\psi\left(t\right) - \overline{\psi}\left(t\right)|, \leqslant \frac{a.g(t)\in_*}{1 - \dfrac{T^{\eta}L_{\theta}}{\Gamma(\eta + 1)}}. \tag{28}$$

Hence, the considered Eq. (5) is UHR stable, and also GUHR stable. This completes the proof.

Equilibrium points and local stability analysis

In order to find basic reproductive number, we consider the Infected classes of concerned model (1), such that

$$\begin{aligned} {}_0{}^{C}D^{\eta}_{t}E(t) &= \beta S(t)I(t) - (\mu + d_2 + \alpha + \gamma_1)E(t), \\ {}_0{}^{C}D^{\eta}_{t}I(t) &= \alpha E(t) - (\mu + \gamma_2 + d_3)I(t). \end{aligned} \tag{29}$$

Now, we have the dominant eigenvalue of next generation matrix $[FV^{-1}]$ is

$$R_0 = \frac{\beta S^0(t)\alpha}{(d_3 + \gamma_2 + \mu)(\alpha + d_2 + \gamma_1 + \mu)}, \tag{30}$$

where

$$F = \begin{bmatrix} 0 & \beta S(t) \\ 0 & 0 \end{bmatrix}, \tag{31}$$

$$V = \begin{bmatrix} \alpha + d_2 + \gamma_1 + \mu & 0 \\ -\alpha & d_3 + \gamma_2 + \mu \end{bmatrix}. \tag{32}$$

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Steady states

There are two types of equilibrium points in population, these are Disease-Free and Endemic Equilibrium points. The Disease-Free Equilibrium of model (1), is denoted by E^0 that is

$$E^0\left(S^0\left(t\right), E^0\left(t\right), I^0\left(t\right), R^0\left(t\right)\right) = E^0\left(\frac{a}{\mu + d_1}, 0, 0, 0\right), \tag{33}$$

while, the Disease-Endemic Equilibrium of model (1) is given as

$$E^*(S^*(t),E^*(t),I^*(t),R^*(t)), \tag{34}$$

where

$$\begin{aligned} S^*(t) &= \frac{a}{\beta I^* + \mu + d_1}, \\ E^*(t) &= I^*\frac{\mu+\gamma_2+d_3}{\alpha}, \\ R^*(t) &= I^*\frac{(\gamma_1(\mu+\gamma_2+d_3)+\gamma_2\alpha)}{\alpha\mu}. \end{aligned} \tag{35}$$

Local stability analysis

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Theorem 0.12. *The disease-free equilibrium point $E^0 = (S^0,0,0,0)$ of model (1) is locally asymptotically stable, if $R_0 < 1$.*

Proof. The Jacobian matrix of model (1) around disease-free equilibrium point E^0 becomes

$$J_1^{|0|} = \begin{bmatrix} -(\mu+d_1) & 0 & -\beta S^0 & 0 \\ 0 & -(\mu+d_2+\alpha+\gamma_1) & \beta S^0 & 0 \\ 0 & \alpha & -(\mu+\gamma_2+d_3) & 0 \\ 0 & \gamma_1 & \gamma_2 & -\mu \end{bmatrix}, \tag{36}$$

Clearly two of its eigenvalues of matrix (36) are negative *i.e* $\lambda_1 = -(\mu+d_1)$ and $\lambda_2 = -\mu$. Now after some matrix operation the reduced matrix becomes.

$$J_2^{|0|} = \begin{bmatrix} -(\mu+d_2+\alpha+\gamma_1) & \beta S^0 \\ 0 & \alpha\beta S^0 - (\mu+d_2+\alpha+\gamma_1)(\mu+\gamma_2+d_3) \end{bmatrix}, \tag{37}$$

now from system (37), clearly we have last two eigenvalues such that $\lambda_3 = -(\mu+d_2+\alpha+\gamma_1)$ and $\lambda_4 = \alpha\beta S^0 - (\mu+d_2+\alpha+\gamma_1)(\mu+\gamma_2+d_3)$, for $\lambda_4 < 0$, such that $\alpha\beta S^0 - (\mu+d_2+\alpha+\gamma_1)(\mu+\gamma_2+d_3) < 0$, *i.e* so that $\frac{\alpha\beta S^0}{(\mu+d_2+\alpha+\gamma_1)(\mu+\gamma_2+d_3)} < 1$, implies that $R_0 < 1$, therefore the disease-free equilibrium point E^0 is locally asymptotically stable for $R_0 < 1$.

Theorem 0.13. *The disease-endemic equilibrium point $E^* = E^*(S^*,E^*,I^*,R^*)$, of model (1) is locally asymptotically stable, if $R_0 > 1$.*

Proof. The Jacobian matrix of model 1 around disease-endemic equilibrium point E^* becomes

$$J_1^{|*|} = \begin{bmatrix} -(\beta I^*+\mu+d_1) & 0 & -\beta S^* & 0 \\ \beta I^* & -(\mu+d_2+\alpha+\gamma_1) & \beta S^* & 0 \\ 0 & \alpha & -(\mu+\gamma_2+d_3) & 0 \\ 0 & \gamma_1 & \gamma_2 & -\mu \end{bmatrix}, \tag{38}$$

Clearly, one of it's eigenvalues of matrix (38) is $\lambda_1 = -\mu$. The reduced matrix becomes

$$J_2^{|*|} = \begin{bmatrix} -(\beta I^*+\mu+d_1) & 0 & -\beta S^* \\ \beta I^* & -(\mu+d_2+\alpha+\gamma_1) & \beta S^* \\ 0 & \alpha & -(\mu+\gamma_2+d_3) \end{bmatrix}, \tag{39}$$

Now multiplying second row by $(\beta I^*+\mu+d_1)$ and multiply first row by βI^* and then adding R_1 and R_2 we have

$$J_3^{|*|} = \begin{bmatrix} -(\beta I^*+\mu+d_1) & 0 & -\beta S^* \\ 0 & -(\mu+d_2+\alpha+\gamma_1)(\beta I^*+\mu+d_1) & \beta S^*(\beta I^*+\mu+d_1) - \beta^2 S^* I^* \\ 0 & \alpha & -(\mu+\gamma_2+d_3) \end{bmatrix}, \tag{40}$$

Clearly, one eigenvalue of matrix(40) is $\lambda_2 = -(\beta I^* + \mu + d_1)$. This leads to the reduced matrix below such that

$$J_4^{|*|} = \begin{bmatrix} -(\mu+d_2+\alpha+\gamma_1)(\beta I^*+\mu+d_1) & \beta S^*(\beta I^*+\mu+d_1) - \beta^2 S^* I^* \\ \alpha & -(\mu+\gamma_2+d_3) \end{bmatrix}, \tag{41}$$

Further, multiply first row by "α" and second row by $(\mu+d_2+\alpha+\gamma_1)(\beta I^*+\mu+d_1)$ and adding R_1 and R_2, we get

$$J_5^{|*|} = \begin{bmatrix} -(\beta I^*+\mu+d_1)(\mu+d_2+\alpha+\gamma_1) & \beta S^*(\beta I^*+\mu+d_1) - \beta^2 S^* I^* \\ 0 & \alpha\beta S^*(\beta I^*+\mu+d_1) - \alpha\beta^2 S^* I^* - (\mu+\gamma_2+d_3)(\beta I^*+\mu+d_1)(\mu+d_2+\alpha+\gamma_1) \end{bmatrix}. \tag{42}$$

Eventually, the last two eigenvalues are

$$\lambda_3 = -(\mu+d_2+\alpha+\gamma_1)(\beta I^*+\mu+d_1)$$

and

$$\lambda_4 = \alpha\beta S^*(\beta I^*+\mu+d_1) - \alpha\beta^2 S^* I^* - (\mu+\gamma_2+d_3)(\beta I^*+\mu+d_1)(\mu+d_2+\alpha+\gamma_1),$$

for $\lambda_4 < 0$, such that

$$\alpha\beta S^*(\beta I^*+\mu+d_1) - \alpha\beta^2 S^* I^* - (\mu+\gamma_2+d_3)(\beta I^*+\mu+d_1)(\mu+d_2+\alpha+\gamma_1) < 0,$$

implies that

$$\frac{\alpha\beta}{(\mu+\gamma_2+d_3)(\mu+d_2+\alpha+\gamma_1)} S^*\left(\beta I^*+\mu+d_1\right) < \alpha\beta^2 S^* I^* + \left(\beta I^*+\mu+d_1\right),$$

therefore

$$R_0 < \frac{\alpha\beta^2 S^* I^*}{S^*(\beta I^*+\mu+d_1)} + \frac{\beta I^*+\mu+d_1}{S^*(\beta I^*+\mu+d_1)},$$

so that this leads to

$$1 < R_0 < \frac{\alpha\beta(I^*1+\mu+d_1)}{S^*(\beta I^*+\mu+d_1)}.$$

Hence, the disease-endemic equilibrium point it locally asymptotically stable, for $R_0 > 1$.

General procedure for construction of approximate solution

In this section, of research work the author's presents the general scheme for analytical solution of consider system given by

$$\begin{cases} {}_0^C D_t^\eta S(t) = a - \beta S(t) I(t) - (\mu + d_1) S(t), \\ {}_0^C D_t^\eta E(t) = \beta S(t) I(t) - (\mu + d_2 + \alpha + \gamma_1) E(t), \\ {}_0^C D_t^\eta I(t) = \alpha E(t) - (\mu + \gamma_2 + d_3) I(t), \\ {}_0^C D_t^\eta R(t) = \gamma_1 E(t) + \gamma_2 I(t) - \mu R(t), \end{cases} \tag{43}$$

By taking Laplace transform on (43) in Caputo fractional derivatives, we obtained

$$\begin{cases} \mathscr{L}\{S(t)\} = \frac{S(0)}{s} + \frac{1}{s^\eta}\mathscr{L}\{a - \beta S(t)I(t) - (\mu + d_1)S(t)\}, \\ \mathscr{L}\{E(t)\} = \frac{E(0)}{s} + \frac{1}{s^\eta}\mathscr{L}\{\beta S(t)I(t) - (\mu + d_2 + \alpha + \gamma_1)E(t)\}, \\ \mathscr{L}\{I(t)\} = \frac{I(0)}{s} + \frac{1}{s^\eta}\mathscr{L}\{\alpha E(t) - (\mu + \gamma_2 + d_3)I(t)\}, \\ \mathscr{L}\{R(t)\} = \frac{R(0)}{s} + \frac{1}{s^\eta}\mathscr{L}\{\gamma_1 E(t) + \gamma_2 I(t) - \mu R(t)\}. \end{cases} \tag{44}$$

Taking inverse Laplace and conditions on (44), we have

$$\begin{cases} S\left(t\right) = S_0 + \mathscr{L}^{-1}\left[\frac{1}{s^\eta}\mathscr{L}\{a - \beta S(t)I(t) - (\mu + d_1)S(t)\}\right], \\ E\left(t\right) = E_0 + \mathscr{L}^{-1}\left[\frac{1}{s^\eta}\mathscr{L}\{\beta S(t)I(t) - (\mu + d_2 + \alpha + \gamma_1)E(t)\}\right], \\ I\left(t\right) = I_0 + \mathscr{L}^{-1}\left[\frac{1}{s^\eta}\mathscr{L}\{\alpha E(t) - (\mu + \gamma_2 + d_3)I(t)\}\right], \\ R\left(t\right) = R_0 + \mathscr{L}^{-1}\left[\frac{1}{s^\eta}\mathscr{L}\{\gamma_1 E(t) + \gamma_2 I(t) - \mu R(t)\}\right]. \end{cases} \tag{45}$$

In form of series, the solution S(t), E(t), I(t) and R(t) are defined as

$$S\left(t\right) = \sum_{k=0}^{\infty} S_k\left(t\right),\ E\left(t\right) = \sum_{k=0}^{\infty} E_k\left(t\right),\ I\left(t\right) = \sum_{k=0}^{\infty} I_k\left(t\right),\ R\left(t\right) = \sum_{k=0}^{\infty} R_k\left(t\right). \tag{46}$$

The consider system contains non-linear term S(t). I(t), which is expressed as

$$S\left(t\right) I\left(t\right) = \sum_{k=0}^{\infty} X_k. \tag{47}$$

where X_k is Adomian's polynomial, defined as

$$X_k = \frac{1}{k!}\cdot\frac{d^k}{d\xi^k}\left[\sum_{n=0}^{k}\xi^n S_n \cdot \sum_{n=0}^{k}\xi^n I_n\right]|_{\xi=0}.$$

Now using (46) and (47) in (44), we have

$$\begin{cases} \mathscr{L}\{S_0\} = \frac{S_0}{s}, \mathscr{L}\{E_0\} = \frac{E_0}{s}, \mathscr{L}\{I_0\} = \frac{I_0}{s}, \mathscr{L}\{R_0\} = \frac{R_0}{s}, \\ \mathscr{L}\{S_1\} = \frac{1}{s^\eta}.\mathscr{L}\{a - \beta X_0 - (\mu + d_1)S_0\}, \\ \mathscr{L}\{E_1\} = \frac{1}{s^\eta}.\mathscr{L}\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\}, \\ \mathscr{L}\{I_1\} = \frac{1}{s^\eta}.\mathscr{L}\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\{ \\ \mathscr{L}\{R_1\} = \frac{1}{s^\eta}.\mathscr{L}\{\gamma_1 E_0 + \gamma_2 I_0 - \mu R_0\}, \\ \mathscr{L}\{S_2\} = \frac{1}{s^\eta}.\mathscr{L}\{a - \beta X_1 - (\mu + d_1)S_1\}, \\ \mathscr{L}\{E_2\} = \frac{1}{s^\eta}.\mathscr{L}\{\beta X_1 - (\mu + d_2 + \alpha + \gamma_1)E_1\}, \\ \mathscr{L}\{I_2\} = \frac{1}{s^\eta}.\mathscr{L}\{\alpha E_1 - (\mu + \gamma_2 + d_3)I_1\{ \\ \mathscr{L}\{R_2\} = \frac{1}{s^\eta}.\mathscr{L}\{\gamma_1 E_1 + \gamma_2 I_1 - \mu R_1\}, \\ \vdots \\ \mathscr{L}\{S_n\} = \frac{1}{s^\eta}.\mathscr{L}\{a - \beta X_{n-1} - (\mu + d_1)S_{n-1}\}, \\ \mathscr{L}\{E_n\} = \frac{1}{s^\eta}.\mathscr{L}\{\beta X_{n-1} - (\mu + d_2 + \alpha + \gamma_1)E_{n-1}\}, \\ \mathscr{L}\{I_n\} = \frac{1}{s^\eta}.\mathscr{L}\{\alpha E_{n-1} - (\mu + \gamma_2 + d_3)I_{n-1}\{ \\ \mathscr{L}\{R_n\} = \frac{1}{s^\eta}.\mathscr{L}\{\gamma_1 E_{n-1} + \gamma_2 I_{n-1} - \mu R_{n-1}\}, \end{cases} \tag{48}$$

Applying inverse Laplace on both side of (48), we get

$$\begin{cases} S_0(t) = S_0, E_0(t) = E_0, I_0(t) = I_0, R_0(t) = R_0, \\ S_1 = \frac{t^\eta}{\eta!}\{a - \beta S_0 I_0 - (\mu + d_1)S_0\}\Big\}, \\ E_1 = \frac{t^\eta}{\eta!}\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\}, \\ I_1 = \frac{t^\eta}{\eta!}\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\}, \\ R_1 = \frac{t^\eta}{\eta!}\{\gamma_1 E_0 + \gamma_2 I_0 - \mu R_0\}, \\ S_2 = \frac{at^\eta}{\eta!} - \frac{t^{2\eta}}{2\eta!}[\beta S_0\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} + \{\beta I_0 + (\mu + d_1)\}\{a - \beta S_0 I_0 - (\mu + d_1)S_0\}], \\ E_2 = \frac{t^{2\eta}}{2\eta!}[\beta S_0\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} + \beta I_0\{a - \beta S_0 I_0 - (\mu + d_1)S_0\} - \{\mu + d_2 + \alpha + \gamma_1\} \\ \{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\}], \\ I_2 = \frac{t^{2\eta}}{2\eta!}[\alpha\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\} - \{\mu + d_2 + d_3\}\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\}], \\ R_2 = \frac{t^{2\eta}}{2\eta!}[\gamma_1\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\} + \gamma_2\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} - \mu\{\gamma_1 E_0 + \gamma_2 I_0 - \mu R_0\}]. \end{cases} \tag{49}$$

Hence by exercising the same process, we can get other terms. Infinite series form up to three terms is given in the following

$$\begin{cases} S_n = S_0 + \frac{t^\eta}{\eta!}\{a - \beta S_0 I_0 - (\mu + d_1)S_0\}\Big\} + \\ \frac{at^\eta}{\eta!} - \frac{t^{2\eta}}{2\eta!}[\beta S_0\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} + \{\beta I_0 + (\mu + d_1)\}\{a - \beta S_0 I_0 - (\mu + d_1)S_0\}] + \ldots, \\ E_n = I_0 + \frac{t^\eta}{\eta!}\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\} + \\ \frac{t^{2\eta}}{2\eta!}[\beta S_0\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} + \beta I_0\{a - \beta S_0 I_0 - (\mu + d_1)S_0\} - \{\mu + d_2 + \alpha + \gamma_1\} \\ \{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\}] + \ldots, \\ I_n = I_0 + \frac{t^\eta}{\eta!}\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} + \frac{t^{2\eta}}{2\eta!}[\alpha\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\} - \{\mu + d_2 + d_3\}\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\}] + \ldots, \\ R_n = R_0 + \frac{t^\eta}{\eta!}\{\gamma_1 E_0 + \gamma_2 I_0 - \mu R_0\} + \\ \frac{t^{2\eta}}{2\eta!}[\gamma_1\{\beta S_0 I_0 - (\mu + d_2 + \alpha + \gamma_1)E_0\} + \gamma_2\{\alpha E_0 - (\mu + \gamma_2 + d_3)I_0\} - \mu\{\gamma_1 E_0 + \gamma_2 I_0 - \mu R_0\}] + \cdots. \end{cases} \tag{50}$$

For the convergence of the series in (46), we establish the following result.

Theorem 0.14. [25] *Let $\mathbb{Z}$ be Banach space and $\mathbb{T}:\mathbb{Z}\rightarrow\mathbb{Z}$ be a contraction operator such that for all $y,\overline{y}\in\mathbb{Z}$, with $||G(y)-G(\overline{y})||\leqslant d_k k^*||y-\overline{y}||, 0<d_k k^*<1$. By Banach result the operator G has a unique fixed point y such that $\mathbb{G}y=y$ the series given in (46) may be express as*

$$Y_n = Gy_{n-1}, Y_{n-1} = \sum_{n=0}^{j-i} Y_n, \ wherej = 1,2,3,\ldots.$$

If $y_0 \in \mathbb{B}_\rho(Y)$ where $\mathbb{B}_\rho(Y) = \{\overline{Y}\in\mathbb{Z} : \left|\left|\overline{Y}-Y\right|\right| < \rho\}$, then

$(i) Y_n \in \mathbb{B}_\rho(Y)$,
$(ii) lim_{n\rightarrow\infty} Y_n = Y$.

Numerical results and discussion

This section, is devoted for the approximation of the parameters used in the concerned model 1. The tables listed here show the assumed values of the parameters. The infinite series solution is obtained up to three terms for the considered model and is based on the assumed values mentioned in the tables for the corresponding country. The assumed values for Pakistan, series solutions and graphs are shown in the following.

For $\eta = 1$, the series solution is obtained as

$$\begin{cases} S_n = 8065518 - 6.83383\times10^9 t + 2.89071\times10^{12}t^2 + \ldots, \\ E_n = 200000 + 6.83151\times10^9 t + 1.21602\times10^{14}t^2 + \ldots, \\ I_n = 28234 + 36447.7t + 1108.1t^2 + \ldots, \\ R_n = 59388.1t + 9.56413\times10^8 t^2 + \ldots. \end{cases}$$

For $\eta = 0.97$, we get the following series solution

$$\begin{cases} S_n = 8065518 - 6.91904\times10^9 t^{0.97} + 2.92675\times10^12t^{1.94} + \ldots, \\ E_n = 200000 + 6.91668\times10^9 t^{0.97} + 1.23118\times10^{14}t^{1.94} + \ldots, \\ I_n = 28234 + 36902.1t^{0.97} + 1121.92t^{1.94} + \ldots, \\ R_n = 60128.6t^{0.97} + 9.68338\times10^8 t^{1.94} + \ldots. \end{cases}$$

Plugging $\eta = 0.93$, the series solution is obtained as

$$\begin{cases} S_n = 8065518 - 7.02782\times10^9 t^{0.93} + 2.97277\times10^{12}t^{1.86} + \ldots, \\ E_n = 200000 + 7.02543\times10^9 t^{0.93} + 1.25054\times10^{14}t^{1.86} + \ldots, \\ I_n = 28234 + 37482.3t^{0.93} + 1139.56t^{1.86} + \ldots, \\ R_n = 61073.9t^{0.93} + 9.83562\times10^8 t^{1.86} + \ldots. \end{cases}$$

Insert $\eta = 89$, we obtained the series solution as

$$\begin{cases} S_n = 8065518 - 7.13061\times10^9 t^{0.89} + 3.01625\times10^{12}t^{1.78} + \ldots, \\ E_n = 200000 + 7.12819\times10^9 t^{0.89} + 1.26883\times10^{14}t^{1.78} + \ldots, \\ I_n = 28234 + 38030.5t^{0.89} + 1156.23t^{1.78} + \ldots, \\ R_n = 61967.2t^{0.89} + 9.97948\times10^8 t^{1.78} + \ldots. \end{cases}$$

Graphs for the corresponding series solution are given by From Fig. 1–4, we observe that as the susceptibility is raising up which results in increase of exposing the papulation for infection. Hence infection is also increasing along with asymptotic infection which raises up. Hence the recovery rate is increasing due to more infection. Therefore density of the removed class is going up. The concerned rate of increase is different at different fractional order, smaller the fractional order shower the growth process, while smaller the fractional order faster the decay process. With smaller fractional order the concerned rate is faster and as the order is enlarging, the corresponding raise is becoming slow.

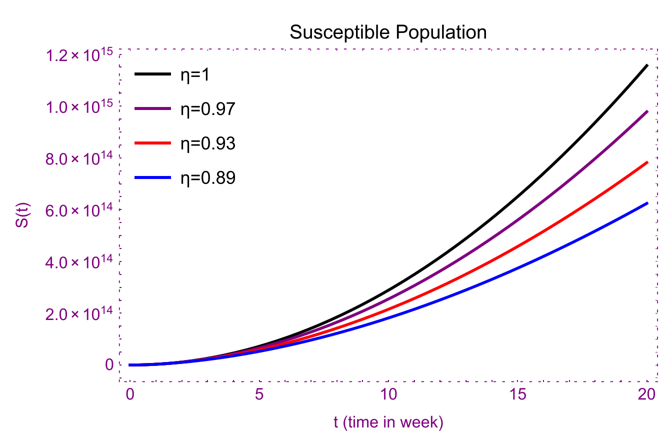

Fig. 1. Plot shows the behavior of $S(t)$ at different values of fractional order η.

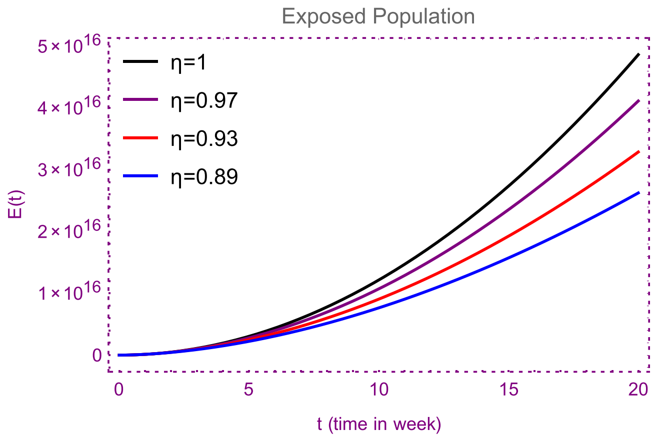

Fig. 2. Plot shows the behavior of $E(t)$ at different values of fractional order η.

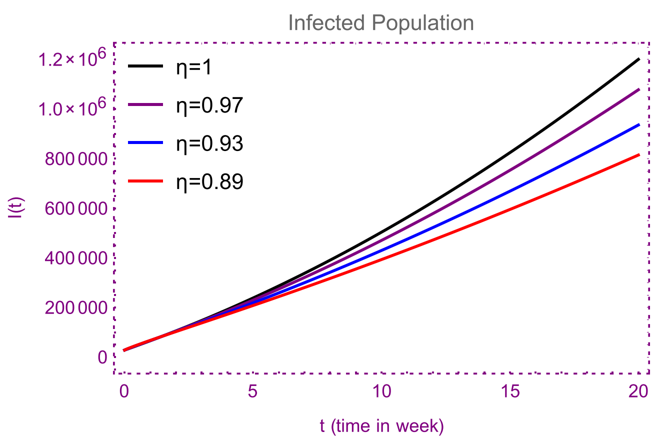

Fig. 3. Plot shows the behavior of $I(t)$ at different values of fractional order η.

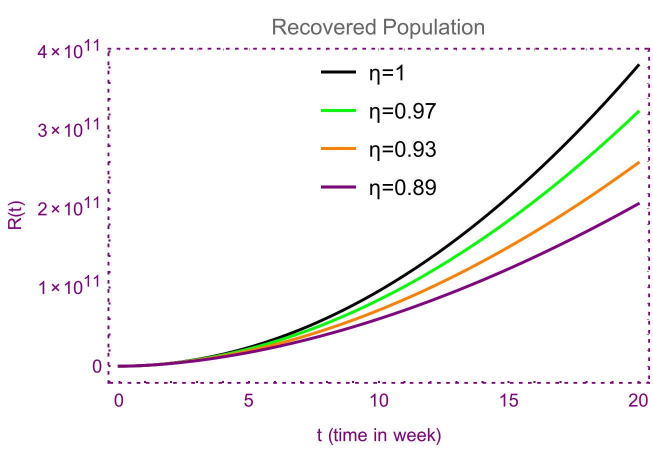

Fig. 4. Plot shows the behavior of $S(t)$ at different values of fractional order η.

Conclusion

In this article, author's focused on extremely important issues related to human health, that realized the dynamic of inflected disease nCOVID-19. We investigated the conditions for qualitative analysis and numerical solutions of capture concerned mathematical model with the help of Caputo fractional derivative. The author's used the tools of non-linear analysis, to obtained the desired results for existence and stability analysis. We thoroughly developed the conditions for local and global asymptotical stabilities for proposed system via the disease free, endemic equilibrium and reproductive number. At final stage, we obtained the approximate solution for concerned model via Laplace Adomian decomposition method. The author's also interpret the dynamics of proposed system graphically via Mathematica, that explained that disease can be either controlled to a large extent or eliminate, if transmission rate is reduced and increase the rate of treatment.

CRediT authorship contribution statement

Amjad Ali: Conceptualization, Data curation, Formal analysis, Investigation, Project administration, Supervision, Visualization, Writing - original draft, Writing - review & editing. **Muhammad Yasin Khan:** Conceptualization, Formal analysis, Investigation, Project administration, Supervision, Visualization, Writing - original draft, Writing - review & editing. **Muhammad Sinan:** Conceptualization, Data curation, Investigation, Methodology, Project administration, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing. **F.M. Allehiany:** Methodology, Resources, Validation, Writing - review & editing. **Emad E. Mahmoud:** Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Software, Validation, Writing - review & editing. **Abdel-Haleem Abdel-Aty:** Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Visualization, Writing - review & editing. **Gohar Ali:** Conceptualization, Data curation, Methodology, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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