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RESEARCH ARTICLE

ANALYTIC APPROXIMATE SOLUTION OF RABIES TRANSMISSION DYNAMICS USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT

In this paper, we consider a mathematical model of Rabies disease which is an infectious disease. The model we are considering is a system of nonlinear ordinary differential equations and it is difficult to find an exact solution. He's Homotopy perturbation method is employed to compute an approximation to the solution of the system of nonlinear ordinary differential equations. The findings obtained by HPM are compared with a nonstandard finite difference (NSFD) and Runge-Kutta fourth order (RK4) methods. Some plots are presented to show the reliability and simplicity of the method.

KEYWORDS

Mathematical model, infectious, nonlinear ordinary, homotopy, equations.

1. INTRODUCTION

Rabies is an infection that mostly affects the brain of an Infected animal or individual, caused by viruses belonging to The genus Lyssavirus of the family Rhabdoviridae and order Mononegavirales (Hayman et al., 2011). This disease has become a global threat and it is also estimated that rabies occurs in more than 150 countries and territories (Rabies fact sheet, 2016). Raccoons, skunks, bats, and foxes are the main animals that transmit the virus in the United States. In Asia, Africa, and Latin America, it is known that dogs are the main source of transmission of the rabies virus into the human population (Zhang et al., 2011; Grace, 2014; Wiraninggih et al., 2015; Global Alliance for rabies control, 2016). When the rabies virus enters the human body or that of an animal, the infection (virus) moves rapidly along the neural pathways to the central nervous system; from there the virus continues to spread to other organs and causes injury by interrupting various nerves (Rabies fact sheet, 2016; CIA, 2016; Birkho and Rota, 1962). The symptoms of rabies are quite similar to those of encephalitis (Rupprecht et al., 2010). Due to the movement of dogs in homes or the surroundings, the risk of being infected by a rabid dog can never be guaranteed. Rabies is a major health problem in many populations dense with dogs, especially in areas where there are less or no preventive measures (vaccination and treatment) for dogs and humans (Lakshmikantham et al., 1994; Diekmann et al., 1990; Pielou, 1969; May, 1973; Martcheva, 2015). Treatment after exposure to the rabies virus is known as post-exposure Prophylaxis (PEP) and vaccination before exposure to the infection is known as pre-exposure prophylaxis.

Since most of the mathematical models raised from biological problems are nonlinear by nature and it is difficult to find the analytical solution of such problems (Yusuf and Benyah, 2012; Hove-Musekwa et al., 2011; Deleon, 2009; Lasalle, 1976). Therefore, it is a great challenge for mathematicians and researchers to find such numerical and perturbation methods which give the best approximation to the solution of such nonlinear problems. Convergence and accuracy are the key concepts while developing and implementing a numerical scheme otherwise results will be inappropriate. As far as the analytical perturbation methods are concerned, a parameter (negligibly small) needs to be exerted in the equation. Exertion and production of such parameter is a difficult task in these methods.

Recent research provided powerful methods like artificial parameter method in which this small parameter is absent. An approximate solution of nonlinear differential equations can be effectively obtained using the well-known Homotopy Analysis Method (HAM) (Zhang et al., 2015; Sharomi and Malik, 2017; Ali et al., 2018; Bushnaq et al., 2018a; Bushnaq et al., 2017; Bushnaq et al., 2018b). The method is used with perturbation methods in recent decades. The basic and fundamental scheme of the method was first introduced by Liao and He. The method uses a free parameter whose appropriate selection yields fast convergence of the algorithm. At the initial stage, He in introduced HPM and applies the procedure to some interesting problems (Rupprecht et al., 2010; Aubert, 1999; Shah et al., 2018; Asamoah et al., 2017; Samia et al., 2018). Another group researchers used optimal Homotopy Analysis Method (OHAM) in order to obtain the solution of multi-point boundary value problem (Ali et al., 2017).

The methods mentioned above are free from the choice of small parameter and have all the advantages of perturbation methods. This work is an extension by considering HPM applied to leptospirosis epidemic model (Aubert, 1999; Anderson and May, 1982; Bohrer et al., 2002; Levin et al., 2012; Coyne et al., 1989; Childs et al., 2000; Hampson et al., 2007; Carroll et al., 2010). We will compare the results obtained by HPM with Runge-Kutta fourth order (RK4) method. Numerous problems of nonlinear nature can be solved accurately and effectively using HPM because of its rapid convergence (Wang and Lou, 2008; Yang and Lou, 2009; Zinsstag et

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al., 2009; Ding et al., 2007; Smith and Cheeseman, 2002; Tchaenche and Bauch, 2012). The rest of the manuscript is as follow. In Section 2, we included the basic concept of HPM. In Section 3, the model is formulated and solved by HPM. In section 4, some numerical result and discussion is given, and the conclusion is presented at the end of the paper.

2. Analysis of Homotopy Perturbation Method (HPM)

To illustrate the basic idea of HPM, consider the general nonlinear differential equation

$$A(\mu) - f(r) = 0, r \in \Omega, \tag{1}$$

with the boundary condition,

$$\beta\left(\mu, \frac{\partial \mu}{\partial n}\right) = 0, r \in \Gamma, \tag{2}$$

where A is a general differential operator, β is a boundary operator, $f(r)$ a known analytic function, Γ is the boundary of the domain Ω. The operator A is divided into linear part L and nonlinear part N. Therefore, equation (1) can be written as,

$$\mathrm{L}(u) + \mathrm{N}(u) - \mathrm{f(r)} = 0 \tag{3}$$

By using the Homotopy technique, one can construct a Homotopy

$$v(r,p): \Omega \times [0,1] \to R \tag{4}$$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(\mu_o)]$$

$$+p[A(v - f(r)] = 0 \tag{5}$$

or

$$H(v,p) = L(v) - L(\mu_0)$$

$$+pL(v_0) + p[N(v) - f(r)] = 0 \tag{6}$$

Where, $p \in [0; 1]$ is an embedding parameter and μ_0 is the initial approximation of the given equation that satisfies the boundary conditions. Clearly, we have

$$H(v,0) = L(v) - L(\mu_0) = 0 \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{8}$$

The changing process of p from zero to one is just that of $v(r,p)$ changing from $\mu_0(r)$ to $\mu(r)$. This is called deformation, and $L(v) - L(\mu_0)$ and $A(v) - f(r)$ are called homotopic in topology. If the embedding parameter $p(0 \le p \le 1)$ is considered as a small parameter, applying the classical perturbation technique, we can naturally assume that the solution of the equation can be given as a power series in p,

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots\dots\dots \tag{9}$$

Setting $p = 1$ results in the approximate solution as

$$v = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots\dots\dots \tag{10}$$

3. The Model of Ordinary Differential Equations

$$\frac{dS_D}{dt} = A_D - (1-\nu_D)\beta_{DD}S_D I_D - (m_D + \nu_D)S_D + \delta\varepsilon_D E_D + \alpha_D R_D,$$
$$\frac{dE_D}{dt} = (1-\nu_D)\beta_{DD}S_D I_D - \left((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)E_D,$$
$$(11)$$
$$\frac{dI_D}{dt} = (1-\rho_D)\delta\gamma_D E_D - (m_D + \mu_D)I_D,$$
$$\frac{dR_D}{dt} = \nu_D S_D + \rho_D E_D - (m_D + \alpha_D)R_D,$$
$$\frac{dS_H}{dt} = B_H - (1-\nu_H)\beta_{DH}S_H I_D - (m_H + \nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H,$$
$$\frac{dE_H}{dt} = (1-\nu_H)\beta_{DH}S_H I_D - \left((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H\right)E_H,$$
$$\frac{dI_H}{dt} = (1-\rho_H)\delta_H\gamma_H E_H - (m_H + \mu_H)I_H,$$
$$\frac{dR_H}{dt} = \nu_H S_H + \rho_H E_H - (m_H + \alpha_H)R_H.$$

With

$$S_D(0) > 0, E_D(0) \ge 0, I_D(0) \ge 0, R_D(0) \ge 0, \tag{12}$$
$$S_H(0) > 0, E_H(0) \ge 0, I_H(0) > 0, R_H(0) > 0.$$

To find out the solution of the given model by Homotopy Perturbation Method (HPM). Let us consider the Homotopy which becomes for the system (1), such that,

$$LS_D(t) - LS_{D0}(t) = p[A_D - (1-\nu_D)\beta_{DD}S_D I_D - (m_D + \nu_D)S_D + \delta\varepsilon_D E_D + \alpha_D R_D - LS_{D0}(t)]$$
$$LE_D(t) - LE_{D0}(t) = p[(1-\nu_D)\beta_{DD}S_D I_D - (1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_D - LE_{D0}(t)]$$
$$LI_D(t) - LI_{D0}(t) = p[(1-\rho_D)\delta\gamma_D E_D - (m_D + \mu_D)I_D - LI_{D0}(t)]$$
$$LR_D(t) - LR_{D0}(t) = p[\nu_D S_D + \rho_D E_D - (m_D + \alpha_D)R_D - LR_{D0}(t)]$$
$$LS_H(t) - LS_{H0}(t) = p[B_H - (1-\nu_H)\beta_{DH}S_H I_D - (m_H + \nu_H)S_H + \delta_H\varepsilon_H E_H + \alpha_H R_H - LS_{H0}(t)]$$
$$LE_H(t) - LE_{H0}(t) = p[(1-\nu_H)\beta_{DH}S_H I_D - \left((1-\rho_H)\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H\right)E_H - LE_{H0}(t)]$$
$$LI_H(t) - LI_{H0}(t) = p[\delta_H\gamma_H E_H - (m_H + \mu_H)I_H - I_{H0}(t)]$$
$$LR_H(t) - LR_{H0}(t) = p[\nu_H S_H + \rho_H E_H - (m_H + \alpha_H)R_H - LR_{H0}(t)]$$

Assume that the solution of model (1) is in the form,

$$S_D(t) = S_{D0} + pS_{D1} + p^2 S_{D2} + \ \dots$$
$$E_D(t) = E_{D0} + pE_{D1} + p^2 E_{D2} + \ \dots$$
$$I_D(t) = I_{D0} + pI_{D1} + p^2 I_{D2} + \ \dots$$
$$R_D(t) = R_{D0} + pR_{D1} + p^2 R_{D2} + \ \dots$$
$$S_H(t) = S_{H0} + pS_{H1} + p^2 S_{H2} + \cdots \tag{13}$$
$$E_H(t) = E_{H0} + pE_{H1} + p^2 E_{H2} + \cdots$$
$$I_H(t) = I_{H0} + pI_{H1} + p^2 I_{H2} + \ \dots$$
$$R_H(t) = R_{H0} + pR_{H1} + p^2 R_{H2} + \ \dots$$

Now using $S_D(t)$, $E_D(t)$, $I_D(t)$, $R_D(t)$, $S_H(t)$, $E_H(t)$, $I_H(t)$, and $R_H(t)$ in equation (3) and compare the coefficients of zeroth order p^0, first order $p^1$and second order p^2 perturbation and so on respectively in each equation of system (13), Comparing the coefficient of $p^0, p^1, p^2, \dots\dots$, such that,

$$p^1: LS_{D1} = A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0}(m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0},$$
$$p^1: LE_{D1} = (1-\nu_D)\beta_{DD}S_{D0} - \left((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)E_{D0},$$
$$p^1: LI_{D1} = (1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do}, \tag{14}$$
$$p^1: LR_{D1} = \nu_D S_{D0} - (m_D + \mu_D)I_{D0},$$
$$p^1: LS_{H1} = B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0},$$
$$p^1: LE_{H1} = (1-\rho_H)\beta_{DH}S_{H0}I_{D0} - \left((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D)\right)E_{H0},$$
$$p^1: LI_{H1} = (1-\rho_H)\delta_H\gamma_H E_{H0} - (m_H + \mu_H)I_{H0},$$
$$p^1: LR_{H1} = \rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0},$$

and

$$p^2: LS_{D2} = -(1-u_D)\beta_{DD}S_{D1}I_{D1} - (m_D + \nu_D)S_{D1} + \delta R_{D1},$$
$$p^2: LE_{D2} = (1-\nu_D)\beta_{DD}S_{D1}I_{D1} - \left((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right),$$
$$p^2: LI_{D2} = (1-\rho_D)\delta\gamma_D E_{D1} - (m_D + \mu_D)I_{D1},$$
$$p^2: LR_{D2} = \nu_D S_{D1} - E_{D1} - (m_D + \alpha_D)R_{D1}, \tag{15}$$
$$p^2: LS_{H2} = -(1-\gamma_H)\beta_{DH}S_{H1}I_{D1} - (m_H + \nu_H)S_{H1} + \varepsilon_H E_{H1} + \alpha_H R_{H1},$$
$$p^2: LE_{H2} = (1-\rho_H)\beta_{DH}S_{H1}I_{D1} - \left((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)E_{H1}\right),$$
$$p^2: LI_{H2} = (1-\rho_H)\delta_H\gamma_H E_{H1} - (m_H + \mu_H)I_{H1},$$
$$p^2: LR_{H2} = \nu_H S_{H1} + \rho_H E_{H1} - (m_H + \alpha_H)R_{H1}$$

In order to obtain the solution of the problem, we consider the following cases,

3.1 Zeroth Order Problem or P^0

$$S_{D0}(t) = 44, E_{D0}(t) = 88,$$
$$I_{D0}(t) = 100, R_{D0}(t) = 110 \tag{16}$$
$$S_{H0}(t) = 105, E_{H0}(t) = 103,$$

$I_{H0}(t) = 90, R_{H0}(t) = 95$

3.2 First Order Problem or P^1

$S_{D1}(t) = (A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})t,$
$E_{D1}(t) = \left((1-\nu_D)\beta_{DD}S_{D0} - \left((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D\right)E_{D0}\right)t,$
$I_{D1}(t) = \left((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do}\right)t,$
$R_{D1}(t) = (\nu_D S_{D0} - (m_D + \mu_D)I_{D0})t, \quad (17)$
$S_{H1}(t) = (B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})t,$
$E_{H1}(t) = ((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})t,$
$I_{H1}(t) = \left((1-\rho_H)\delta_H\gamma_H E_{H0} - (m_H + \mu_H)I_{H0}\right)t,$
$R_{H1}(t) = (\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})t,$

3.3 Second Order Problem or P^2

$S_{D2}(t) = -(1-u_D)\beta_{DD}((A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2}) \quad (18)$

$E_{D2}(t) = (1-\nu_D)\beta_{DD}((A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2})$
$((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D) \quad (19)$

$I_{D2}(t) = (1-\rho_D)\delta\gamma_D\left((1-\nu_D)\beta_{DD}S_{D0} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_{D0}\right)\frac{t^2}{2} - (m_D + \mu_D)((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2} \quad (20)$

$R_{D2}(t) = \nu_D(A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2} - \left((1-\nu_D)\beta_{DD}S_{D0} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_{D0}\right)\frac{t^2}{2} - (m_D + \alpha_D)(\nu_D S_{D0} - (m_D + \mu_D)I_{D0})\frac{t^2}{2} \quad (21)$

$S_{H2}(t) = -(1-\gamma_H)\beta_{DH}((B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2})(((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) - (m_H + \nu_H)(((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2}) + \varepsilon_H\left(((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0}\right)\frac{t^2}{2}) + \alpha_H(\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})\frac{t^2}{2} \quad (22)$

$E_{H2}(t) = (1-\rho_H)\beta_{DH}((B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2})(((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2}) \quad (23)$

$I_{H2}(t) = (1-\rho_H)\delta_H\gamma_H\left((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0}\right)\frac{t^2}{2} - (m_H + \mu_H)((1-\rho_H)\delta_H\gamma_H E_{H0} - (m_H + \mu_H)I_{H0})\frac{t^2}{2}$
(24)

$R_{H2}(t) = \nu_H(B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2} + \rho_H\left((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0}\right)\frac{t^2}{2} - (m_H + \alpha_H)(\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})\frac{t^2}{2} \quad (25)$

To find the solution we consider $p = 1$ in the system (14), we get

$$
\begin{aligned}
S_D(t) &= S_{D0} + S_{D1} + S_{D2} + \dots \\
E_D(t) &= E_{D0} + E_{D1} + E_{D2} + \dots \\
I_D(t) &= I_{D0} + I_{D1} + I_{D2} + \dots \\
R_D(t) &= R_{D0} + R_{D1} + R_{D2} + \dots \\
S_H(t) &= S_{H0} + S_{H1} + S_{H2} + \dots \quad (26) \\
E_H(t) &= E_{H0} + E_{H1} + E_{H2} + \dots \\
I_H(t) &= I_{H0} + I_{H1} + I_{H2} + \dots \\
R_H(t) &= R_{H0} + R_{H1} + R_{H2} + \dots
\end{aligned}
$$

Thus, we have a solution to the model below, i.e

$S_D(t) = 44 + (A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})t + (-(1-u_D)\beta_{DD}((A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2})(((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) - (m_D + \nu_D)(A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2} + \delta(\nu_D S_{D0} - (m_D + \mu_D)I_{D0})\frac{t^2}{2}) + \cdots \quad (27)$

$E_D(t) = 88 + ((1-\nu_D)\beta_{DD}S_{D0} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_{D0})t + ((1-\nu_D)\beta_{DD}((A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2})((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)) + \cdots \quad (28)$

$I_D(t) = 100 + ((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})t + ((1-\rho_D)\delta\gamma_D\left((1-\nu_D)\beta_{DD}S_{D0} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_{D0}\right)\frac{t^2}{2} - (m_D + \mu_D)((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) + \cdots \quad (29)$

$R_D(t) = 110 + (\nu_D S_{D0} - (m_D + \mu_D)I_{D0})t + (\nu_D(A_D - (1-\nu_D)\beta_{D0}S_{D0}I_{D0} - (m_D - \nu_D)S_{D0} + \delta\varepsilon_D R_{D0} + \alpha_D R_{D0})\frac{t^2}{2} - ((1-\nu_D)\beta_{DD}S_{D0} - ((1-\rho_D)\delta\gamma_D + m_D + \rho_D + \delta\varepsilon_D + C_D)E_{D0})\frac{t^2}{2} - (m_D + \alpha_D)(\nu_D S_{D0} - (m_D + \mu_D)I_{D0})\frac{t^2}{2}) + \cdots \quad (30)$

$S_H(t) = 105 + (B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})t + (-(1-\gamma_H)\beta_{DH}((B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2})(((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) - (m_H + \nu_H)(((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2}) + \varepsilon_H(((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2}) + \alpha_H(\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})\frac{t^2}{2}) + \cdots \quad (31)$

$E_H(t) = 103 + \left((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0}\right)t + ((1-\rho_H)\beta_{DH}((B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2})(((1-\rho_D)\delta\gamma_D E_{D0} - (m_D + \mu_D)I_{Do})\frac{t^2}{2}) - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_H)((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2})) + \cdots \quad (32)$

$I_H(t) = 90 + ((1-\rho_H)\delta_H\gamma_H E_{H0} - (m_H + \mu_H)I_{H0})t + ((1-\rho_H)\delta_H\gamma_H\left((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0}\right)\frac{t^2}{2} - (m_H + \mu_H)((1-\rho_H)\delta_H\gamma_H E_{H0} - (m_H + \mu_H)I_{H0})\frac{t^2}{2}) + \cdots$
(33)

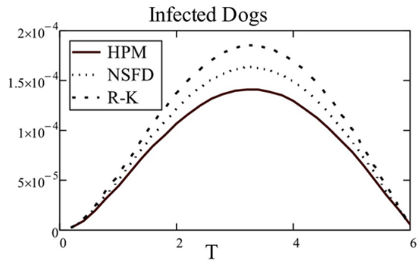
$R_H(t) = 95 + (\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})t + (\nu_H(B_H - (1-\gamma_H)\beta_{DH}\delta_H I_{D0} - (m_H + \gamma_H)S_{H0} + \beta_H + \varepsilon_H E_{H0} + \alpha_H R_{H0})\frac{t^2}{2} + \rho_H((1-\rho_H)\beta_{DH}S_{H0}I_{D0} - ((1-\rho_H)(\delta_H\gamma_H + m_H + \rho_H + \delta_H\varepsilon_D))E_{H0})\frac{t^2}{2} - (m_H + \alpha_H)(\rho_H S_{H0} + \rho_H E_{H0} - (m_H + \alpha_H)R_{H0})\frac{t^2}{2}) + \cdots \quad (34)$

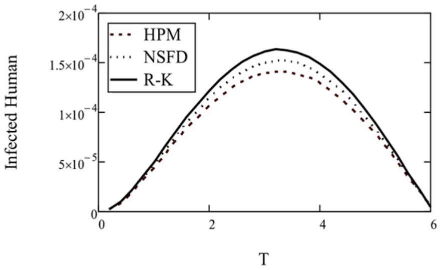
4. NUMERICAL RESULTS AND DISCUSSION

Table 1: Description of Parameter and its Value.

Notation	Description of Parameters	Values
A_D	Rate at which $S_D(t)$ is increased by recruitment	1.35
ν_D	The control strategy due to public education and vaccination in the dogs	2.50
β_{DD}	Contact rate of the rabid dog into the dog	3.000
ν_H	The preexposure prophylaxis(vaccination)	4.008
ρ_D	The rate at which exposed dogs are treated by their owners	9.020
C_D	The rate at which exposed dogs die due to culling	12.223
μ_D	The death rate associated with rabies infection in dogs	13.322
B_H	Birth or immigration rate into the susceptible human population	14.444

β_{DH}	Contact rate of infectious dogs to the human population	16.666
α_H	The rate of losing immunity in the human population	18.888
α_D	The rate of losing immunity in dogs' population	17.777
m_D	The natural death rate of dogs	19.999
m_H	The mortality rate of humans	20.222
ρ_H	The rate at which administrating treatment to affected humans	15.555
μ_H	The disease induces death in humans	23.333





5. Conclusion

In this article, we used a semi-analytical approach that is Homotopy Perturbation Method (HPM), compared with other kind of methods like Runge-Kutta fourth order (RK4) and NSFD methods for the solution of optimal control model of rabies transmission dynamics in dogs and the best way of reducing death rate of rabies in humans. We concluded that the Homotopy perturbation method is a powerful technique to solve non-linear differential equations.

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