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Analysis of the mathematical model of cutaneous Leishmaniasis disease

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Abstract Mathematical models are powerful tools to study various real-world problems from different perspectives. This branch has been given much more popularity over the last several decades. Various mathematical models corresponding to different diseases have been studied so far. Keeping these details in mind, the present manuscript is devoted to present a detailed mathematical analysis of the Cutaneous Leishmaniasis disease model. Some basic properties of the model are studied including positivity, the existence of equilibrium points, and reproductive number. The existence and uniqueness of the solution for the model under consideration are also investigated. Local and global stability analyses of equilibrium points are also studied. For the required results, we use the Lyapunov function method and the third additive compound matrix technique based on the Metzler procedure. Sensitivity analysis is also investigated by using some tools from the numerical-functional analysis. A numerical analysis of the proposed model is performed by using a nonstandard finite difference scheme. Moreover, for the justification of our results, we give some graphical presentation of the model for each class in the model. Also, we present some graphical presentations related to the sensitivity analysis along with the tables for its various indices.

1. Introduction

A parasitic disease called Leishmaniasis is commonly spread through the bite of phlebotomine sandflies, phlebotomus, and Lutzomyia. The disease mostly occurs in the tropics and sub-tropics of Africa, Asia, the Americas, and southern

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Europe. There are three ways of the disease: cutaneous, mucocutaneous, and visceral. While some risk factors are connected with a particular eco-epidemiological substance, others influence all types of leishmaniasis [1]. Cutaneous leishmaniasis (CL) [2] is the most common form of leishmaniasis and causes skin lesions, mainly ulcers, on exposed parts of the body, leaving life-long scars and serious disability or stigma. About 95% of CL cases occur in the Americas, the Mediterranean basin, the Middle East, and Central Asia. In 2020, over 85% of new CL cases occurred in 10 countries which included Afghanistan, Algeria, Brazil, Colombia, Iraq, Libya, Pakistan, Peru, the Syrian Arab Republic, and Tunisia. It is estimated that between 600 000 to 1 million new cases occur worldwide annually. In Pakistan, Anthroponotic Cutaneous Leishmaniasis has a vast distribution and commonly occurs in populated regions of Punjab, Azad Kashmir, and Baluchistan. In 1997, some cases of Anthroponotic Cutaneous Leishmaniasis were reported in the northwestern province of Pakistan. More cases were reported in refugee camp Timargara (Dir) [3]. The author [4] noted the increase in leishmaniasis risk factors. According to the report, there is a dearth of amenities including housing, sanitation, and drinking water in populous areas, which creates a favorable environment for leishmaniosis to spread. Phlebotomine sand flies, which feed on blood to create eggs, bites of infected females carry the Leishmaniasis parasite. Leishmaniasis epidemiology is influenced by the parasite and sandflies species. The local ecological features of the transmission locations, the human population's historical and present exposure to the infection, and human behavior are common sources. Leishmaniasis parasites have been identified to naturally inhabit about 70 animal species, including humans (see [5]).

With variations in transmission cycles, reservoir hosts, sand fly vectors, clinical symptoms, therapeutic response, and several circulating Leishmaniasis species in the same geographic location, the epidemiology of cutaneous leishmaniasis in the Americas is extremely complex. More than 97% of the VL cases in the region in 2020 were reported in Brazil. In this area, both visceral and cutaneous leishmaniasis are common (see [6]). A total of 199 cases, mostly from Africa and the Americas, were imported in 2020. Leishmaniasis outbreaks, both cutaneous and visceral, are frequently linked to migration and the movement of non-immune people into regions where there are already active transmission cycles. Both widespread deforestation and occupational exposure are still significant issues. Early detection and timely, efficient treatment lower illness prevalence and avoid disability and death. They are also responsible. As well as monitoring the prevalence and burden of disease, it aids in reducing transmission. Although they can be challenging to use, there are currently extremely effective and safe anti-leishmaniasis medications, especially for visceral leishmaniasis. Thanks to a WHO-negotiated price plan and a drug donation program, access to medications has considerably improved. By reducing the number of sand flies, vector management aids in slowing down or stopping the spread of disease. Spraying insecticide, using nets treated with insecticide, managing the environment, and using personal protection are all examples of control strategies. Effective disease surveillance is essential to quickly monitor and act during epidemics and circumstances with high case fatality rates while being treated. Animal reservoir hosts are difficult to control and need to be handled locally. Social mobilization and partnership building: community education and mobilization with successful behavioral change interventions must always be locally tailored. Collaboration and partnership with other vector-borne illness management programs and other stakeholders are essential.

Human illnesses caused by parasites, viruses, and bacteria that are spread by vectors are known as vector-borne diseases. Diseases like malaria, dengue, schistosomiasis, human African trypanosomiasis, leishmaniasis, Chagas disease, yellow fever, Japanese encephalitis, and onchocerciasis cause more than 700,000 deaths annually. These diseases disproportionately afflict the poorest populations and are more prevalent in tropical and subtropical regions. Numerous nations have experienced large epidemics of dengue, malaria, chikungunya, yellow fever, and Zika since 2014, which have affected populations, claimed lives, and taxed health systems. Other illnesses, including leishmaniasis, lymphatic filariasis, and chikungunya, inflict lifelong morbidity, chronic misery, disabilities, and sporadic stigmatization, which are also counted. A complex combination of demographic, environmental, and societal factors affects the distribution of vector-borne diseases. Global travel and trade, unplanned urbanization, and so on. For such discussion, we refer some work as [7–9].

There is a wealth of literature on modeling infectious diseases. It has been stated in various research articles that traditional models have been established. As opposed to statistical models, the main concept of transmission models is a mechanical explanation of the transmission of infection between two individuals. By connecting the individual mechanism of transmission with a population description of the incidence and prevalence of infectious diseases, this mechanistic explanation enables one to mathematically characterize the time evolution of an epidemic. For more details about the said disease see [10–12]. The meticulous mathematical formulation of these connections necessitates a thorough analysis of all the dynamic mechanisms involved in disease transmission. As a result, creating a mathematical model aids in concentrating thought on the crucial processes that shape the epidemiology of infectious diseases and identifies the parameters that have the most influence and are most amenable to control. Thus, mathematical modeling is integrative in that it brings together knowledge from very disparate fields, such as microbiology, the social sciences, and clinical sciences (see [13]). Here in Fig. 1, we present a scenario of the basic reproduction number.

According to the diagram 1, the first generation's infection rate rises by a factor equal to the reproduction rate R_0. During the outbreak, the pool of susceptible people becomes increasingly smaller. The epidemic ends, when the final diseased person fails to spread the disease to any vulnerable individuals.

Here, we remark that recently researchers have worked very well on the area of mathematical models using different concepts. For instance, COVID-19 has been investigated very well by using mathematical models concepts. In this regards large numbers of articles have been published. Here we refer some work like [14–20]. In the same way some researchers have also developed frequently very useful work like [21–23]. In additions, for some more analysis on models and detail application of R_0, we refer here [24,25].

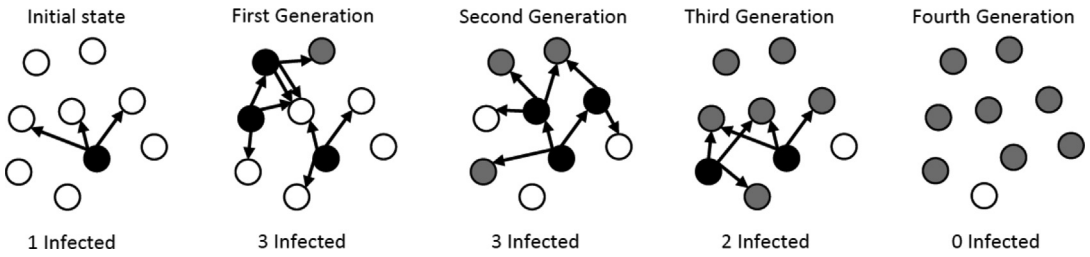

Fig. 1 Diagrammatical scenario of basic reproduction number R_0 in a population.

Since the aforesaid disease is a common threat in numerous societies. Its proper investigation and precautionary measures will help the public and society to save their lives from the mentioned infection. Further, with the help of mathematical models and their analysis, we will be in position to know the transmission mechanism of the diseases and their control procedure. This analysis will help health department in making precautionary measures to save people from catching the infection. Inspired from this we will establish a detailed analysis for the said disease through an updated mathematical model been formulated in the next section.

2. Cutaneous Leishmaniasis disease mathematical model

The following model (1) is considered from [26] for the study in this project by extending with a new class of hospitalization denoted by $H(t)$ as,

$$\begin{aligned}
\tfrac{dS_\hbar}{dt} &= \alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar,\\
\tfrac{dE_\hbar}{dt} &= v\theta_1 I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar,\\
\tfrac{dI_\hbar}{dt} &= \eta_1 E_\hbar - (\gamma_1 + \delta_1 + \Omega)I_\hbar,\\
\tfrac{dR_\hbar}{dt} &= \gamma_1 I_\hbar - \delta_1 R_\hbar,\\
\tfrac{dS_v}{dt} &= \alpha_2 - v\theta_2 I_\hbar S_v - \delta_2 S_v,\\
\tfrac{dE_v}{dt} &= v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v,\\
\tfrac{dI_v}{dt} &= \eta_2 E_v - \delta_2 I_v,
\end{aligned} \tag{1}$$

where, v is the sand fly biting rate, α_1 is the new recruitment ratio of humans to the class at risk, α_2 is the new recruitment ratio of vectors to the class at risk, γ_1 denoted the rate of recovery of the infected human population, δ_1 stands for the natural death rate in the human population. δ_2 denoted the natural death rate in the vector population, θ_1 represents the transfer probability of ACL from sand fly to human, θ_2 represents the CL probability of spread from human to sand fly. Ω stands for the death rate of infected humans due to disease. ρ presents the death rate of exposed humans due to disease. η_1 represents the rate of infection of exposed humans, η_2 is used for the rate of infection of the exposed vector. While the classes, $S_\hbar(t)$ stands for Susceptible human population, $E_\hbar(t)$ Exposed human population, $I_\hbar(t)$ Infected human population, $R_\hbar(t)$ Recovered human population, $S_v(t)$ Susceptible vector population, $E_v(t)$ Exposed vector population, and $I_v(t)$ Infected human population. During the modification of the model (1), the class H denoted the hospitalization density and the parameters γ_2 which is the rate of recovery of infected human population in hospital, and $\hbar$ shows the transfer rate of infected humans to hospital. Therefore, the modified model from (1) is given as

$$\begin{aligned}
\tfrac{dS_\hbar}{dt} &= \alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar,\\
\tfrac{dE_\hbar}{dt} &= v\theta_1 I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar,\\
\tfrac{dI_\hbar}{dt} &= \eta_1 E_\hbar - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar,\\
\tfrac{dH}{dt} &= \hbar I_\hbar - (\delta_1 + \gamma_2)H,\\
\tfrac{dR_\hbar}{dt} &= \gamma_1 I_\hbar + \gamma_2 H - \delta_1 R_\hbar,\\
\tfrac{dS_v}{dt} &= \alpha_2 - v\theta_2 I_\hbar S_v - \delta_2 S_v,\\
\tfrac{dE_v}{dt} &= v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v,\\
\tfrac{dI_v}{dt} &= \eta_2 E_v - \delta_2 I_v.
\end{aligned} \tag{2}$$

Where the initial conditions are described as,

$$\begin{aligned}
S_\hbar(t) > 0, E_\hbar(t) \geqslant 0, I_\hbar(t) \geqslant 0, H(t) \geqslant 0, R_\hbar(t)\\
\geqslant 0, S_v(t) > 0, E_v(t) \geqslant 0, I_v(t) \geqslant 0.
\end{aligned} \tag{3}$$

Some basic properties of the model (2) are investigated including positivity, the existence of equilibrium points, and reproduction number. The existence and uniqueness of the solution of the model under consideration are also investigated. Local stability and global stability analysis of equilibrium points are also studied. Here we will follow the procedures given in [27–31]. For the required results, we use the Lyapunov function the third additive compound matrix technique. Researchers have used some analysis to deal various real world problems of models on the same way like in [32–38]. Sensitivity Analysis is also investigated by using some tools from the Numerical functional analysis. Numerical analysis of the proposed model is performed nonstandard finite difference scheme. Moreover, for the justification of our results, we give a graphical presentation of the model for each class in the model. The mentioned tool has been used in [39–45]. Also, we present some graphical presentations related to the sensitivity analysis along with the tables for its sensitivity indices.

The paper is organized as: Section first is devoted to introduction. Section 2 is related to formulation of the model. Section 3 is connected with basic results. Section 4 is devoted to the existence theory and stability analysis. Further, in Section 5, we give global stability analysis. In addition, Section 6 is related to sensitivity analysis. Also, the numerical scheme and its implementation is performed in Section 7. Further, last section is devoted to conclusion.

3. Basic properties of Leishmaniasis model

The dynamics of total individuals and vector population is represented as:

$$\frac{dN_\hbar}{dt} = \alpha_1 - \delta_1 N_\hbar - \frac{\Omega}{\delta_1} I_\hbar - \frac{\rho}{\delta_1} E_\hbar \leqslant \alpha_1 - \delta_1 N_\hbar. \tag{4}$$

The above inequality lead to $N_\hbar \to \alpha_1/\delta_1$ as $t \to \infty$. Next

$$\frac{dN_v}{dt} = \alpha_2 - \delta_2 N_v. \tag{5}$$

Biologically, feasible region Δ is given by

$$\Delta = \left\{ (S_\hbar, E_\hbar, I_\hbar, H, R_\hbar, S_v, E_v, I_v) \in R^{\not\infty}, N_\hbar \leqslant \frac{\alpha_1}{\delta_1}, N_v \leqslant \frac{\alpha_2}{\delta_2} \right\}.$$

From the model (2) and (4)–(5), we get

$$\limsup_{t\to\infty} N_\hbar = \frac{\alpha_1}{\delta_1}, \text{and} \quad \limsup_{t\to\infty} N_v = \frac{\alpha_2}{\delta_2}. \tag{6}$$

When $t \to \infty$, also depicts the wellposedness of the model (2) with positively invariant domain Δ.

Lemma 3.1. The orthant R^8_+ is invariant positively for model (2).

Proof. Suppose, $z = (S_\hbar, E_\hbar, I_\hbar, H, R_\hbar, S_v, E_v, I_v)^T$, and assume that

$$\ell = \begin{pmatrix} -\delta_1 - I_v^* v\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & -S_\hbar^* v\theta_1 \\ I_v^* v\theta_1 & -\delta_1 - \eta_1 - \rho & 0 & 0 & 0 & 0 & 0 & S_\hbar^* v\theta_1 \\ 0 & \eta_1 & -\Omega - \delta_1 - \gamma_1 - \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hbar & -\delta_1 - \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_1 & \gamma_2 & -\delta_1 & 0 & 0 & 0 \\ 0 & 0 & -S_v^* v\theta_2 & 0 & 0 & -\delta_2 - I_\hbar^* v\theta_2 & 0 & 0 \\ 0 & 0 & S_v^* v\theta_2 & 0 & 0 & I_\hbar^* v\theta_2 & -\delta_2 - \eta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta_2 & -\delta_2 \end{pmatrix}, \tag{7}$$

and

$$b = \begin{pmatrix} v\theta_1 S_\hbar^* - v\theta_1 S_\hbar^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v\theta_2 S_v^* - v\theta_2 S_v^* \end{pmatrix}, \tag{8}$$

satisfying the following equation,

$$\frac{dx}{dt} = \ell z + b. \tag{9}$$

As we observed that the Metzler matrix (ℓ) has non–negative entries on its off–diagonal and $b \geqslant 0$. Thus, this concludes that the model (2) is positively invariant in R^8_+.

Lemma 3.2. If solutions of the model (2) exist, then under the conditions (3) they are positive for $t > 0$.

Proof. Assume that, solutions to model (2) exist in $\Pi, \forall t \in \Pi \subset [0, \infty)$. Furthermore, consider the second equation of the model (2) with solution in the following form:

$$\begin{aligned} E_\hbar(t) &= E_\hbar(0)\exp\left(-[(\eta_1+\rho+\delta_1)E_\hbar]t\right) + \exp\left(-[(\eta_1+\rho+\delta_1)E_\hbar]t\right) \\ &\quad \times \int_0^t v\theta_1 I_v(x) S_\hbar(x) \exp\left((\eta_1+\rho+\delta_1)x\right)dx. \end{aligned} \tag{10}$$

Similarly, the next equation of the model (2):

$$\begin{aligned} I_\hbar(t) &= I_\hbar(0)\exp\left(-[(\gamma_1+\delta_1+\Omega+\hbar)I_\hbar]t\right) + \exp\left(-[(\gamma_1+\delta_1+\Omega+\hbar)I_\hbar]t\right) \\ &\quad \times \int_0^t \eta_1 E_\hbar(y) \exp\left((\gamma_1+\delta_1+\Omega+\hbar)y\right)dy. \end{aligned} \tag{11}$$

So, from Eqs. (10) and (11) we observe strictly positive solution to model (2). In the same manner, we can obtain the non–negative solutions to $S_\hbar, H, R_\hbar, S_v, E_v, I_v$.

3.1. Existence of Equilibrium Points and Calculation of Basic Reproduction Number $\boldsymbol{R_0}$

Suppose, in the model (2) we set $dS_\hbar/dt = dE_\hbar/dt = dI_\hbar/dt = dH/dt = dR_\hbar/dt = dS_v/dt = dE_v/dt = dI_v/dt = 0$. Then the model becomes:

$$\alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar = 0, \tag{12}$$
$$v\theta_1 I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar = 0, \tag{13}$$
$$\eta_1 E_\hbar - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar = 0, \tag{14}$$
$$\hbar I_\hbar - (\delta_1 + \gamma_2)H = 0, \tag{15}$$
$$\gamma_1 I_\hbar + \gamma_2 H - \delta_1 R_\hbar = 0, \tag{16}$$
$$\alpha_2 - v\theta_2 I_\hbar S_v - \delta_2 S_v = 0, \tag{17}$$
$$v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v = 0, \tag{18}$$
$$\eta_2 E_v - \delta_2 I_v = 0. \tag{19}$$

Furthermore, let there is no infection in the population, therefore we neglect the infected compartments in the model, thus the system of equations (12)–(19) becomes:

$$\alpha_1 - \delta_1 S_\hbar^{\star} = 0, \tag{20}$$
$$\alpha_2 - \delta_2 S_v^{\star} = 0. \tag{21}$$

Next, from Eq. (20) implies that:

$$S_\hbar^{\star} = \frac{\alpha_1}{\delta_1}. \tag{22}$$

While, from Eq. (21) we obtain

$$S_v^{\star} = \frac{\alpha_2}{\delta_2}. \tag{23}$$

Hence, the positive disease–free equilibrium point is given by:

$$\mathscr{E}_0 = \left(\frac{\alpha_1}{\delta_1}, 0, 0, 0, 0, 0, 0, \frac{\alpha_2}{\delta_2}\right). \tag{24}$$

Furthermore, let along with infection, consider the whole system of Eqs. (12)–(19) and calculating the endemic equilibrium point $\mathscr{E}^*$ by adding Eqs. 12,13, we obtain:

$$\alpha_1 - \delta_1 S_\hbar^* - (\eta_1 + \rho + \delta_1)E_\hbar^* = 0, \tag{25}$$

which implies that

$$S_\hbar^* = \frac{\alpha_1 - (\eta_1 + \rho + \delta_1)E_\hbar^*}{\delta_1}. \tag{26}$$

From Eq. (14), we have

$$\eta_1 E_\hbar^* - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar^* = 0, \tag{27}$$

which implies that

$$I_\hbar^* = \frac{\eta_1}{(\gamma_1 + \delta_1 + \Omega + \hbar)}E_\hbar^*. \tag{28}$$

From the Eq. (15):

$$\hbar I_\hbar^* - (\delta_1 + \gamma_2)H^* = 0 \tag{29}$$

implies that

$$H^* = \frac{\hbar}{\delta_1 + \gamma_2}I^*. \tag{30}$$

Furthermore, from Eqs. (28) and (30), one has

$$H^* = \frac{\hbar\eta_1}{(\delta_1 + \gamma_2)(\gamma_1 + \delta_1 + \Omega + \hbar)}E_\hbar^*. \tag{31}$$

From Eq. (16), we have

$$\gamma_1 I_\hbar^* + \gamma_2 H^* - \delta_1 R_\hbar^* = 0, \tag{32}$$

which yields

$$R_\hbar^* = \frac{\gamma_1}{\delta_1}I_\hbar^* + \gamma_2 H^*. \tag{33}$$

Now, using Eqs. (28), and (30) in Eq. (34), we get

$$R_\hbar^* = \frac{\gamma_1\eta_1}{\delta_1(\gamma_1 + \delta_1 + \Omega + \hbar)}E_\hbar^* + \frac{\gamma_2\hbar\eta_1}{\delta_1(\delta_1 + \gamma_2)(\gamma_1 + \delta_1 + \Omega + \hbar)}E_\hbar^*. \tag{34}$$

Next, we add Eqs. (17) and (18) to get

$$\alpha_2 - \delta_2 S_v^* - (\delta_2 + \eta_2)E_v^* = 0, \tag{35}$$

which implies that

$$S_v^* = \frac{\alpha_2}{\delta_2} - \frac{(\delta_2 + \eta_2)}{\delta_2}E_v^*. \tag{36}$$

Eventually, from Eq. (19), we have

$$I_v^* = \frac{\eta_2}{\delta_2}E_v^*. \tag{37}$$

Hence, the positive disease–endemic equilibrium point $\mathscr{E}^*$ is given by:

$$\mathscr{E}^* = \left(S_\hbar^*, E_\hbar^*, I_\hbar^*, H^*, R_\hbar^*, S_v^*, E_v^*, I_v^*\right), \tag{38}$$

where

$$\begin{aligned}
S_\hbar^* &= \tfrac{\alpha_1}{\delta_1} - \tfrac{(\eta_1+\rho+\delta_1)E_\hbar^*}{\delta_1},\\
I_\hbar^* &= \tfrac{\eta_1}{(\gamma_1+\delta_1+\Omega+\hbar)}E_\hbar^*,\\
H^* &= \tfrac{\hbar\eta_1}{(\delta_1+\gamma_2)(\gamma_1+\delta_1+\Omega+\hbar)}E_\hbar^*,\\
R_\hbar^* &= \tfrac{\gamma_1\eta_1}{\delta_1(\gamma_1+\delta_1+\Omega+\hbar)}E_\hbar^* + \tfrac{\gamma_2\hbar\eta_1}{\delta_1(\delta_1+\gamma_2)(\gamma_1+\delta_1+\Omega+\hbar)}E_\hbar^*,\\
S_v^* &= \tfrac{\alpha_2}{\delta_2} - \tfrac{(\delta_2+\eta_2)}{\delta_2}E_v^*,\\
I_v^* &= \tfrac{\eta_2}{\delta_2}E_v^*.
\end{aligned}$$

To calculate the basic reproduction number R_0, we consider the infected compartments in the model (2), such that

$$\begin{aligned}
\breve{f}_1 &= v\theta_1 I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar,\\
\breve{f}_2 &= \eta_1 E_\hbar - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar,\\
\breve{f}_3 &= \hbar I_\hbar - (\delta_1 + \gamma_2)H,\\
\breve{f}_4 &= v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v,\\
\breve{f}_5 &= \eta_2 E_v - \delta_2 I_v.
\end{aligned} \tag{39}$$

Now, we take the non–linear terms from system (39) as f while linear terms as v such that:

$$f = \begin{pmatrix} v\theta_1 I_v S_\hbar \\ 0 \\ 0 \\ v\theta_2 I_\hbar S_v \\ 0 \end{pmatrix}, \text{and} \quad v = \begin{pmatrix} (\eta_1 + \rho + \delta_1)E_\hbar \\ (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar - \eta_1 E_\hbar \\ (\delta_1 + \gamma_2)H \\ (\delta_2 + \eta_2)E_v \\ \delta_2 I_v - \eta_2 E_v \end{pmatrix} \tag{40}$$

satisfying

$$\frac{dx}{dt} = f - v. \tag{41}$$

Next, the Jacobian matrices of f and v are given by:

$$\mathsf{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & S_\hbar^{\star} v\theta_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S_v^{\star} v\theta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{and}$$

$$V = \begin{pmatrix} \delta_1 + \eta_1 + \rho & 0 & 0 & 0 & 0 \\ -\eta_1 & \Omega + \delta_1 + \gamma_1 + \hbar & 0 & 0 & 0 \\ 0 & -\hbar & \delta_1 + \gamma_2 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 + \eta_2 & 0 \\ 0 & 0 & 0 & -\eta_2 & \delta_2 \end{pmatrix}. \tag{42}$$

Also

$$V^{-1} = \begin{pmatrix} v_{11} & 0 & 0 & 0 & 0 \\ v_{21} & \frac{1}{\Omega+\delta_1+\gamma_1+\hbar} & 0 & 0 & 0 \\ v_{31} & \frac{\hbar}{(\delta_1+\gamma_2)(\Omega+\delta_1+\gamma_1+\hbar)} & \frac{1}{\delta_1+\gamma_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\delta_2+\eta_2} & 0 \\ 0 & 0 & 0 & \frac{\eta_2}{\delta_2(\delta_2+\eta_2)} & \frac{1}{\delta_2} \end{pmatrix}, \text{and } e$$

$$mG = \begin{pmatrix} 0 & 0 & 0 & \frac{S_\hbar^{\star}\eta_2 v\theta_1}{\delta_2(\delta_2+\eta_2)} & \frac{S_\hbar^{\star} v\theta_1}{\delta_2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ g_{31} & \frac{S_v^{\star} v\theta_2}{\Omega+\delta_1+\gamma_1+\hbar} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} v_{11} &= 1/(\delta_1+\eta_1+\rho),\ v_{21} = \eta_1/((\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)),\\ v_{31}+ &\ \eta_1\hbar/((\delta_1+\gamma_2)(\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)),\\ g_{31} &= S_v^{\star}\eta_1 v\theta_2/((\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)). \end{aligned}$$

The non–zero eigenvalues of G are given by:

$$\begin{aligned} \breve{\lambda}_1 &= v\sqrt{\frac{\theta_1\eta_1 S_{\hbar}^{\star}\theta_2\eta_2 S_v^{\star}}{\delta_2(\delta_2+\eta_2)(\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)}},\\ \breve{\lambda}_2 &= -v\sqrt{\frac{\theta_1\eta_1 S_{\hbar}^{\star}\theta_2\eta_2 S_v^{\star}}{\delta_2(\delta_2+\eta_2)(\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)}}. \end{aligned} \tag{44}$$

So, the spectral radius of the next generation matrix is given by $R_0 = \rho(G)$, that is

$$R_0 = v\sqrt{\frac{\theta_1\eta_1 S_{\hbar}^{\star}\theta_2\eta_2 S_v^{\star}}{\delta_2(\delta_2+\eta_2)(\delta_1+\eta_1+\rho)(\Omega+\delta_1+\gamma_1+\hbar)}}, \tag{45}$$

where, $S_{\hbar}^{\star}$ and $S_v^{\star}$ are given as

$$S_{\hbar}^{\star} = \frac{\alpha_1}{\delta_1}, \text{and} \quad S_v^{\star} = \frac{\alpha_2}{\delta_2}. \tag{46}$$

Thus, basic reproduction number R_0 is obtained in Eq. (45).

We conclude the above discussion for uniqueness of the concerned equilibrium points by following [21,22] as.

Remark 3.3.

- The disease-free equilibrium $\mathscr{E}_0$ will be a unique biological feasible steady state if and only if $R_0 = 1$ and $Rc > 1$;
- a co-existence between the disease-free equilibrium $\mathscr{E}_0$ and two endemic equilibrium points as the only possible three steady states if $R_0 < 1$, and $Rc < 1$;
- a co-existence between the disease-free equilibrium $\mathscr{E}_0$ and an endemic equilibrium as the only two steady states if $R_0 = 1$ and $Rc > 1$;
- a co-existence between the disease-free equilibrium $\mathscr{E}_0$ and an endemic equilibrium point as the two only possible steady states if $R_0 > 1$.

4. Existence, uniqueness of the model and local stability analysis

Using the Theorem (9) from [27], also one can read [28], we prove the existence and uniqueness of the model (2), such that we consider the model (2) and re–write in the form of $\dot{z} = g(z)$.

$$\dot{z} = \begin{pmatrix} g_1(t,z)\\ g_2(t,z)\\ g_3(t,z)\\ g_4(t,z)\\ g_5(t,z)\\ g_6(t,z)\\ g_7(t,z)\\ g_8(t,z) \end{pmatrix} = \begin{pmatrix} \alpha_1 - v\theta_1 I_v S_{\hbar} - \delta_1 S_{\hbar}\\ v\theta_1 I_v S_{\hbar} - (\eta_1+\rho+\delta_1)E_{\hbar}\\ \eta_1 E_{\hbar} - (\gamma_1+\delta_1+\Omega+\hbar)I_{\hbar}\\ \hbar I_{\hbar} - (\delta_1+\gamma_2)H\\ \gamma_1 I_{\hbar} + \gamma_2 H - \delta_1 R_{\hbar}\\ \alpha_2 - v\theta_2 I_{\hbar} S_v - \delta_2 S_v\\ v\theta_2 I_{\hbar} S_v - (\delta_2+\eta_2)E_v\\ \eta_2 E_v - \delta_2 I_v \end{pmatrix}. \tag{47}$$

Taking the first partial derivatives w.r.t the state variables of $\dot{z}$, which is given by:

$$\frac{\partial}{\partial p}\dot{z} = \frac{\partial}{\partial p}g(z), \tag{48}$$

where p is used in general implies that

$$\frac{\partial}{\partial p}g(z) = \frac{\partial}{\partial p}g(t,z) = \begin{pmatrix} \frac{\partial}{\partial S_{\hbar}}g(t,z)\\ \frac{\partial}{\partial E_{\hbar}}g(t,z)\\ \frac{\partial}{\partial I_{\hbar}}g(t,z)\\ \frac{\partial}{\partial H}g(t,z)\\ \frac{\partial}{\partial R_{\hbar}}g(t,z)\\ \frac{\partial}{\partial S_v}g(t,z)\\ \frac{\partial}{\partial E_v}g(t,z)\\ \frac{\partial}{\partial I_v}g(t,z) \end{pmatrix} = \begin{pmatrix} -v\theta_1 I_v - \delta_1\\ -(\eta_1+\rho+\delta_1)\\ -(\gamma_1+\delta_1+\Omega+\hbar)\\ -(\delta_1+\gamma_2)\\ -\delta_1\\ -v\theta_2 I_{\hbar} - \delta_2\\ -(\delta_2+\eta_2)\\ -\delta_2 \end{pmatrix}. \tag{49}$$

Clearly, we see that the partial derivatives are continues such that

$$\frac{\partial g}{\partial p} \to \text{continuousin}\mathbb{R}^n. \tag{50}$$

Thus, according to [[27]: Theorem 9], there exists a unique continuous solution to the model (2).

Theorem 4.1. The Leishmaniasis disease model (2) is locally asymptotically stable at disease-free equilibrium $E^{\star}$, when $R_0 < 1, \Delta > 0, \Delta_2 > 0$ and $\Delta_3 > 0$, otherwise unstable.

Proof. The Jacobian matrix $J(\mathscr{E}^{\star})$ of the model (2) at disease–free equilibrium point $E^{\star}$ is given by

$$J(\mathscr{E}^{\star}) = \begin{pmatrix} -\delta_1 & 0 & 0 & 0 & 0 & 0 & 0 & -S_{\hbar}^{\star}v\theta_1\\ 0 & -\delta_1-\eta_1-\rho & 0 & 0 & 0 & 0 & 0 & S_{\hbar}^{\star}v\theta_1\\ 0 & \eta_1 & -\Omega-\delta_1-\gamma_1-\hbar & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \hbar & -\delta_1-\gamma_2 & 0 & 0 & 0 & 0\\ 0 & 0 & \gamma_1 & \gamma_2 & -\delta_1 & 0 & 0 & 0\\ 0 & 0 & -S_v^{\star}v\theta_2 & 0 & 0 & -\delta_2 & 0 & 0\\ 0 & 0 & S_v^{\star}v\theta_2 & 0 & 0 & 0 & -\delta_2-\eta_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \eta_2 & -\delta_2 \end{pmatrix}. \tag{51}$$

The characteristic equation of the Jacobian matrix (51) in $\breve{\lambda}$ is given by:

$$(\breve{\lambda}+\delta_1)(\breve{\lambda}+\delta_2)(\breve{\lambda}+\delta_1)(\breve{\lambda}+\delta_1+\gamma_2)(\breve{\lambda}^4+\breve{\sigma}_{11}\breve{\lambda}^3+\breve{\sigma}_{12}\breve{\lambda}^2 + \breve{\sigma}_{13}\breve{\lambda}+\breve{\sigma}_{14})=0. \tag{52}$$

where, the coefficients are given with some simplifications $Q_1=(\delta_2+\eta_2), Q_2=(\delta_1+\eta_1+\rho), \quad Q_3=(\Omega+\delta_1+\gamma_1+\hbar)$, and $Q_4=(\Omega+2\delta_1+2\delta_2+\eta_1+\eta_2+\gamma_1+\hbar+\rho)$.

$$\breve{\sigma}_{11}=Q_4, \tag{53}$$
$$a_{12}=Q_1Q_2+\delta_2Q_2+Q_1Q_3+\delta_2Q_3+(Q_2Q_3+\delta_2Q_1, \tag{54}$$
$$\breve{\sigma}_{13}=Q_1Q_2Q_3+\delta_2Q_2Q_3+\delta_2Q_1Q_2+\delta_2Q_1Q_3-a_{31}\eta_2v\theta_1S_{\hbar}^{\star}, \tag{55}$$
$$\breve{\sigma}_{14}=-S_v^{\star}\eta_1\eta_2\theta_1\theta_2v^2S_{\hbar}^{\star}-a_{31}\eta_2\theta_1Q_3vS_{\hbar}^{\star}+\delta_2Q_1Q_2Q_3. \tag{56}$$

According to Routh-Hurwitz criterion [29–31], $\breve{\sigma}_{11}>0, \breve{\sigma}_{12}>0, \breve{\sigma}_{13}>0$ and $\breve{\sigma}_{14}>0$, also

$$\Delta_1>0, \quad \Delta_2=\begin{vmatrix}\breve{\sigma}_{11} & 1\\ \breve{\sigma}_{13} & \breve{\sigma}_{12}\end{vmatrix}>0, \text{and} \quad \Delta_3=\begin{vmatrix}\breve{\sigma}_{11} & 1 & 0\\ \breve{\sigma}_{13} & \breve{\sigma}_{12} & \breve{\sigma}_{11}\\ \breve{\sigma}_{15} & \breve{\sigma}_{14} & \breve{\sigma}_{13}\end{vmatrix}>0. \tag{57}$$

Furthermore

$$\begin{aligned}\Delta_2 &= \breve{\sigma}_{11}\breve{\sigma}_{12}-\breve{\sigma}_{13},\\ &= Q_4(Q_1Q_2+\delta_2Q_2+Q_1Q_3+\delta_2Q_3+Q_2Q_3+\delta_2Q_1)-Q_1Q_2Q_3-\delta_2Q_2Q_3-\delta_2Q_1Q_2-\delta_2Q_1Q_3.\end{aligned} \tag{58}$$

$$\begin{aligned}\breve{G}_1 &= (Q_1Q_2Q_3+\delta_2Q_2Q_3+\delta_2Q_1Q_2+\delta_2Q_1Q_3)(Q_1Q_2+\delta_2Q_2+Q_1Q_3+\delta_2Q_3+Q_2Q_3+\delta_2Q_1)\\ &\quad +S_hS_v\eta_1\eta_2v^2\theta_1\theta_2Q_4,\\ \breve{G}_2 &= -\delta_2Q_1Q_2Q_3Q_4.\end{aligned} \tag{63}$$

This implies that

$$\breve{G}=\breve{G}_1-\breve{G}_2, \tag{64}$$

such that, $\breve{\sigma}_{12}\breve{\sigma}_{13}-\breve{\sigma}_{14}\breve{\sigma}_{11}>0$, if $\breve{G}_1>\breve{G}_2$, while, $\breve{\sigma}_{11}\breve{\sigma}_{15}-\breve{\sigma}_{13}^2>0$, if $Q_1Q_2Q_3+\delta_2Q_2Q_3+\delta_2Q_1Q_2+\delta_2Q_1Q_3<0$. Thus, the eigenvalues $\breve{\lambda}_1=-\delta_1, \breve{\lambda}_2=-\delta_1, \breve{\lambda}_3=-\delta_2$, and $\breve{\lambda}_4=-Q_1$ are negative while according to Routh-Hurwitz theorem the rest of the eigenvalues of the Jacobian matrix (51) are negative and hence the model (2) is locally asymptotically stable around disease–free equilibrium point $E^{\star}$.

Theorem 4.2. The Leishmaniasis disease model (2) is locally asymptotically stable at disease-endemic equilibrium E^*, when $R_0>1, \Delta_2>0, \Delta_3>0$, and $\Delta_4>0$, otherwise unstable.

Proof. The Jacobian matrix $J(\mathscr{E}^*)$ of the model (2) at disease–endemic equilibrium point E^* is given by:

The characteristics equation of the matrix (65) is given by

$$J(\mathscr{E}^*)=\begin{pmatrix}-\delta_1-I_v^*v\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & -S_{\hbar}^*v\theta_1\\ I_v^*v\theta_1 & -\delta_1-\eta_1-\rho & 0 & 0 & 0 & 0 & 0 & S_{\hbar}^*v\theta_1\\ 0 & \eta_1 & -\Omega-\delta_1-\gamma_1-\hbar & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \hbar & -\delta_1-\gamma_2 & 0 & 0 & 0 & 0\\ 0 & 0 & \gamma_1 & \gamma_2 & -\delta_1 & 0 & 0 & 0\\ 0 & 0 & -S_v^*v\theta_2 & 0 & 0 & -\delta_2-I_{\hbar}^*v\theta_2 & 0 & 0\\ 0 & 0 & S_v^*v\theta_2 & 0 & 0 & I_{\hbar}^*v\theta_2 & -\delta_2-\eta_2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \eta_2 & -\delta_2\end{pmatrix}. \tag{65}$$

Let

$$\begin{aligned}\breve{F}_1 &= Q_4((Q_1Q_2+\delta_2Q_2+Q_1Q_3+\gamma_1+\hbar)+\delta_2Q_3+Q_2Q_3+\delta_2Q_1),\\ \breve{F}_2 &= -Q_1Q_2Q_3-\delta_2Q_2Q_3-\delta_2Q_1Q_2-\delta_2Q_1Q_3.\end{aligned} \tag{59}$$

This implies that

$$\breve{F}=\breve{F}_1-\breve{F}_2. \tag{60}$$

Here, we see that $\breve{F}>0$, if $\breve{F}_1>\breve{F}_2$. Next, $\Delta_3>0$, then $\breve{\sigma}_{11}(\breve{\sigma}_{12}\breve{\sigma}_{13}-\breve{\sigma}_{14}\breve{\sigma}_{11})+(\breve{\sigma}_{11}\breve{\sigma}_{15}-\breve{\sigma}_{13}^2)>0$

$$\begin{aligned}\breve{\sigma}_{12}\breve{\sigma}_{13}-\breve{\sigma}_{14}\breve{\sigma}_{11} &= (Q_1Q_2Q_3+\delta_2Q_2Q_3+\delta_2Q_1Q_2+\delta_2Q_1Q_3)(Q_1Q_3+\delta_2Q_2\\ &\quad +(Q_1Q_3+\delta_2Q_3+Q_2Q_3+\delta_2Q_1)-Q_4(\delta_2Q_1Q_2Q_3-S_hS_v\eta_1\eta_2v^2\theta_1\theta_2),\end{aligned} \tag{61}$$

and

$$\breve{\sigma}_{11}\breve{\sigma}_{15}-\breve{\sigma}_{13}^2 = -(Q_1Q_2Q_3+\delta_2Q_2Q_3+\delta_2Q_1Q_2+\delta_2Q_1Q_3)^2. \tag{62}$$

Let

$$(\breve{\lambda}+\delta_1)(\breve{\lambda}_2+\delta_1+\gamma_2)(\breve{\lambda}^6+\breve{b}_{11}\breve{\lambda}^5+\breve{b}_{12}\breve{\lambda}^4+\breve{b}_{13}\breve{\lambda}^3+\breve{b}_{14}\breve{\lambda}^2 + \breve{b}_{15}\breve{\lambda}+\breve{b}_{16})=0. \tag{66}$$

Where, the coefficients are given with some simplifications $Q_1=(\delta_2+\eta_2), Q_2=(\delta_1+\eta_1+\rho)$,
$Q_3=(\Omega+\delta_1+\gamma_1+\hbar), \sigma_1=\delta_1+I_v^*v\theta_1$, and $\sigma_2=\delta_2-I_{\hbar}^*v\theta_2$ as

$$\begin{aligned}\breve{b}_{11} &= \Omega-\delta_1-\delta_2-\eta_1-\eta_2+\gamma_1+\hbar-\rho-I_{\hbar}^*v\theta_2-I_v^*v\theta_1,\\ \breve{b}_{12} &= \delta_2\sigma_1-\delta_2\sigma_2+Q_1Q_2-\sigma_2Q_2+\sigma_1Q_2+\delta_2Q_2-Q_1Q_3+\sigma_2Q_3-\sigma_1Q_3\\ &\quad -\delta_2Q_3-Q_1\sigma_2+Q_1\sigma_1-Q_2Q_3+\delta_2Q_1-\sigma_2\sigma_1.\end{aligned} \tag{67}$$

$$\begin{aligned}\breve{b}_{13} &= Q_1Q_2Q_3+Q_1\sigma_2\sigma_1-\sigma_2Q_2Q_3+\sigma_1Q_2Q_3+\delta_2Q_1\sigma_2-\delta_2Q_1\sigma_1+\delta_2Q_2Q_3\\ &\quad +\delta_2\sigma_2\sigma_1+Q_1\sigma_2Q_2-Q_1\sigma_1Q_2-\delta_2Q_1Q_2+\sigma_2\sigma_1Q_2+\delta_2\sigma_2Q_2-\delta_2\sigma_1Q_2\\ &\quad -Q_1\sigma_2Q_3+Q_1\sigma_1Q_3+\delta_2Q_1Q_3-\sigma_2\sigma_1Q_3-\delta_2\sigma_2Q_3+\delta_2\sigma_1Q_3\end{aligned} \tag{68}$$

$$\begin{aligned}\breve{b}_{14} &= Q_1\sigma_2\sigma_1Q_3 + \delta_2Q_1\sigma_2Q_3 - \delta_2Q_1\sigma_1Q_3 + \delta_2\sigma_2\sigma_1Q_3 + Q_1\sigma_2Q_2Q_3\\ &-Q_1\sigma_1Q_2Q_3 - \delta_2Q_1Q_2Q_3 + \sigma_2\sigma_1Q_2Q_3 - \delta_2Q_1\sigma_2\sigma_1 + \delta_2\sigma_2Q_2Q_3 - \delta_2\sigma_1Q_2Q_3\\ &-Q_1\sigma_2\sigma_1Q_2 - \delta_2Q_1\sigma_2Q_2 + \delta_2Q_1\sigma_1Q_2 - \delta_2\sigma_2\sigma_1Q_2 - S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^2\theta_1\theta_2.\end{aligned} \tag{69}$$

$$\begin{aligned}\breve{b}_{15} &= \delta_2Q_1\sigma_2\sigma_1Q_2 - \delta_2Q_1\sigma_2\sigma_1Q_3 - Q_1\sigma_2\sigma_1Q_2Q_3 - \delta_2Q_1\sigma_2Q_2Q_3 + \delta_2Q_1\sigma_1Q_2Q_3\\ &-\delta_2\sigma_2\sigma_1Q_2Q_3 + I_{\hbar}^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^3\theta_1\theta_2^2 + I_v^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^3\theta_1^2\theta_2 - S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^2\theta_1\theta_2\sigma_2\\ &+S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^2\theta_1\theta_2\sigma_1\end{aligned} \tag{70}$$

$$\begin{aligned}\breve{b}_{16} &= \delta_2Q_1\sigma_2\sigma_1Q_2Q_3 + S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^2\theta_1\theta_2\sigma_2\sigma_1 - I_{\hbar}^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^3\theta_1\theta_2^2\sigma_1\\ &+I_v^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^3\theta_1^2\theta_2\sigma_2 - I_{\hbar}^{*}I_v^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^4\theta_1^2\theta_2^2\end{aligned} \tag{71}$$

According to Routh-Hurwitz criterion [29,30], $\breve{b}_{11} > 0, \breve{b}_{12} > 0, \breve{b}_{13}, \breve{b}_{14} > 0, \breve{b}_{15} > 0$ and $\breve{b}_{16} > 0$ also,

$$\Delta_2 = \begin{vmatrix} \breve{b}_{11} & 1 \\ \breve{b}_{13} & \breve{b}_{12} \end{vmatrix} > 0, \quad \Delta_3 = \begin{vmatrix} \breve{b}_{11} & 1 & 0 \\ \breve{b}_{13} & \breve{b}_{12} & \breve{b}_{11} \\ \breve{b}_{15} & \breve{b}_{14} & \breve{b}_{13} \end{vmatrix} > 0, \text{and}$$

$$\Delta_4 = \begin{vmatrix} \breve{b}_{11} & 1 & 0 & 0 \\ \breve{b}_{13} & \breve{b}_{12} & \breve{b}_{11} & 1 \\ \breve{b}_{15} & \breve{b}_{14} & \breve{b}_{13} & \breve{b}_{12} \\ 0 & \breve{b}_{16} & \breve{b}_{15} & \breve{b}_{13} \end{vmatrix} > 0. \tag{72}$$

Furthermore

$$\begin{aligned}\Delta_2 &= \breve{b}_1\breve{b}_2 - \breve{b}_3,\\ &= -Q_1^2Q_2 + Q_1^2Q_3 - Q_1^2\delta_2 - Q_1^2\sigma_1 + Q_1^2\sigma_2 - Q_1Q_2^2 + 2Q_1Q_2Q_3 - 2Q_1Q_2\delta_2\\ &-2Q_1Q_2\sigma_1 + 2Q_1Q_2\sigma_2 - Q_1Q_3^2 + 2Q_1Q_3\delta_2 + 2Q_1Q_3\sigma_1 - 2Q_1Q_3\sigma_2 - Q_1\delta_2^2 - 2Q_1\delta_2\sigma_1\\ &+2Q_1\delta_2\sigma_2 - Q_1\sigma_1^2 + 2Q_1\sigma_1\sigma_2 - Q_1\sigma_2^2 + Q_2^2Q_3 - Q_2^2\delta_2 - Q_2^2\sigma_1 + Q_2^2\sigma_2\\ &-Q_2Q_3^2 + 2Q_2Q_3\delta_2 + 2Q_2Q_3\sigma_1 - 2Q_2Q_3\sigma_2 - Q_2\delta_2^2 - 2Q_2\delta_2\sigma_1 + 2Q_2\delta_2\sigma_2 - Q_2\sigma_1^2\\ &+2Q_2\sigma_1\sigma_2 - Q_2\sigma_2^2 - Q_3^2\delta_2 - Q_3^2\sigma_1 + Q_3^2\sigma_2 + Q_3\delta_2^2 + 2Q_3\delta_2\sigma_1 - 2Q_3\delta_2\sigma_2\\ &+Q_3\sigma_1^2 - 2Q_3\sigma_1\sigma_2 + Q_3\sigma_2^2 - \delta_2^2\sigma_1 + \delta_2^2\sigma_2 - \delta_2\sigma_1^2 + 2\delta_2\sigma_1\sigma_2\\ &-\delta_2\sigma_2^2 + \sigma_1^2\sigma_2 - \sigma_1\sigma_2^2.\end{aligned} \tag{73}$$

Furthermore, assume that

$$\begin{aligned}\breve{H}_1 &= Q_1^2Q_3 + Q_1^2\sigma_2 + 2Q_1Q_2\sigma_2 + 2Q_1Q_3\sigma_1 + 2Q_1Q_3\delta_2 + 2Q_1\delta_2\sigma_2 + Q_2^2Q_3 + 2Q_1\sigma_1\sigma_2\\ &+Q_2^2\sigma_2 + 2Q_2Q_3\delta_2 + 2Q_2Q_3\sigma_1 + 2Q_2\delta_2\sigma_2 + 2Q_2\sigma_1\sigma_2 + Q_3^2\sigma_2 + Q_3\delta_2^2 + 2Q_3\delta_2\sigma_1\\ &+2\delta_2\sigma_1\sigma_2 + \delta_2^2\sigma_2 + Q_3\sigma_1^2 + Q_3\sigma_2^2 + \sigma_1^2\sigma_2 + 2Q_1Q_2Q_3,\end{aligned} \tag{74}$$

$$\begin{aligned}\breve{H}_2 &= -Q_1^2Q_2 - Q_1^2\delta_2 - Q_1^2\sigma_1 - Q_1Q_2^2 - 2Q_1Q_2\delta_2 - 2Q_1Q_2\sigma_1 - Q_1Q_3^2 - 2Q_1Q_3\sigma_2 - Q_1\delta_2^2\\ &-2Q_1\delta_2\sigma_1 - Q_1\sigma_1^2 - Q_1\sigma_2^2 - Q_2^2\delta_2 - Q_2^2\sigma_1 - Q_2Q_3^2 - 2Q_2Q_3\sigma_2 - Q_2\delta_2^2 - 2Q_2\delta_2\sigma_1\\ &-Q_2\sigma_1^2 - Q_2\sigma_2^2 - Q_3^2\delta_2 - Q_3^2\sigma_1 - 2Q_3\delta_2\sigma_2 - 2Q_3\sigma_1\sigma_2 - \delta_2^2\sigma_1 - \delta_2\sigma_1^2\\ &-\delta_2\sigma_2^2 - \sigma_1\sigma_2^2.\end{aligned} \tag{75}$$

This implies that

$$\breve{H} = \breve{H}_1 - \breve{H}_2. \tag{76}$$

Therefore, $\breve{H} > 0$, if $\breve{H}_1 > \breve{H}_2$. Next, $\Delta_3 > 0$ such that $\breve{b}_{11}(\breve{b}_{12}\breve{b}_{13} - \breve{b}_{14}\breve{b}_{11}) - (\breve{b}_{13}^2 - \breve{b}_{15}\breve{b}_{11}) > 0$ if $\breve{b}_{12}\breve{b}_{13} - \breve{b}_{14}\breve{b}_{11} > 0$ and $\breve{b}_{13}^2 - \breve{b}_{15}\breve{b}_{11} < 0$, where:

$$\begin{aligned}\breve{b}_{12}\breve{b}_{13} - \breve{b}_{14}\breve{b}_{11} &= -(Q_1 + Q_2 - Q_3 + \delta_2 + \sigma_1 - \sigma_2)(S_{\hbar}S_v\eta_1\eta_2\theta_1\theta_2v^2 + Q_1Q_2Q_3\delta_2 + Q_1Q_2Q_3\sigma_1\\ &-Q_1Q_2Q_3\sigma_2 - Q_1Q_2\delta_2\sigma_1 + Q_1Q_2\delta_2\sigma_2 + Q_1Q_3\delta_2\sigma_1 - Q_1Q_3\delta_2\sigma_2 + Q_2Q_3\delta_2\sigma_1\\ &-Q_2Q_3\delta_2\sigma_2 + Q_1Q_2\sigma_1\sigma_2 - Q_1Q_3\sigma_1\sigma_2 - Q_2Q_3\sigma_1\sigma_2 + Q_1\delta_2\sigma_1\sigma_2 + Q_2\delta_2\\ &\times\sigma_1\sigma_2 - Q_3\delta_2\sigma_1\sigma_2) - (Q_1Q_3 - Q_1Q_2 + Q_2Q_3 - Q_1\delta_2 - Q_2\delta_2 + Q_3\delta_2 - Q_1\sigma_1 + Q_1\sigma_2\\ &-Q_2\sigma_1 + Q_2\sigma_2 + Q_3\sigma_1 - Q_3\sigma_2 - \delta_2\sigma_1 + \delta_2\sigma_2 + \sigma_1\sigma_2)(Q_1Q_2Q_3 - Q_1Q_2\delta_2 + Q_1Q_3\delta_2\\ &+Q_2Q_3\delta_2 - Q_1Q_2\sigma_1 + Q_1Q_2\sigma_2 + Q_1Q_3\sigma_1 - Q_1Q_3\sigma_2 + Q_2Q_3\sigma_1 - Q_2Q_3\sigma_2 - Q_1\delta_2\sigma_1\\ &+Q_1\delta_2\sigma_2 - Q_2\delta_2\sigma_1 + Q_2\delta_2\sigma_2 + Q_3\delta_2\sigma_1 - Q_3\delta_2\sigma_2 + Q_1\sigma_1\sigma_2 + Q_2\sigma_1\sigma_2\\ &-Q_3\sigma_1\sigma_2 + \delta_2\sigma_1\sigma_2).\end{aligned} \tag{77}$$

and

$$\begin{aligned}\breve{b}_{13}^2 - \breve{b}_{15}\breve{b}_{11} &= (Q_1Q_2Q_3 - Q_1Q_2\delta_2 + Q_1Q_3\delta_2 + Q_2Q_3\delta_2 - Q_1Q_2\sigma_1 + Q_1Q_2\sigma_2 + Q_1Q_3\sigma_1\\ &-Q_1Q_3\sigma_2 + Q_2Q_3\sigma_1 - Q_2Q_3\sigma_2 - Q_1\delta_2\sigma_1 + Q_1\delta_2\sigma_2 - Q_2\delta_2\sigma_1 + Q_2\delta_2\sigma_2 + Q_3\delta_2\sigma_1\\ &-Q_3\delta_2\sigma_2 + Q_1\sigma_1\sigma_2 + Q_2\sigma_1\sigma_2 - Q_3\sigma_1\sigma_2 + \delta_2\sigma_1\sigma_2)^2 + (Q_1 + Q_2 - Q_3 + \delta_2 + \sigma_1 - \sigma_2)\\ &\times(Q_1Q_2Q_3\delta_2\sigma_1 - Q_1Q_2Q_3\delta_2\sigma_2 - Q_1Q_2Q_3\sigma_1\sigma_2 + Q_1Q_2\delta_2\sigma_1\sigma_2 - Q_1Q_3\delta_2\sigma_1\sigma_2\\ &-Q_2Q_3\delta_2\sigma_1\sigma_2 + I_{\hbar}^{*}S_{\hbar}^{*}S_v^{*}\eta_1\eta_2v^3\theta_1\theta_2^2 + I_v^{*}S_{\hbar}S_v\eta_1\eta_2v^3\theta_1^2\theta_2 + S_{\hbar}S_v^{*}\eta_1\eta_2v^2\\ &\times\sigma_1\theta_1\theta_2 - S_{\hbar}^{*}S_v\eta_1\eta_2v^2\sigma_2\theta_1\theta_2).\end{aligned} \tag{78}$$

Furthermore, $\Delta_4 > 0$, such that

$$\begin{aligned}\Delta_4 &= \breve{b}_{12}^2\breve{b}_{13}^2 - \breve{b}_{12}^3\breve{b}_{15} - \breve{b}_{11}\breve{b}_{12}\breve{b}_{14}\breve{b}_{13} + \breve{b}_{11}\breve{b}_{12}^2\breve{b}_{15} - \breve{b}_{12}\breve{b}_{14}\breve{b}_{15}\\ &+\breve{b}_{12}\breve{b}_{13}\breve{b}_{16} - \breve{b}_{13}^3 - \breve{b}_{12}\breve{b}_{13}^2\breve{b}_{15} + \breve{b}_{11}\breve{b}_{13}\breve{b}_{15} - \breve{b}_{15}^2.\end{aligned} \tag{79}$$

Let

$$\begin{aligned}\breve{G}_1 &= \breve{b}_{12}^2\breve{b}_{13}^2 + \breve{b}_{11}\breve{b}_{12}^2\breve{b}_{15} + \breve{b}_{12}\breve{b}_{13}\breve{b}_{16} + \breve{b}_{11}\breve{b}_{13}\breve{b}_{15},\\ \breve{G}_2 &= -\breve{b}_{12}^3\breve{b}_{15} - \breve{b}_{11}\breve{b}_{12}\breve{b}_{14}\breve{b}_{13} - \breve{b}_{12}\breve{b}_{14}\breve{b}_{15} - \breve{b}_{13}^3 - \breve{b}_{12}\breve{b}_{13}^2\breve{b}_{15} - \breve{b}_{15}^2,\end{aligned} \tag{80}$$

which implies that

$$\breve{G} = \breve{G}_1 - \breve{G}_2. \tag{81}$$

So, $\breve{G} >$, if $\breve{G}_1 > \breve{G}_2$. Therefore, the model (2) is locally asymptotically stable around disease–free equilibrium point E^{*}, that is the eigenvalues $\breve{\lambda} = -\delta_1$ and $\breve{\lambda}_2 = -(\delta_1 + \gamma_2)$ are both negative, while the rest of the eigenvalues are negative based on the Routh–Hurwitz criterion.

5. Global stability analysis

Here we follow the same method as used in [32–36] to derive stability results.

Theorem 5.1 (*Lyaponov Function*). If $R_0 > 1$, the endemic equilibrium point $\mathscr{E}^{*}$ of the model (2) is globally asymptotically stable, otherwise unstable.

Proof. For the global stability of the model (2), the Lyapunov function can be written as,

$$\begin{aligned}&L(S_{\hbar}^{*}, E_{\hbar}^{*}, I_{\hbar}^{*}, H^{*}, R_{\hbar}^{*}, S_v^{*}, E_v^{*}, I_v^{*})\\ &= \left(S_{\hbar} - S_{\hbar}^{*} - S_{\hbar}^{*}\log\frac{S_{\hbar}^{*}}{S_{\hbar}}\right) + \left(E_{\hbar} - E_{\hbar}^{*} - E_{\hbar}^{*}\log\frac{E_{\hbar}^{*}}{E_{\hbar}}\right)\\ &+\left(I_{\hbar} - I_{\hbar}^{*} - I_{\hbar}^{*}\log\frac{I_{\hbar}^{*}}{I_{\hbar}}\right) + \left(H - H^{*} - H^{*}\log\frac{H_{\hbar}^{*}}{H}\right)\\ &+\left(R_{\hbar} - R_{\hbar}^{*} - R_{\hbar}^{*}\log\frac{R_{\hbar}^{*}}{R_{\hbar}}\right) + \left(S_v - S_v^{*} - S_v^{*}\log\frac{S_v^{*}}{S_v}\right)\\ &+\left(E_v - E_v^{*} - E_v^{*}\log\frac{E_v^{*}}{E_v}\right) + \left(I_v - I_v^{*} - I_v^{*}\log\frac{I_v^{*}}{I_v}\right).\end{aligned} \tag{82}$$

Taking the first derivative of the Lyapunov function (82), such that,

$$\begin{aligned}\frac{dL}{dt} &= \left(\frac{S_{\hbar}-S_{\hbar}^{*}}{S_{\hbar}}\right)\frac{dS_{\hbar}}{dt} + \left(\frac{E_{\hbar}-E_{\hbar}^{*}}{E_{\hbar}}\right)\frac{dE_{\hbar}}{dt} + \left(\frac{I_{\hbar}-I_{\hbar}^{*}}{I_{\hbar}}\right)\frac{dI_{\hbar}}{dt}\\ &+\left(\frac{H-H^{*}}{H}\right)\frac{dH}{dt} + \left(\frac{R_{\hbar}-R_{\hbar}^{*}}{R_{\hbar}}\right)\frac{dR_{\hbar}}{dt} + \left(\frac{S_v-S_v^{*}}{S_v}\right)\frac{dS_v}{dt}\\ &+\left(\frac{E_v-E_v^{*}}{E_v}\right)\frac{dE_v}{dt} + \left(\frac{I_v-I_v^{*}}{I_v}\right)\frac{dI_v}{dt}.\end{aligned} \tag{83}$$

Eq. (83) implies that

$$\begin{aligned}\frac{dL}{dt} &= \left(\frac{S_\hbar - S_\hbar^*}{S_\hbar}\right)(\alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar) + \left(\frac{E_\hbar - E_\hbar^*}{E_\hbar}\right)(v\theta I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar)\\ &+\left(\frac{I_\hbar - I_\hbar^*}{I_\hbar}\right)(\eta_1 E_\hbar - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar) + \left(\frac{H - H^*}{H}\right)(\hbar I_\hbar - (\delta_1 + \gamma_2)H)\\ &+\left(\frac{R_\hbar - R_\hbar^*}{R_\hbar}\right)(\gamma_1 I_\hbar + \gamma_2 H - \delta_1 R_\hbar) + \left(\frac{S_v - S_v^*}{S_v}\right)(\alpha_2 - v\theta_2 I_\hbar S_v - \delta_2 S_v)\\ &+\left(\frac{E_v - E_v^*}{E_v}\right)(v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v) + \left(\frac{I_v - I_v^*}{I_v}\right)(\eta_2 E_v - \delta_2 I_v)\end{aligned} \tag{84}$$

$$\begin{aligned}\frac{dL}{dt} &= \alpha_1 - \alpha_1\frac{S_\hbar^*}{S_\hbar} - \frac{v\theta_1 I_v}{S_\hbar}(S_\hbar - S_\hbar^*)^2 - \frac{v\theta_1 I_v^*}{S_\hbar}(S_\hbar - S_\hbar^*)^2 - \frac{\delta_1}{S_\hbar}(S_\hbar - S_\hbar^*)^2\\ &+\frac{v\theta_1 E_\hbar I_v S_\hbar}{E_\hbar} - \frac{v\theta_1 E_\hbar I_v^* S_\hbar}{E_\hbar} - \frac{v\theta_1 E_\hbar^* I_v S_\hbar}{E_\hbar} + \frac{v\theta_1 E_\hbar^* I_v^* S_\hbar}{E_\hbar} - \frac{v\theta_1 E_\hbar I_v S_\hbar^*}{E_\hbar}\\ &+\frac{v\theta_1 E_\hbar I_v^* S_\hbar^*}{E_\hbar} + \frac{v\theta_1 E_\hbar^* I_v S_\hbar^*}{E_\hbar} - \frac{v\theta_1 E_\hbar^* I_v^* S_\hbar^*}{E_\hbar} - \frac{(\eta_1+\rho+\delta_1)}{E_\hbar}(E_\hbar - E_\hbar^*)^2\\ &+\frac{\eta_1}{I_\hbar}E_\hbar I_\hbar - \frac{\eta_1}{I_\hbar}E_\hbar I_\hbar^* - \frac{\eta_1}{I_\hbar}E_\hbar^* I_\hbar + \frac{\eta_1}{I_\hbar}E_\hbar^* I_\hbar^* - \frac{(\gamma_1+\delta_1+\Omega+\hbar)}{I_\hbar}(I_\hbar - I_\hbar^*)^2\\ &+\frac{\hbar}{H}I_\hbar H - \frac{\hbar}{H}I_\hbar H^* + \frac{\hbar}{H}I_\hbar^* H^* - \frac{(\delta_1+\gamma_2)}{H}(H - H^*)^2 + \frac{\gamma_1}{R_\hbar}I_\hbar R_\hbar - \frac{\gamma_1}{R_\hbar}I_\hbar R_\hbar^*\\ &-\frac{\gamma_1}{H}I_\hbar^* R_\hbar^* + \frac{\gamma_2}{R_\hbar}HR_\hbar - \frac{\gamma_2}{R_\hbar}HR_\hbar^* - \gamma_2\frac{H}{R_\hbar}R_\hbar^* + \gamma_2 H^*\frac{R_\hbar^*}{R_\hbar} - \delta_1\frac{(R_\hbar - R_\hbar^*)^2}{R_\hbar}\\ &+\alpha_2 - \frac{\alpha_2 S_v^*}{S_v} - \frac{v\theta_2 I_\hbar}{S_v}(S_v - S_v^*)^2 + \frac{v\theta_2 I_\hbar^*}{S_v}(S_v - S_v^*)^2 - \frac{\delta_2}{S_v}(S_v - S_v^*)^2\\ &+\frac{v\theta_2 E_v I_\hbar S_v}{E_v} - \frac{v\theta_2 E_v I_\hbar^* S_v}{E_v} - \frac{v\theta_2 E_v^* I_\hbar S_v}{E_v} + \frac{v\theta_2 E_v^* I_\hbar^* S_v}{E_v} - \frac{v\theta_2 E_v I_\hbar S_v^*}{E_v}\\ &+\frac{v\theta_2 E_v I_\hbar^* S_v^*}{E_v} + \frac{v\theta_2 E_v^* I_\hbar S_v^*}{E_v} - \frac{v\theta_2 E_v^* I_\hbar^* S_v^*}{E_v} - \frac{\delta_2(E_v - E_v)^2}{E_v} - \eta_2\frac{(E_v - E_v^*)^2}{E_v}\\ &+\eta_2 E_v\frac{(I_v - I_v^*)^2}{I_v} - \frac{\eta_2 E_v^*(I_v - I_v^*)^2}{I_v} - \delta_2\frac{(I_v - I_v^*)^2}{I_v}.\end{aligned} \tag{85}$$

Now, let

$$\begin{aligned}l_1 &= \alpha_1 + \alpha_2 + \frac{v\theta_1 E_\hbar I_v S_\hbar}{E_\hbar} + \frac{v\theta_1 E_\hbar I_v^* S_\hbar^*}{E_\hbar} + \frac{v\theta_1 E_\hbar^* I_v S_\hbar^*}{E_\hbar} + \frac{\eta_1}{I_\hbar}E_\hbar I_\hbar + \frac{\hbar}{H}I_\hbar H\\ &+\frac{v\theta_2 E_v I_\hbar S_v}{E_v} + \eta_2 E_v\frac{(I_v - I_v^*)^2}{I_v} + \frac{v\theta_2 E_v^* I_\hbar^* S_v}{E_v} + \frac{v\theta_2 E_v^* I_\hbar S_v^*}{E_v} + \frac{v\theta_2 E_v I_\hbar^* S_v^*}{E_v}\\ &+\frac{\eta_1}{I_\hbar}E_\hbar^* I_\hbar^* + \frac{\gamma_1}{R_\hbar}I_\hbar R_\hbar + \frac{v\theta_2 I_\hbar^*}{S_v}(S_v - S_v^*)^2 + \frac{v\theta_1 E_\hbar^* I_v^* S_\hbar}{E_\hbar} + \frac{\hbar}{H}I_\hbar^* H^* + \frac{\gamma_2}{R_\hbar}HR_\hbar\\ &+\gamma_2 H^*\frac{R_\hbar^*}{R_\hbar},\end{aligned} \tag{86}$$

and

$$\begin{aligned}l_2 &= -\alpha_1\frac{S_\hbar^*}{S_\hbar} - \frac{v\theta_1 I_v}{S_\hbar}(S_\hbar - S_\hbar^*)^2 - \frac{v\theta_1 I_v^*}{S_\hbar}(S_\hbar - S_\hbar^*)^2 - \frac{\delta_1}{S_\hbar}(S_\hbar - S_\hbar^*)^2 - \frac{v\theta_1 E_\hbar I_v^* S_\hbar}{E_\hbar}\\ &-\frac{v\theta_1 E_\hbar^* I_v S_\hbar}{E_\hbar} - \frac{v\theta_1 E_\hbar I_v S_\hbar^*}{E_\hbar} - \frac{v\theta_1 E_\hbar^* I_v^* S_\hbar^*}{E_\hbar} - \frac{(\eta_1+\rho+\delta_1)}{E_\hbar}(E_\hbar - E_\hbar^*)^2 - \frac{\eta_1}{I_\hbar}E_\hbar I_\hbar^*\\ &-\frac{\eta_1}{I_\hbar}E_\hbar^* I_\hbar - \frac{(\gamma_1+\delta_1+\Omega+\hbar)}{I_\hbar}(I_\hbar - I_\hbar^*)^2 - \frac{\hbar}{H}I_\hbar H^* - \frac{(\delta_1+\gamma_2)}{H}(H - H^*)^2\\ &-\frac{\gamma_1}{R_\hbar}I_\hbar R_\hbar^* - \frac{\gamma_1}{H}I_\hbar^* R_\hbar^* - \frac{\gamma_2}{R_\hbar}HR_\hbar^* - \gamma_2\frac{H}{R_\hbar}R_\hbar^* - \delta_1\frac{(R_\hbar - R_\hbar^*)^2}{R_\hbar} - \frac{\alpha_2 S_v^*}{S_v}\\ &-\frac{v\theta_2 I_\hbar}{S_v}(S_v - S_v^*)^2 - \frac{\delta_2}{S_v}(S_v - S_v^*)^2 - \frac{v\theta_2 E_v I_\hbar^* S_v}{E_v} - \frac{v\theta_2 E_v^* I_\hbar S_v}{E_v} - \frac{v\theta_2 E_v I_\hbar S_v^*}{E_v}\\ &-\frac{v\theta_2 E_v^* I_\hbar^* S_v^*}{E_v} - \frac{\delta_2(E_v - E_v)^2}{E_v} - \eta_2\frac{(E_v - E_v^*)^2}{E_v} - \frac{\eta_2 E_v^*(I_v - I_v^*)^2}{I_v} - \delta_2\frac{(I_v - I_v^*)^2}{I_v},\end{aligned} \tag{87}$$

such that, dL/dt can be written as

$$\frac{dL}{dt} = l_1 - l_2. \tag{88}$$

Eventually, if $l_1 < l_2$, then $\frac{dL}{dt} < 0$, while using $S_\hbar = S_\hbar - S_\hbar^*, E_\hbar = E_\hbar - E_\hbar^*, I_\hbar = I_\hbar - I_\hbar^*, H = H - H^*, R_\hbar = R_\hbar - R_\hbar^*, S_v = S_v - S_v^*, E_v = E_v - E_v^*$, and $I_v = I_v - I_v^*$, then $0 = l_1 - l_2$ implies that $\frac{dL}{dt} = 0$. Also, for the suggested model (2), we are looking the largest compact invariant set $\left\{\left(S_\hbar^*, E_\hbar^*, I_\hbar^*, H^*, R_\hbar^*, S_v^*, E_v^*, I_v^*\right) \in \prod : \frac{dL}{dt} = 0\right\}$ is the endemic equilibrium point $\mathscr{E}^* = \left(S_\hbar^*, E_\hbar^*, I_\hbar^*, H^*, R_\hbar^*, S_v^*, E_v^*, I_v^*\right)$ of the considered model. Further, it is clearly holds that $L(0) = 0$ and $L > 0$, at $\mathscr{E}^*$. Thus L satisfied all the properties of positive definite operator.

Thus, the model (2) is stable in $\prod$, if $R_0 > 1$, and $l_1 < l_2$.

Theorem 5.2. If $I_\hbar^*\theta_2 > \frac{S_\hbar}{S_v}S_\hbar^*\theta_1, I_v^* > \frac{S_\hbar}{E_v}S_\hbar^*, \theta_2\frac{E_\hbar}{S_\hbar}I_\hbar^* > I_v^*\theta_1$, $I_\hbar^* > \frac{S_v}{E_\hbar}S_v^*, I_v^*\theta_1 > \frac{S_v}{S_\hbar}S_v^*\theta_2, I_\hbar^*\theta_2 > \frac{E_v}{S_v}I_v^*\theta_1$ and $R_0 > 1$, then the model (2) is globally asymptotically stable at endemic equilibrium E^* and unstable otherwise.

Proof. For global stability analysis, we consider the non–linear equations from the proposed model (2), such that

$$\left.\begin{aligned}\frac{dS_\hbar}{dt} &= \alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar,\\ \frac{dE_\hbar}{dt} &= v\theta_1 I_v S_\hbar - (\eta_1 + \rho + \delta_1)E_\hbar,\\ \frac{dS_v}{dt} &= \alpha_2 - v\theta_2 I_\hbar S_v - \delta_2 S_v,\\ \frac{dE_v}{dt} &= v\theta_2 I_\hbar S_v - (\delta_2 + \eta_2)E_v.\end{aligned}\right\} \tag{89}$$

Now, for the generic matrix J, such that

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}. \tag{90}$$

We consider the third additive compound matrix $J^{[3]}$ from [37,38], that is

$$J^{[3]} = \begin{pmatrix} a_{11} + a_{22} + a_{33} & a_{34} & -a_{24} & a_{14}\\ a_{43} & a_{11} + a_{22} + a_{44} & a_{23} & -a_{13}\\ -a_{42} & a_{32} & a_{11} + a_{33} + a_{44} & a_{12}\\ a_{41} & -a_{31} & a_{21} & a_{22} + a_{33} + a_{44}\end{pmatrix}. \tag{91}$$

The Jacobian of the non–linear sub–system (89) at disease endemic equilibrium point (E^*) is given by

$$J = \begin{pmatrix} -\delta_1 - I_v^* v\theta_1 & 0 & 0 & -S_\hbar^* v\theta_1\\ I_v^* v\theta_1 & 0 & 0 & S_\hbar^* v\theta_1\\ 0 & -S_v^* v\theta_2 & -\delta_2 - I_\hbar^* v\theta_2 & 0\\ 0 & S_v^* v\theta_2 & I_\hbar^* v\theta_2 & 0\end{pmatrix}. \tag{92}$$

Based on (92), the third additive compound matrix is obtained as

$$J^{[3]} = \begin{pmatrix} -(\delta_1+\delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1) & 0 & -S_\hbar^* v\theta_1 & -S_\hbar^* v\theta_1\\ I_\hbar^* v\theta_2 & -\delta_1 - I_v^* v\theta_1 & 0 & 0\\ -S_v^* v\theta_2 & -S_v^* v\theta_2 & -(\delta_1+\delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1) & 0\\ 0 & 0 & I_v^* v\theta_1 & -\delta_2 - I_\hbar^* v\theta_2\end{pmatrix}. \tag{93}$$

Next, the function $q(y) = \mathrm{diag}\{S_\hbar(t), E_\hbar(t), S_v(t), E_v(t)\}$, while the inverse of function $q(y)$ is given as $q^{-1}(y) = \mathrm{diag}\{1/S_\hbar(t), 1/E_\hbar(t), 1/S_v(t), 1/E_v(t)\}$, the time derivative is given by $q_f(y) = \mathrm{diag}\{\dot{S}_\hbar(t), \dot{E}_\hbar(t), \dot{S}_v(t), \dot{E}_v(t)\}$, while

$q_f q^{-1} = \mathrm{diag}\{\dot{S}_\hbar(t)/S_\hbar(t), \dot{E}_\hbar(t)/E_\hbar(t), \dot{S}_v(t)/S_v(t), \dot{E}_v(t)/E_v(t)\}$ and

$$qJ^{[3]}q^{-1} = \begin{pmatrix} b_1 & 0 & -\frac{S_\hbar(t)}{S_v(t)}S_\hbar^* v\theta_1 & -\frac{S_\hbar(t)}{E_v(t)}S_\hbar^* v\theta_1\\ \frac{E_\hbar(t)}{S_\hbar(t)}I_\hbar^* v\theta_2 & b_2 & 0 & 0\\ -\frac{S_v(t)}{S_\hbar(t)}S_v^* v\theta_2 & -\frac{S_v(t)}{E_\hbar(t)}S_v^* v\theta_2 & b_3 & 0\\ 0 & 0 & \frac{E_v(t)}{S_v(t)}I_v^* v\theta_1 & b_4\end{pmatrix}, \tag{94}$$

where, $b_1 = -(\delta_1 + \delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1), b_2 = -\delta_1 - I_v^* v\theta_1$, $b_3 = -(\delta_1 + \delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1)$, and $b_4 = -\delta_2 - I_\hbar^* v\theta_2$. Furthermore, the matrix $Q = q_f q^{-1} + qJ^{[3]}q^{-1}$ is obtained in the following manner

$$Q = \begin{pmatrix} \frac{\dot{S}_\hbar(t)}{S_\hbar(t)} + b_1 & 0 & -\frac{S_\hbar(t)}{S_v(t)}S_\hbar^* v\theta_1 & -\frac{S_\hbar(t)}{E_v(t)}S_\hbar^* v\theta_1\\ \frac{E_\hbar(t)}{S_\hbar(t)}I_\hbar^* v\theta_2 & \frac{\dot{E}_\hbar(t)}{E_\hbar(t)} + b_2 & 0 & 0\\ -\frac{S_v(t)}{S_\hbar(t)}S_v^* v\theta_2 & -\frac{S_v(t)}{E_\hbar(t)}S_v^* v\theta_2 & \frac{\dot{S}_v(t)}{S_v(t)} + b_3 & 0\\ 0 & 0 & \frac{E_v(t)}{S_v(t)}I_v^* v\theta_1 & \frac{\dot{E}_v(t)}{E_v(t)} + b_4\end{pmatrix}. \tag{95}$$

Now, consider the entries of the matrix (95) as q_{jk}, then, we have to calculate $z_j(t)$ for $j = 1, 2, 3, 4$ such that

$$\begin{aligned} z_1(t) &= q_{11} + \sum_{k\neq 1\wedge k=2}^{4} |q_{1k}|, \\ z_2(t) &= q_{22} + \sum_{k\neq 2\wedge k=1}^{4} |q_{2k}|, \\ z_3(t) &= q_{33} + \sum_{k\neq 3\wedge k=1}^{4} |q_{3k}|, \\ z_4(t) &= q_{44} + \sum_{k\neq 4\wedge k=1}^{4} |q_{4k}|. \end{aligned} \tag{96}$$

Now, to evaluate $z_j(t)$ for $j = 1, 2, 3, 4$ such that, if $I_\hbar^*\theta_2 > \frac{S_\hbar}{S_v} S_\hbar^*\theta_1$ and $I_v^* > \frac{S_\hbar}{E_v} S_\hbar^*$ then $z_1(t)$ becomes

$$\begin{aligned} z_1(t) &= \tfrac{\dot{S}_\hbar(t)}{S_\hbar(t)} - (\delta_1+\delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1) + \left|-\tfrac{S_\hbar(t)}{S_v(t)} S_\hbar^* v\theta_1\right| + \left|-\tfrac{S_\hbar(t)}{E_v(t)} S_\hbar^* v\theta_1\right|, \\ &= \tfrac{\dot{S}_\hbar(t)}{S_\hbar(t)} - (\delta_1+\delta_2) - v\left(I_\hbar^*\theta_2 - \tfrac{S_\hbar}{S_v} S_\hbar^*\theta_1\right) - v\theta_1\left(I_v^* - \tfrac{S_\hbar}{E_v} S_\hbar^*\right), \\ &= \tfrac{\dot{S}_\hbar(t)}{S_\hbar(t)} - (\delta_1+\delta_2). \end{aligned} \tag{97}$$

For $z_2(t)$, if $\theta_2 \frac{E_\hbar}{S_\hbar} I_\hbar^* > I_v^*\theta_1$, then

$$\begin{aligned} z_2(t) &= \tfrac{\dot{I}_\hbar(t)}{I_\hbar(t)} - \delta_1 - I_v^* v\theta_1 + \left|\tfrac{E_\hbar(t)}{S_\hbar(t)} I_\hbar^* v\theta_2\right|, \\ &= \tfrac{\dot{I}_\hbar(t)}{I_\hbar(t)} - \delta_1 - v\left(\theta_2 \tfrac{E_\hbar}{S_\hbar} I_\hbar^* - I_v^*\theta_1\right), \\ &= \tfrac{\dot{I}_\hbar(t)}{I_\hbar(t)} - \delta_1. \end{aligned} \tag{98}$$

For $z_3(t)$, if $I_\hbar^* > \frac{S_v}{E_\hbar} S_v^*$ and $I_v^*\theta_1 > \frac{S_v}{S_\hbar} S_v^*\theta_2$, then

$$\begin{aligned} z_3(t) &= \tfrac{\dot{S}_v(t)}{S_v(t)} - (\delta_1+\delta_2) - (I_\hbar^* v\theta_2 + I_v^* v\theta_1) + \left|-\tfrac{S_v(t)}{S_\hbar(t)} S_v^* v\theta_2\right| + \left|-\tfrac{S_v(t)}{E_\hbar(t)} S_v^* v\theta_2\right|, \\ &= \tfrac{\dot{S}_v(t)}{S_v(t)} - (\delta_1+\delta_2) - v\theta_2\left(I_\hbar^* - \tfrac{S_v}{E_\hbar} S_v^*\right) - v\left(I_v^*\theta_1 - \tfrac{S_v}{S_\hbar} S_v^*\theta_2\right), \\ &= \tfrac{\dot{S}_v(t)}{S_v(t)} - (\delta_1+\delta_2). \end{aligned} \tag{99}$$

Finally, for $z_4(t)$, if $I_\hbar^*\theta_2 > \frac{E_v}{S_v} I_v^*\theta_1$, then

$$\begin{aligned} z_4(t) &= \tfrac{\dot{I}_v(t)}{I_v(t)} - \delta_2 - I_\hbar^* v\theta_2 + \left|\tfrac{E_v(t)}{S_v(t)} I_v^* v\theta_1\right|, \\ &= \tfrac{\dot{I}_v(t)}{I_v(t)} - \delta_2 - v\left(I_\hbar^*\theta_2 - \tfrac{E_v}{S_v} I_v^*\theta_1\right), \\ &= \tfrac{\dot{I}_v(t)}{I_v(t)} - \delta_2. \end{aligned} \tag{100}$$

Furthermore, let a vector $x = (\xi_1, \xi_2, \xi_3, \xi_4)$, while the Lozinski measure $\ell(Q)$ is defined as $\ell(Q) = z_i, i = 1, 2, 3, 4$. To integrate the $\ell(Q)$ with in limits $t \to \infty$ lead to the following equations

$$\begin{aligned} &\lim_{t\to\infty} \sup \sup \tfrac{1}{t} \int_0^t z_1(t)dt \leqslant \tfrac{1}{t} \tfrac{S_\hbar(t)}{S_\hbar(0)} - (\delta_1+\delta_2), \\ &< -(\delta_1+\delta_2). \end{aligned} \tag{101}$$

$$\begin{aligned} &\lim_{t\to\infty} \sup \sup \tfrac{1}{t} \int_0^t z_2(t)dt \leqslant \tfrac{1}{t} \tfrac{E_\hbar(t)}{E_\hbar(0)} - \delta_1, \\ &< -\delta_1. \end{aligned} \tag{102}$$

$$\begin{aligned} &\lim_{t\to\infty} \sup \sup \tfrac{1}{t} \int_0^t z_3(t)dt \leqslant \tfrac{1}{t} \tfrac{S_v(t)}{S_v(0)} - (\delta_1+\delta_2), \\ &< -(\delta_1+\delta_2), \end{aligned} \tag{103}$$

and

$$\begin{aligned} &\lim_{t\to\infty} \sup \sup \tfrac{1}{t} \int_0^t z_4(t)dt \leqslant \tfrac{1}{t} \tfrac{E_v(t)}{E_v(0)} - \delta_2, \\ &< -\delta_2. \end{aligned} \tag{104}$$

Now, the combination of the last four inequalities give

$$\bar{x} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \ell(Q)dt < 0. \tag{105}$$

The system containing only four non-linear equations of model (2) is globally asymptotically stable around its interior equilibrium $\left(S_\hbar^*, I_\hbar^*, S_v^*, E_v^*\right)$. Furthermore, calculating the rest of the equations can lead to $I_\hbar(t) \to I_\hbar^*, H(t) \to H^*$, $R_\hbar(t) \to E_\hbar^*, I_v(t) \to I_v^*$ as $t \to \infty$. Thus, E^* is globally asymptotically stable.

6. Sensitivity analysis

Based on the procedure used in [39], we present sensitivity analysis of the parameters in the considered model.

$$\mathbf{s}_\tau^{\mathscr{R}_0} = \frac{\tau}{\mathscr{R}_0}\left[\frac{\partial \mathscr{R}_0}{\partial \tau}\right].$$

Now, according to the above relation, we have

$$\mathbf{s}_v^{R_0} = \tfrac{v}{R_0}\left[\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}\right] > 0, \tag{106}$$

$$\mathbf{s}_{\alpha_1}^{R_0} = \tfrac{\alpha_1}{R_0}\left[\frac{\alpha_2\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0, \tag{107}$$

$$\mathbf{s}_{\alpha_2}^{R_0} = \tfrac{\alpha_2}{R_0}\left[\frac{\alpha_1\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_2 Q_2 Q_3\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0, \tag{108}$$

$$\mathbf{s}_{\delta_1}^{R_0} = \tfrac{\delta_1}{R_0}\left[-\frac{v\left(\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3^2} + \tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2^2 Q_3} + \tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1^2\delta_2^2 Q_1 Q_2 Q_3}\right)}{2\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{109}$$

$$\mathbf{s}_{\delta_2}^{R_0} = \tfrac{\delta_2}{R_0}\left[-\frac{v\left(\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1^2 Q_2 Q_3} + \tfrac{2\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^3 Q_1 Q_2 Q_3}\right)}{2\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{110}$$

$$\mathbf{s}_{\eta_1}^{R_0} = \tfrac{\eta_1}{R_0}\left[\frac{v\left(\tfrac{\alpha_1\alpha_2\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3} - \tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2^2 Q_3}\right)}{2\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0, \tag{111}$$

$$\mathbf{s}_{\eta_2}^{R_0} = \tfrac{\eta_2}{R_0}\left[\frac{v\left(\tfrac{\alpha_1\alpha_2\eta_1\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3} - \tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^3 Q_1^2 Q_2 Q_3}\right)}{2\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0 \tag{112}$$

$$\mathbf{s}_{\theta_1}^{R_0} = \tfrac{\alpha_2}{R_0}\left[\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0, \tag{113}$$

$$\mathbf{s}_{\theta_2}^{R_0} = \tfrac{\delta_1}{R_0}\left[\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_1}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3\sqrt{\tfrac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] > 0, \tag{114}$$

$$\mathbf{s}_{\rho}^{R_0} = \frac{\delta_2}{R_0}\left[-\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2^2 Q_3\sqrt{\frac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{115}$$

$$\mathbf{s}_{\Omega}^{R_0} = \frac{\eta_1}{R_0}\left[-\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3^2\sqrt{\frac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{116}$$

$$\mathbf{s}_{\gamma_1}^{R_0} = \frac{\eta_2}{R_0}\left[-\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3^2\sqrt{\frac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{117}$$

$$\mathbf{s}_{\hbar}^{R_0} = \frac{\eta_2}{R_0}\left[-\frac{\alpha_1\alpha_2\eta_1\eta_2 v\theta_1\theta_2}{2\delta_1\delta_2^2 Q_1 Q_2 Q_3^2\sqrt{\frac{\alpha_1\alpha_2\eta_1\eta_2\theta_1\theta_2}{\delta_1\delta_2^2 Q_1 Q_2 Q_3}}}\right] < 0, \tag{118}$$

where, $Q_1 = (\delta_2 + \eta_2), Q_2 = (\delta_1 + \eta_1 + \rho)$, and $Q_3 = (\Omega + \delta_1 + \gamma_1 + \hbar)$.

7. Numerical results and discussion

In this section, we investigate numerical solution of the model (2) using Non standard finite difference scheme (NSFD)[40,41] which can replicate the dynamics of the Leishmaniasis disease. Assume, $Y = (S_\hbar, E_\hbar, I_\hbar, H, R_\hbar, S_v, E_v, I_v)^T$, which approximates $X(t_i)$, where $t_i = i\Delta t$, with $i \in N, h = \Delta t > 0$ be a step size then, consider the first equation of the model (2) such that

$$\frac{dS_\hbar}{dt} = \alpha_1 - v\theta_1 I_v S_\hbar - \delta_1 S_\hbar \tag{119}$$

and use $dS_\hbar/dt = (S_\hbar^{i+1} - S_\hbar^i)/\varpi(h)$,$S_\hbar = S_\hbar^{i+1}$, and $I_v = I_v^i$ which implies that

$$\frac{S_\hbar^{i+1} - S_\hbar^i}{\varpi(h)} = \alpha_1 - v\theta_1 I_v^i S_\hbar^{i+1} - \delta_1 S_\hbar^{i+1}. \tag{120}$$

Consider the second equation of model (2), and use $dE_\hbar/dt = (E_\hbar^{i+1} - E_\hbar^i)/\varpi(h)$,$E_\hbar = E_\hbar^{i+1}, S_\hbar = S_\hbar^i$ and $I_v = I_v^i$ which implies that:

$$\frac{E_\hbar^{i+1} - E_\hbar^i}{\varpi(h)} = v\theta_1 I_v^i S_\hbar^{i+1} - (\eta_1 + \rho + \delta_1)E_\hbar^{i+1} \tag{121}$$

Consider the third equation of model (2), and use $dI_\hbar/dt = (I_\hbar^{i+1} - I_\hbar^i)/\varpi(h)$, and $I_\hbar = I_\hbar^{i+1}$ which implies that:

$$\frac{I_\hbar^{i+1} - I_\hbar^i}{\varpi(h)} = \eta_1 E_\hbar^{i+1} - (\gamma_1 + \delta_1 + \Omega + \hbar)I_\hbar^{i+1}. \tag{122}$$

Consider the fourth equation of model (2), and use $dH/dt = (H^{i+1} - H^i)/\varpi(h)$, and $H = H^{i+1}$ which implies that:

$$\frac{H^{i+1} - H^i}{\varpi(h)} = \hbar I_\hbar^{i+1} - (\delta_1 + \gamma_2)H^{i+1}. \tag{123}$$

Consider the fifth equation of model (2), and use $dR_\hbar/dt = (R_\hbar^{i+1} - R_\hbar^i)/\varpi(h)$, and $R = R^{i+1}$ which implies that:

$$\frac{R^{i+1} - R^i}{\varpi(h)} = \gamma_1 I_\hbar^{i+1} + \gamma_2 H^{i+1} - \delta_1 R^{i+1}. \tag{124}$$

Consider the sixth equation of model (2), and use $dS_v/dt = (S_v^{i+1} - S_v^i)/\varpi(h)$, $I_\hbar = I_\hbar^i$, and $S_v = S_v^{i+1}$ which implies that:

$$\frac{S_v^{i+1} - S_v^i}{\varpi(h)} = \alpha_2 - v\theta_2 I_\hbar^i S_v^{i+1} - \delta_2 S_v^{i+1}. \tag{125}$$

Consider the seventh equation of model (2), and use $dE_v/dt = (E_v^{i+1} - E_v^i)/\varpi(h)$, and $E = E^{i+1}$ which implies that:

$$\frac{E_v^{i+1} - E_v^i}{\varpi(h)} = v\theta_2 I_\hbar^i S_v^{i+1} - (\delta_2 + \eta_2)E_v^{i+1}. \tag{126}$$

and finally, the last equation of model (2), and use $dI_v/dt = (I_v^{i+1} - I_v^i)/\varpi(h)$, and $I_v = I_v^{i+1}$ which implies that:

$$\frac{I_v^{i+1} - I_v^i}{\varpi(h)} = \eta_2 E_v^{i+1} - \delta_2 I_v^{i+1}. \tag{127}$$

where $\varpi(h)$ [42]

$$\varpi(h) = 1 - e^{-h} \tag{128}$$

is a real-valued function satisfying the condition $\varpi(h) \longrightarrow 0$ as $h \longrightarrow 0$.

7.1. Analysis of the scheme

Theorem 7.1. The NSFD scheme (120)–(127) [41,43] is used to numerically interpret a dynamical system on the biological feasible domain Δ of the continuous model (2).

Proof. At first, let us consider the system of Eqs. (4) and descritize them, such that:

$$\begin{aligned} \frac{dN_\hbar}{dt} &= \alpha_1 - \delta_1 N_\hbar - \frac{\Omega}{\delta_1} I_\hbar - \frac{\rho}{\delta_1} E_\hbar, \\ \frac{dN_v}{dt} &= \alpha_2 - \delta_2 N_v. \end{aligned} \tag{129}$$

For the system of Eqs. (129), now the descritization become:

$$\frac{N_\hbar^{i+1} - N_\hbar^i}{\varpi(h)} = \alpha_1 - \delta_1 N_\hbar^{i+1} - \frac{\Omega}{\delta_1} I_\hbar^i - \frac{\rho}{\delta_1} E_\hbar^i, \tag{130}$$

$$\frac{N_v^{i+1} - N_v^i}{\varpi(h)} = \alpha_2 - \delta_2 N_v^{i+1}. \tag{131}$$

We prove the condition of positivity of the scheme (120)–(127), which can be represented by NSFD scheme as: from Eq. (120), we have

$$\begin{aligned} &\tfrac{S_\hbar^{i+1} - S_\hbar^i}{\varpi(h)} = \alpha_1 - v\theta_1 I_v^i S_\hbar^{i+1} - \delta_1 S_\hbar^{i+1}, \\ &S_\hbar^{i+1} + v\theta_1(1 + e^{-h})S_\hbar^{i+1} I_v^i + \delta_1 S_\hbar^{i+1} = S_\hbar^i + (1 - e^{-h})\alpha_1, \\ &S_\hbar^{i+1} = \tfrac{S_\hbar^i + (1 - e^{-h})\alpha_1}{[1 + v\theta_1(1 - e^{-h})I_v^i + \delta_1]}. \end{aligned} \tag{132}$$

From Eq. (121):

$$\begin{aligned} &\tfrac{E_\hbar^{i+1} - E_\hbar^i}{\varpi(h)} = v\theta_1 I_v^i S_\hbar^{i+1} - (\eta_1 + \rho + \delta_1)E_\hbar^{i+1}, \\ &E_\hbar^{i+1} + (1 - e^{-h})(\eta_1 + \rho + \delta_1)E_\hbar^{i+1} = E_\hbar^i + v\theta_1(1 - e^{-h})S_\hbar^{i+1} I_v^i, \\ &E_\hbar^{i+1} = \tfrac{E_\hbar^i + v\theta_1(1 - e^{-h})S_\hbar^{i+1} I_v^i}{[1 + (1 - e^{-h})(\eta_1 + \rho + \delta_1)]}. \end{aligned} \tag{133}$$

Similarly, for the rest of the equations:

$$I_{\hbar}^{i+1} = \frac{I_{\hbar}^{i} + \eta_1(1-e^{-h})E_{\hbar}^{i+1}}{[1+(1-e^{-h})(\gamma_1+\delta_1+\Omega+\hbar)]}, \tag{134}$$

$$H^{i+1} = \frac{H^{i} + \hbar(1-e^{-h})I_{\hbar}^{i+1}}{[1+(1-e^{-h})(\delta_1+\gamma_2)]}, \tag{135}$$

$$R_{\hbar}^{i+1} = \frac{R_{\hbar}^{i} + (1-e^{-h})[\gamma_1 I_{\hbar}^{i+1} + \gamma_2 H^{i+1}]}{[1+\delta_1(1-e^{-h})]}, \tag{136}$$

$$S_{v}^{i+1} = \frac{S_{v}^{i} + \alpha_2(1-e^{-h})}{[1+(1-e^{-h})(v\theta_2 I_{\hbar}^{i}+\delta_2)]}, \tag{137}$$

$$E_{v}^{i+1} = \frac{E_{v}^{i} + v\theta_2(1-e^{-h})I_{\hbar}^{i}S_{v}^{i+1}}{[1+(1-e^{-h})(\delta_2+\eta_2)]}, \tag{138}$$

$$I_{v}^{i+1} = \frac{I_{v}^{i} + \eta_2(1-e^{-h})E_{v}^{i+1}}{[1+\delta_2(1-e^{-h})]}. \tag{139}$$

Thus, the NSFD scheme of the model (2) is obtained as (132)–(139) [44]. From Eqs. (120)–(124) implies that

$$\frac{N_{\hbar}^{i+1} - N_{\hbar}^{i}}{\varpi(h)} = \alpha_1 - \delta_1 N_{\hbar}^{i+1} - \frac{\Omega}{\delta_1} I_{\hbar}^{i} - \frac{\rho}{\delta_1} E_{\hbar}^{i}, \tag{140}$$

which is the exact finite difference of Eq. (130) while from Eqs. (125)–(124) implies that

$$\frac{N_{v}^{i+1} - N_{v}^{i}}{\varpi(h)} = \alpha_2 - \delta_2 N_{v}^{i+1}. \tag{141}$$

which is the exact finite scheme for the Eq. (131). The system of difference equations given in Eqs. (132)–(139) are not in a form suitable for computation. Since each of these equations is linear in $S_{\hbar}^{i+1}, E_{\hbar}^{i+1}, I^{i+1}, H^{i+1}, R_{\hbar}^{i+1}, S_{v}^{i+1}, E_{v}^{i+1}$, and I_{v}^{i+1} a rather long but elementary calculation gives:

$$\begin{aligned}
S_{\hbar}^{i+1} &= \tfrac{S_{\hbar}^{i}+(1-e^{-h})\alpha_1}{[1+v\theta_1(1-e^{-h})I_{v}^{i}+\delta_1]}, \\
E_{\hbar}^{i+1} &= \tfrac{E_{\hbar}^{i}+v\theta_1 I_{v}^{i}(1-e^{-h})(S_{\hbar}^{i}+(1-e^{-h})\alpha_1)}{[1+(1-e^{-h})(\eta_1+\rho+\delta_1)][1+v\theta_1(1-e^{-h})I_{v}^{i}+\delta_1]}, \\
I_{\hbar}^{i+1} &= \tfrac{G_1}{G_2}, \\
H^{i+1} &= \tfrac{G_3}{G_4}, \\
R_{\hbar}^{i+1} &= \tfrac{R_{\hbar}^{i}}{[1+\delta_1(1-e^{-h})]} + \tfrac{G_5}{[1+\delta_1(1-e^{-h})]} + \tfrac{G_6}{[1+\delta_1(1-e^{-h})]}, \\
S_{v}^{i+1} &= \tfrac{S_{v}^{i}+\alpha_2(1-e^{-h})}{[1+(1-e^{-h})(v\theta_2 I_{\hbar}^{i}+\delta_2)]}, \\
E_{v}^{i+1} &= \tfrac{E_{v}^{i}+v\theta_2(1-e^{-h})I_{\hbar}^{i}(S_{v}^{i}+\alpha_2(1-e^{-h}))}{[1+(1-e^{-h})(\delta_2+\eta_2)][1+(1-e^{-h})(v\theta_2 I_{\hbar}^{i}+\delta_2)]}, \\
I_{v}^{i+1} &= \tfrac{I_{v}^{i}+\eta_2(1-e^{-h})(E_{v}^{i}+v\theta_2(1-e^{-h})I_{\hbar}^{i}(S_{v}^{i}+\alpha_2(1-e^{-h})))}{[1+\delta_2(1-e^{-h})][1+(1-e^{-h})(\delta_2+\eta_2)][1+(1-e^{-h})(v\theta_2 I_{\hbar}^{i}+\delta_2)]},
\end{aligned} \tag{142}$$

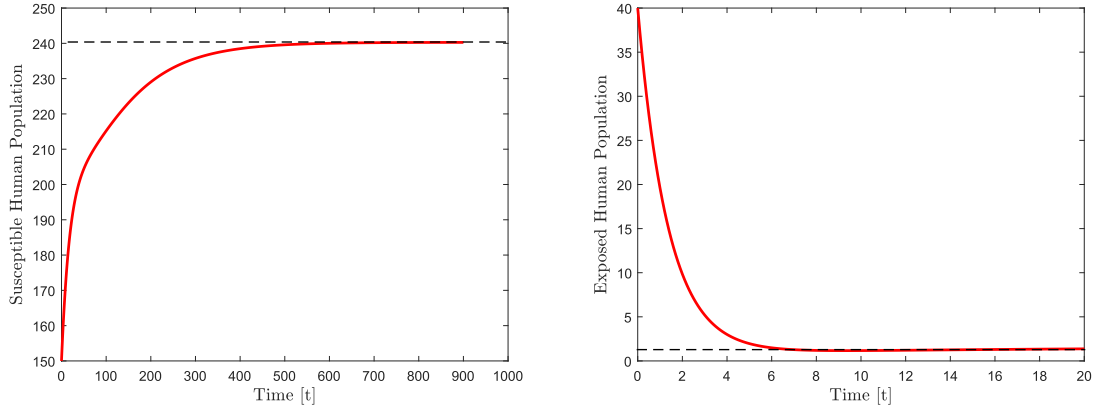

Fig. 2 Profiles of Susceptible $S_{\hbar}(t)$ and Exposed $E_{\hbar}(t)$ human population with a dashed line for equilibrium position, left and right hand figures respectively.

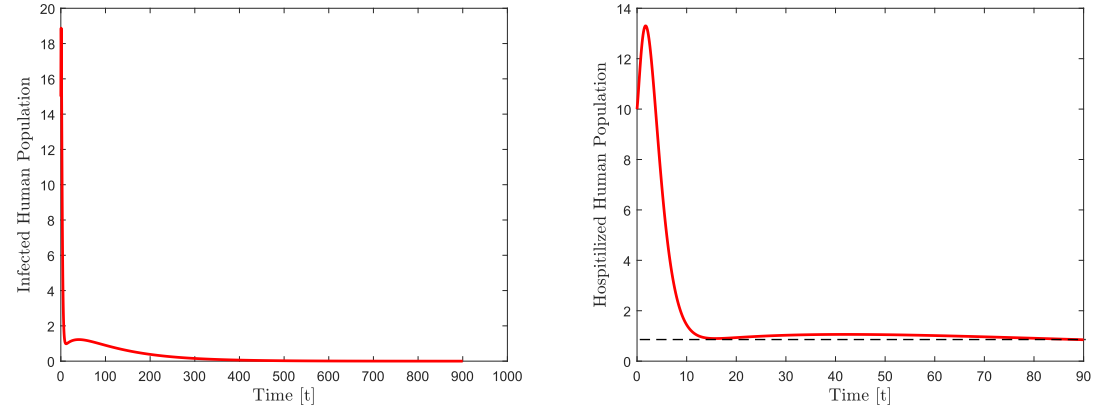

Fig. 3 Profiles of Infected $I_{\hbar}(t)$ and Hospitalized $H(t)$ human population with a dashed line for equilibrium position, left and right hand figures respectively.

where, $G_1 = I_{\hbar}^i + \eta_1(1-e^{-h})(E_{\hbar}^i + v\theta_1 I_v^i(1-e^{-h})(S_{\hbar}^i + (1-e^{-h})\quad \alpha_1)), G_2 = [1+(1-e^{-h})(\gamma_1+\delta_1+\Omega+\hbar)][1+(1-e^{-h})(\eta_1+\rho+\delta_1)][1+v\theta_1(1-e^{-h})I_v^i+\delta_1]$, $G_3 = H^i + \bar{\hbar}(1-e^{-h})(I_{\bar{\hbar}}^i + \eta_1(1-e^{-h})(E_{\bar{\hbar}}^i + v\theta_1 I_v^i(1-e^{-h})(S_{\bar{\hbar}}^i + (1-e^{-h})\alpha_1)))$, $G_4 = [1+(1-e^{-h})(\delta+\gamma_2)][1+(1-e^{-h})(\gamma_1+\delta_1+\Omega+\bar{\hbar})][1+(1-\ e^{-h})(\eta_1+\rho+\ \delta_1)][1+v\theta_1(1-e^{-h})I_v^i+\delta_1]$, $G_5 = (1-e^{-h})\gamma_1(G_1/G_2)$, and

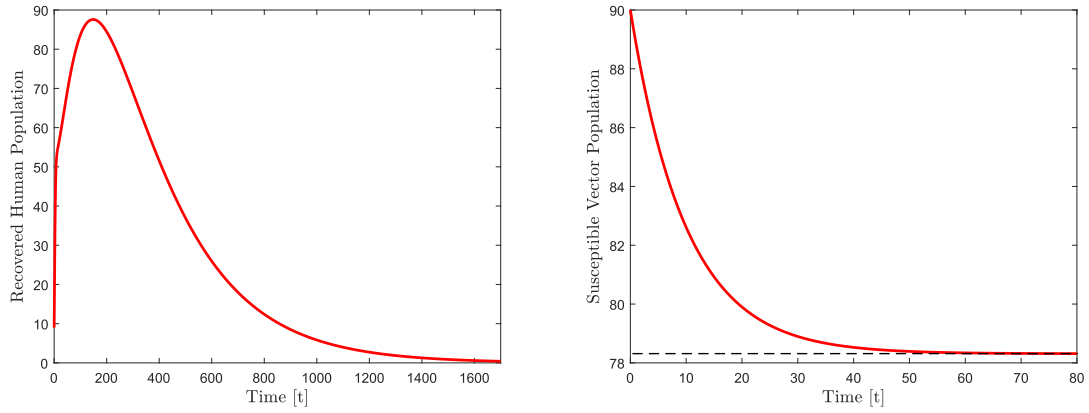

Fig. 4 Profiles of Recovered $R_{\hbar}(t)$ Human and Susceptible $S_v(t)$ vector population with a dashed line for equilibrium position, left and right hand figures respectively.

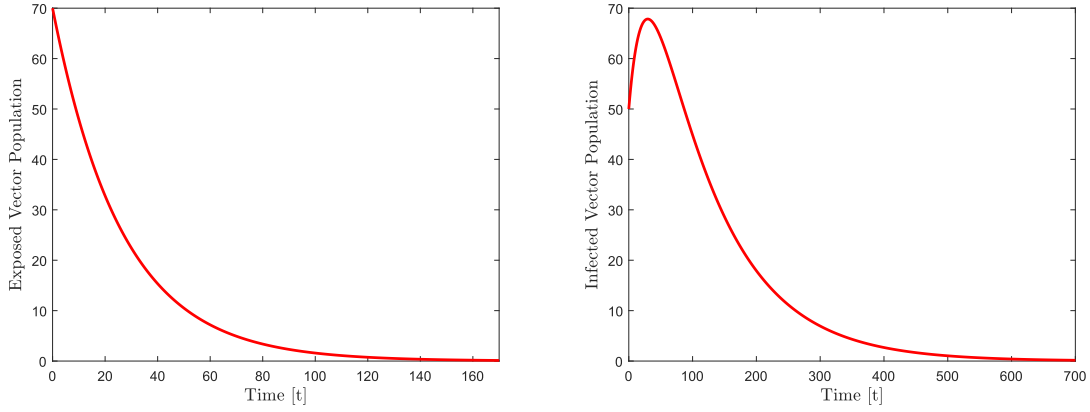

Fig. 5 Profiles of Exposed $E_v(t)$ and Infected $I_v(t)$ vector population with a dashed line for equilibrium position, left and right hand figures respectively.

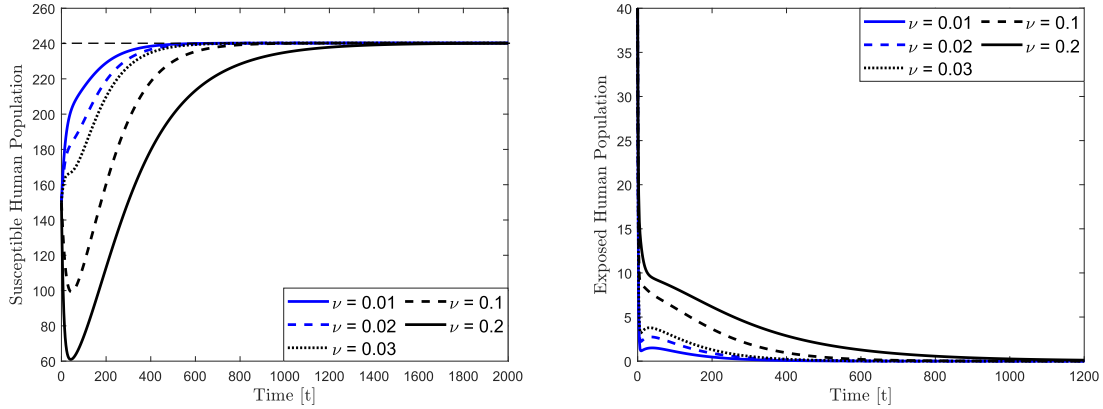

Fig. 6 Profiles of Susceptible $S_{\hbar}(t)$ and Exposed $E_{\hbar}(t)$ human population with the effect of a parameter v, left and right hand figures respectively.

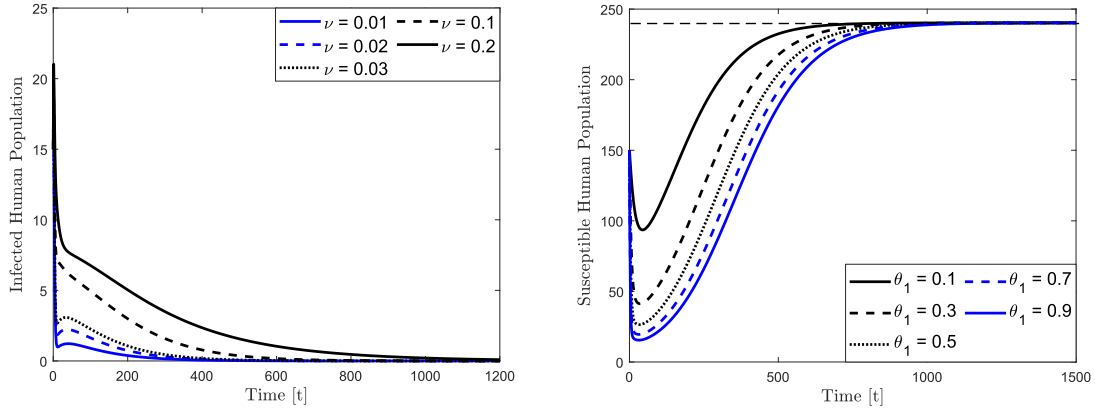

Fig. 7 Profiles of Infected $I_\hbar(t)$ and Susceptible $S_\hbar(t)$ human population with the effect of parameters v and θ_1, left and right hand figures respectively.

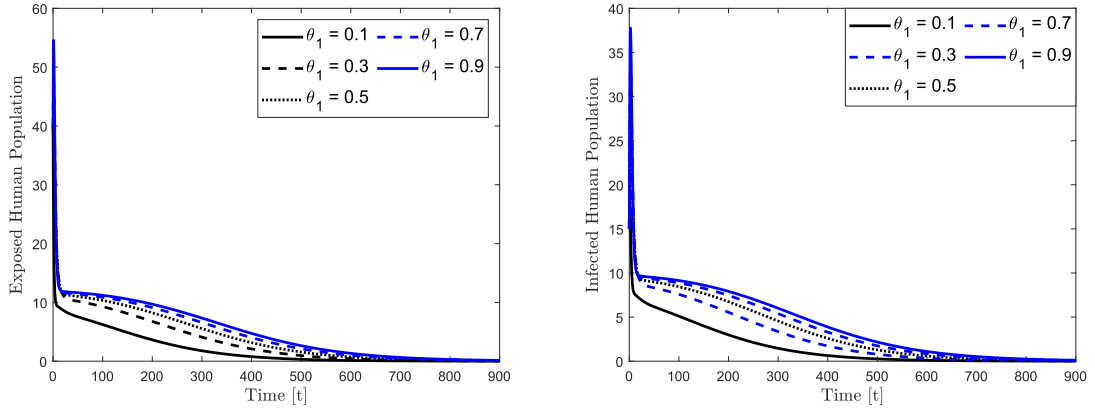

Fig. 8 Profiles of Exposed $E_\hbar(t)$ and Infected $I_\hbar(t)$ human population with the effect of a parameter θ_1, left and right hand figures respectively.

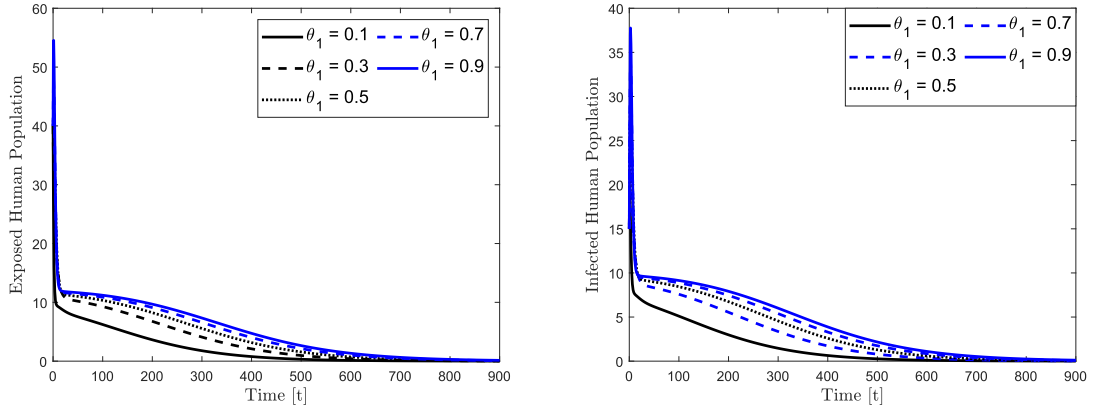

Fig. 9 Profiles of Recovered $R_\hbar(t)$ and Infected $I_\hbar(t)$ human population with the effect of a parameter $\hbar$, left and right hand figures respectively.

$G_6 = (1 - e^{-h})\gamma_2 G_3 / G_4$. Consider the system of Eqs. (142), such that the right hand side of the system is positive for all $i > 0$, while for step size $h > 0$. There are also no limitations on the step size h for the above general NSFD scheme (142) of the model (2) which is the powerful method been discussed in [40]. Next, the significant of the NSFD scheme are the following:

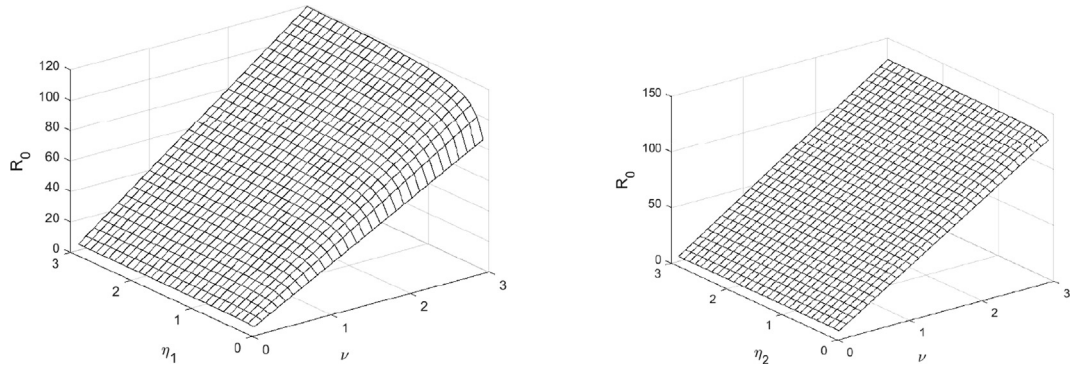

Fig. 10 Profile of the effect of parameter η_1 and v and the other hand η_2 and v on basic reproduction number R_0, left and right hand figures respectively.

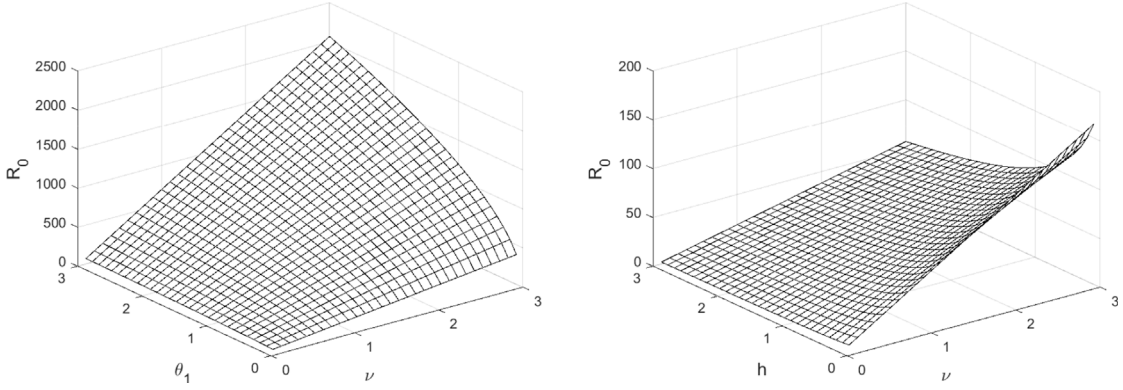

Fig. 11 The effect on basic reproduction number R_0 by parameters v and θ_1 while v and $\hbar$, left and right hand figures respectively.

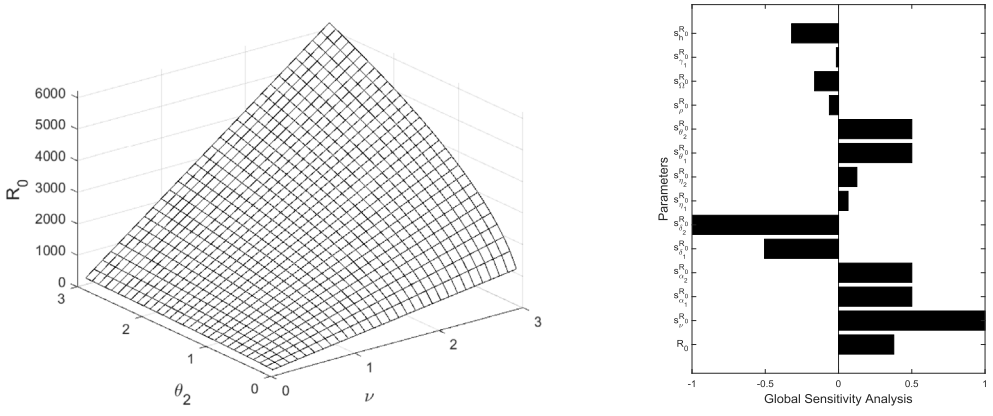

Fig. 12 The effect on basic reproduction number R_0 by parameters v and θ_2 while on the other hand, Bar graph of Sensitivity Analysis is presented based on sensitivity indices given in 1. Also, the basic reproduction number is shown in the bar graph which is computed from the model (2) as 0.3771.

- The NSFD solution $(S_{\hbar}^{i+1}, E_{\hbar}^{i+1}, I_{\hbar}^{i+1}, H^{i+1}, R_{\hbar}^{i+1}, S_v^{i+1}, E_v^{i+1}, I_v^{i+1})$ to model (2) is only determined by $(S_{\hbar}^{i}, E_{\hbar}^{i}, I_{\hbar}^{i}, H^{i}, R_{\hbar}^{i}, S_v^{i}, E_v^{i}, I_v^{i})$, with a step size h and non–negative parameters in 3.
- The denominator function does not require a particular specialized form and is explicitly determined while the scheme solution satisfies the positivity requirement for the step size h.

The 2–6, the proposed model of Cutaneous Leishmaniasis disease is simulated for sixteen hundred (1600) days to investigate the equilibrium state of the solution curves while the stability of the curve we then make the x–axis adjusted to see the best behavior in the span of sixteen hundred (1600) days. The sub-figures in profile 2, Susceptible $S_{\hbar}(t)$ and Exposed $E_{\hbar}(t)$ population are presented. The Susceptible $S_{\hbar}(t)$ population is increasing in time and then stays in the equilibrium position while the Exposed $E_{\hbar}(t)$ population is decreasing and stays in the equilibrium position. In 3, the Infected $I_{\hbar}(t)$ and Hospitalized $H(t)$ population are shown which are both decreasing in time and also staying in equilibrium position. This also means that the disease is stable and the basic reproduction number is less than one, in other words, $R_0 < 1$. In the sub-figures in 5, the Recovered human population is quickly increased and then takes a step to decrease in time before 200 days. Also, the Susceptible Vector population is decreasing in time and then in an equilibrium position. Sub–figures in 6,

Table 1 Sensitivity of the R_0 versus proposed parameters.

Parameter	$\mathbf{s}_{\tau}^{R_0}$	Value	Parameter	$\mathbf{s}_{\tau}^{R_0}$	Value	Parameter	$\mathbf{s}_{\tau}^{R_0}$	Value
v	$\mathbf{s}_{v}^{R_0}$	1	α_1	$\mathbf{s}_{\alpha_1}^{R_0}$	0.5000	α_2	$\mathbf{s}_{\alpha_2}^{R_0}$	0.5000
δ_1	$\mathbf{s}_{\delta_1}^{R_0}$	−0.5048	δ_2	$\mathbf{s}_{\delta_2}^{R_0}$	−1.1250	η_1	$\mathbf{s}_{\eta_1}^{R_0}$	0.0647
θ_1	$\mathbf{s}_{\theta_1}^{R_0}$	0.5000	θ_2	$\mathbf{s}_{\theta_2}^{R_0}$	0.5000	ρ	$\mathbf{s}_{\rho}^{R_0}$	−0.0622
γ_1	$\mathbf{s}_{\gamma_1}^{R_0}$	−0.0146	$\hbar$	$\mathbf{s}_{\hbar}^{R_0}$	−0.3201	η_2	$\mathbf{s}_{\eta_2}^{R_0}$	0.1250
Ω	$\mathbf{s}_{\Omega}^{R_0}$	−0.1630	R_0	–	0.3771			

Table 2 Description of compartments and the initial conditions.

No#	Parameter	Description of Compartment	Initial Condition
1.	$S_{\hbar}$	Susceptible Human population.	150
2.	$E_{\hbar}$	Infected Human population.	40
3.	$I_{\hbar}$	Exposed Human population.	15
4.	H	Hospitalized Human population.	10
5.	$R_{\hbar}$	Recovered Human population.	9
6.	S_v	Susceptible Vector population.	90
7.	E_v	Infected Vector population.	70
8.	I_v	Exposed Vector population.	50

Table 3 Description, value and dimension of each parameter.

No#	Parameter	Description of Parameter	Value	Source
1.	v	Sand fly biting rate	0.01	0.01
2.	α_1	The new recruitment ratio of humans to the class at risk	10.1	[26]
3.	α_2	The new recruitment ratio of vector to the class at risk	8.23	[26]
4.	γ_1	Rate of Recovery of Infected human population	0.025	[26]
5.	γ_2	Rate of Recovery of Infected human Population in hospital	0.63	Assumed
6.	δ_1	Natural Death Rate in human population	0.004	[26]
7.	δ_2	Natural Death Rate in vector population	0.01	[26]
8.	θ_1	The transfer probability of ACL from sandfly to human	0.009	[26]
9.	θ_2	The CL probability of spread from human to sand fly	0.001	[26]
10.	Ω	Death rate of infected human due to disease	0.28	Assumed
11.	ρ	Death rate of exposed human due to disease	0.10	Assumed
12.	η_1	Rate of infection of exposed human	0.7	[26]
13.	η_2	Rate of infection of exposed vector	0.03	[26]
14.	$\hbar$	Transfer of infected human to hospital	0.55	Assumed

Exposed Vector population is decreasing and on the other hand, infection in the vector population is also decreased after a few weeks which is a consequence of the $R_0 < 1$.

Similarly, we also want to investigate the effect of some parameters on the population such that in sub–figures of 7 and 8, the effect of Sand fly biting rate v causes a difference in the Susceptible, Exposed and Infected human population. As the value of v increasing the population takes long time to reach the equilibrium states. Also, due to the effect of The CL probability of spread from sand fly to human θ_1 the population leads to equilibrium state after taking long time and so in Exposed and Infected human population. As a result of the rate of hospitalization $\hbar$ of Infected human population, the infection in the population is decreased and this is the advantage of our model in which we included the effect of hospitalization in the $I(t)$ compartment, also the recovery is increased due to the hospitalization of Infected population. Therefore, our proposed model showing best results and is better than the existing model.9.

The sub-figures in 10–12, we check the effect of parameters associated with R_0 on R_0. We used several parameters such as $\eta_1, \eta_2, \theta_1, h$ as$\hbar$, and θ_2 along with v in each figure. We took a range of 0 to 3 for each parameter for the simulation.

Eventually, in the bar graph in 12, we present the sensitivity analysis based on the 1 in which the parameter that can cause the disease and the spread of the infection the most are v, α_1, α_2, and θ_2. Also, these parameters should be controlled so that the infection can not spread again in the population. The R_0 computed from the model is 0.3771 which is of course less than one and needs to keep it less than one by controlling the above-mentioned parameters.2

8. Conclusion

We have presented a detailed mathematical analysis of the Cutaneous Leishmaniasis disease model. The concerned analysis is devoted to the formulation of the model, the existence theory of solution. Moreover, global and local stability together with the computation of R_0 have been performed by using various mathematical tools. In the concerned tools we have used Lyapunov function method, and the compound matrix method based on the Metzler procedure. Applying the tools of numerical-functional analysis, we have presented the sensitivity analysis of the model also. On using NSFD scheme, an algorithm has been developed. With the help of this algorithm, we have presented the models corresponding to some reals values of the parameters graphically. The obtained results and their graphical analysis have been explained in details.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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