H. PETRYK
Institute of Fundamental Technological Research
Warsaw, Poland

ON ENERGY CRITERIA OF PLASTIC INSTABILITY

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1. INTRODUCTION

An observation based on long-standing experience is that a plastic body deformed under conditions which could result in a quasi-static, macroscopically uniform deformation process, in certain circumstances begins to deform in a quite different way. The term instability is commonly used in such cases though more in an intuitive than in a specified sense. Typical examples of so-understood instability are the buckling, bulging and necking phenomena as "geometric" instabilities, snap-through phenomena as "dynamic" instabilities, and shear band localization of deformation as a "local" instability. Similar phenomena may naturally appear in the course of non-uniform deformations.

The different forms of instability in plastic solids have been investigated in the literature by using various theoretical approaches. The basic tool in analysis of "geometric" instabilities is nowadays the bifurcation theory due to Hill [1÷4], supplemented by studies of post-bifurcation behaviour and imperfection sensitivity (cf. [5÷8]). Appearing of uncontrolled, dynamic deformations is usually attributed to an instability of equilibrium and investigated on the basis of an energy-type criterion of instability of equilibrium, as e.g. those in [1,9,10]. The localization of deformation, either in thin metal sheets or within a three-dimensional body, is usually examined by considering a possible bifurcation within a band [11,12] or by studying the growth of initial imperfections as in Marciniak's approach [13] (cf. [14÷17]). In many cases, an intuitive instability criterion is formulated for the particular problem considered, as in [18,19].

None of the approaches mentioned above allows to investigate all the "geometric", "dynamic" and "local" instabilities in a unified way. In this paper we explore a possibility that the observed distinct forms of plastic instability are merely different symptoms of instability of the fundamental deformation process in the energy sense. Such approach, based on a single energy-type postulate of stability of a quasi-static deformation process, has been proposed recently by the author [20÷22]. In the assumed definition of stability the persistent disturbances are considered for which an energy measure is adopted. For our present purposes, we use a simplified and less restrictive version of the postulate, starting from a stability criterion which is a consequence of the basic stability definition and not from the definition itself. To make the paper self-contained, the stability criterion is introduced here as an independent hypothesis. In the criterion it is assumed that a sufficiently small deformation increment in a stable fundamental process requires less energy to be supplied from external sources to the system consisting of the body and loading device than any other kinematically admissible deformation increment. The deformations are caused by varying loading conditions; in particular, if the loading conditions do not vary in time then the criterion of stability of a deformation process reduces to the familiar concept of stability of equilibrium in the energy sense. The criterion is proposed here for loadings which are conservative in an overall sense, generalizing in this way the previous criteria [21,22] formulated for the surface tractions which were conservative at each surface point separately.

In formulating the stability postulate, the mechanical properties of the material are assumed to be time-independent but are otherwise arbitrary. However, the postulate itself imposes certain symmetry restrictions on the general form of constitutive relations (which will be studied elsewhere). Implications of the assumed stability criterion will be examined below assuming a class of nonlinear constitutive rate equations which admit a potential provided they are expressed in terms of rates of work-conjugate stress and strain measures [23,4]. This class of constitutive relations contains the rate equations for conventional elastic-plastic solids obeying the normality flow rule as a special case. Existence of potential assures that the incremental moduli have the needed principal symmetry property. The evolution equations for the moduli are left unspecified though their form is also expected to be restricted by the stability requirement.

It is not the aim of the present paper to analyze particular examples, rather, to provide a synthesis of the known results by specifying possibly general circumstances under which the proposed energy criterion yields the known criteria of plastic instability.

2. DEFINITIONS AND ASSUMPTIONS

2.1. Notation

We are concerned with isothermal, quasi-static deformations of a continuous body of time-independent material which in a fixed reference configuration occupied a space domain V bounded by a piecewise-regular surface S. $d\xi$ and da are infinitesimal elements of volume and of surface area, respectively, in the reference configuration in which space or surface integrations will be performed. The position vector of a material element in the reference or current configuration is denoted by $\underline{\xi}$ or $\underline{x}$, respectively. All vector or tensor components are for simplicity taken relative to a fixed rectangular basis and denoted by Latin subscripts which range from 1 to 3. A natural time does not appear at all throughout the paper and its role plays a scalar parameter t which is called time for simplicity. A deformation process is described by the equation $\underline{x} = \underline{x}(\underline{\xi},t)$. For any local quantity φ, the derivatives

$$\varphi_{,i} \equiv \frac{\partial\varphi(\underline{\xi},t)}{\partial\xi_i} , \qquad \dot{\varphi} \equiv \frac{\partial\varphi(\underline{\xi},t)}{\partial t} ,$$

and likewise the higher-order ones, are assumed to be at least piecewise continuous functions of $(\underline{\xi},t)$. A field defined on V will be distinguished from its value at $\underline{\xi}$ by a superimposed tilda, when needed.

$\underline{u} = \underline{x} - \underline{\xi}$, $\underline{v} = \dot{\underline{x}} = \dot{\underline{u}}$ and $\dot{\underline{F}} = \partial\underline{v}/\partial\underline{\xi}$ denote the displacement, velocity and velocity gradient in the reference configuration, respectively. The deformation and stress are measured by the deformation gradient tensor $\underline{F} = \partial\underline{x}/\partial\underline{\xi}$ and the nominal stress tensor $\underline{s}$, respectively, which form a work-conjugate pair in the sense that the deformation work in the body is equal to

$$W = \int_V \int s_{ij}\, dF_{ji}\, d\xi = \int_V \int s_{ij}\, v_{j,i}\, dt\, d\xi , \tag{1}$$

with the summation convention for repeated indices. $\underline{s}$ is related to the symmetric Cauchy stress $\underline{\sigma}$ by the formula $F_{ij}\, s_{jk} = \det(\underline{F})\, \sigma_{ik}$. We assume that $\underline{u}$ is a continuous function of $(\underline{\xi},t)$, $\underline{F}$ varies continuously in time while $\underline{v}$ is a continuous function of place.

2.2. Constitutive relations

A general assumption is made that the mechanical properties of the material do not depend in any way on a natural time. Starting from the Section 3 we will assume that the constitutive relations for the material can be expressed in the general rate form proposed by Hill [1,2] :

$$\dot{s}_{ij} = \frac{\partial U}{\partial(v_{j,i})} , \qquad 2U = \dot{s}_{ij}\, v_{j,i} , \tag{2}$$

no matter what is their original (objective) form (for the formulae relating (2) to the constitutive rate equations written in terms of objective work-conjugate measures, see [4]). U is a continuous, continuously differentiable and piecewise-continuously twice differentiable function of $\dot{\underline{F}}$, dependent also on the deformation history. Since the material is time-independent, the potential U is homogeneous of degree two and can be thus written in the form $(2)_2$. A value of U will be denoted equivalently as $U(\dot{\underline{F}}) = U(v_{j,i}) = U[\underline{v}]$. The homogeneous relationship (2) between $\dot{\underline{s}}$ and $\dot{\underline{F}}$ can also be written, by the Euler theorem, as

$$\dot{s}_{ij} = C_{ijkl}(\dot{\underline{F}})\, v_{l,k} , \qquad C_{ijkl} = C_{klij} , \tag{3}$$

provided the second derivatives of U

$$\frac{\partial^2 U(\dot{\underline{F}})}{\partial(v_{l,k})\,\partial(v_{j,i})} \equiv C_{ijkl}(\dot{\underline{F}}) \tag{4}$$

are continuous at $\dot{\underline{F}}$ so that the incremental moduli tensor $\underline{C}(\dot{\underline{F}})$ is well defined. The notation $\underline{C}(\dot{\underline{F}})$ indicates the dependence of the moduli on the actual rate of strain, present in any model of elastic-plastic response. The dependence is homogeneous of degree zero and in general only piecewise-continuous, for instance, piecewise constant when the relation (3) is piecewise-linear. We will denote $\underline{C}(\dot{\underline{F}})$ equivalently as $\underline{C}[\underline{v}]$. In general, the body may be inhomogeneous so that U and $\underline{C}$ are tacitly assumed below to depend also expticitly on $\underline{\xi}$ (in a piecewise smooth manner).

2.3. Loading device

We consider the body placed in a loading device which applies body forces and surface tractions and constraints surface displacements in a manner dependent on a scalar loading parameter λ varying in time. The surface tractions are allowed to be configuration-dependent and may even be functionals (dependent on λ) of displacement field on the whole body surface. However, the surface loading is assumed to be conservative in the following overall sense (cf. [3,24]): For each fixed value of λ, the total work done by the applied surface tractions vanishes for all closed virtual displacement paths which are compatible with geometric constraints and restore the material surface points to their starting positions. If we denote by $\underline{T}$ the nominal surface tractions (per unit reference area) then this assumption is written as

$$\int_S \oint T_j \, du_j \, da = 0 \, , \qquad \lambda = \text{const} \, . \tag{5}$$

As examples of such loadings, we can mention (i) the tractions acting on the contact surface with a deformable hyperelastic continuum, (ii) the loading by uniform pressure on a surface part of fixed perimeter,(iii) the loading by smoothly distributed springs, or (iv) the dead loading. At a certain $\underline{\xi}$ and fixed λ, $\underline{T}$ is (i) a functional of the surface displacement field, (ii) a function of the displacement and its surface gradient at $\underline{\xi}$, (iii) a function of the displacement at $\underline{\xi}$, or (iv) independent of another factors, respectively. Surface displacement may be constrained by contact with a rigid tool, however, if slipping on the tool-material interface is allowed then it must occur without friction. The nominal body forces $\underline{b}$ (per unit reference volume) are taken for simplicity to depend only on $\underline{\xi}$ and λ, though conservative configuration-dependent body forces could be considered as well.

From the above assumptions it follows that the total work done by the body forces and surface tractions in a virtual motion compatible with geometric constraints and leading from $\underline{\tilde{u}}^1$ to $\underline{\tilde{u}}^2$ at fixed λ (the right hand side expression in (6)) is path-independent. (This work is path-dependent if λ is varying during the motion). Hence, for each λ we may define the potential energy of the loading device Ω as a functional defined on a class of admissible displacement fields $\underline{\tilde{u}}$ such that

$$\Omega[\underline{\tilde{u}}^1;\lambda] - \Omega[\underline{\tilde{u}}^2;\lambda] = \int_V \int_{\underline{u}^1}^{\underline{u}^2} b_j \, du_j \, d\xi + \int_S \int_{\underline{u}^1}^{\underline{u}^2} T_j \, du_j \, da \, , \quad \lambda = \text{const} \, . \tag{6}$$

Formally, Ω is defined in this way to within an additive function of λ which may be chosen arbitrarily with no influence on the stability criteria discussed below. The choice may be done once the loading device is specified in order to give the term "potential energy" a physical meaning. Note that if λ is varying along a deformation path then the increment of the value of Ω is still path-independent but is no longer equal to the work done by the body forces and surface tractions with negative sign.

2.4. General stability criterion

For a class of kinematically admissible deformation processes, introduce the energy functional E defined by

$$E = W + \Omega \ . \tag{7}$$

In general, E is a functional of the whole deformation history due to the path-dependence of the deformation work W. With an appropriate specification of Ω (cf. the remark following the formula (6)), the increment of the value of E in a quasi-static deformation process can be interpreted as the amount of energy which has to be supplied in that process from external sources to the system consisting of the body and the loading device.

We are concerned with stability of an idealized, isothermal, quasi-static deformation process, called the fundamental process, which is intended to describe sufficiently slow deformations of a real body subject to varying loading. The value of any quantity in the fundamental process will be distinguished by the superscript " ^{0} ". At every stage of the deformation along the fundamental path, we consider a class of kinematically admissible branching paths. Note that a branching path need not satisfy here the conditions of continuing equilibrium; when it does then the term "bifurcation" will be used. We will consider only such deformation paths along which the increment δE of the value of the energy functional (7) from that at the branching point can be developed into a Taylor series with respect to a small time increment δt at least to the second order,

$$\delta E = \dot{E}\ \delta t + \frac{1}{2}\ddot{E}\ (\delta t)^2 + \dots , \tag{8}$$

where $\dot{E}$ and $\ddot{E}$ are the first and second time derivatives of E taken along a branching path at the instant of branching. We postulate now the following stability criterion (cf. [21,22]).

Criterion 1. The fundamental process is stable only if

$$\delta E \geq \delta E^0 \tag{9}$$

for all kinematically admissible branching paths and for sufficiently small δt.

Note that this criterion gives only necessary condition for stability. If (9) does not hold for some branching mode then the fundamental process is unstable in the energy sense, and it is conjectured that this corresponds to some observable form of instability.

Suppose now that the stresses, surface tractions and body forces vary continuously in time in any deformation process. This assumption is generally not satisfied for rigid-plastic solids or for unilateral constraints, therefore both are excluded from now on. Denote by the prefix Δ the difference of any corresponding quantities in the branching and fundamental processes at the instant of branching. From (1), (6) and (7) we have

$$\dot{E} - \dot{E}^0 \equiv \Delta\dot{E} = \int_V \{s^0_{ij}\ \Delta v_{j,i} - b_j\ \Delta v_j\}\ d\xi - \int_S T^0_j\ \Delta v_j\ da \tag{10}$$

for all $\Delta\bar{v}$ compatible with the kinematical constraints. Since we have excluded unilateral constraints, from (8) and the Criterion 1 it follows that the expression (10) vanishes in a stable fundamental process for all admissible $\Delta\tilde{v}$. This is just the virtual work principle which is satisfied a fortiori if the fundamental process corresponds to a quasi-static solution to the problem. From (8) and the Criterion 1 we obtain the following Criterion 2. The fundamental process is stable only if

$$\Delta\ddot{E} \geq 0 \tag{11}$$

for all kinematically admissible branching paths.

3. MINIMUM PRINCIPLE FOR VELOCITIES

To find an explicit form of (11), we specify now the surface data in an incremental, linearized form. The material body surface is split into two complementary parts, S_u and S_T. On the part S_u the displacement increments $\delta\underline{u}$ are given functions of λ. On the part S_T the first-order expression for the increment $\delta\underline{T}$ of the nominal traction from its starting value $\underline{T}^0$ is assumed to consist, as in [3,4], of two parts

$$\delta T = \frac{\partial T}{\partial \lambda}\,\delta\lambda + \underline{f}[\delta\underline{u}] + \dots \;. \tag{12}$$

The first term on the right hand side of (12) represents the part of the traction increment which is independent of the deformation increment, and the second term is the deformation-sensitive part. Both terms may depend on $\underline{\xi}$ and λ. $\underline{f}[\underline{w}]$ is assumed to be a linear homogeneous function of a space vector $\underline{w}$ and its surface gradient at the surface point considered ($\underline{f}[\underline{w}]$ could depend in the same way also on higher order surface gradients of $\underline{w}$ with no influence on the following considerations). The assumption (5) of conservative loading impliess Hill's "self-adjointness" condition [3,4]

$$\int_{S_T} \left\{ f_j[\underline{v}^2]\, v_j^1 - f_j[\underline{v}^1]\, v_j^2 \right\} da = 0 \;, \tag{13}$$

where $\tilde{v}^1$ and $\tilde{v}^2$ are any pair of continuous and piecewise-continuously differentiable vector fields whose difference vanishes over S_u.

Identifying the displacement fields $\tilde{u}^2$ and $\tilde{u}^1$ in the expression (6) with those on the branching and fundamental paths and using (12) and (13), we obtain the second-order formula

$$-\,\hat{\Delta}\Omega = \int_V b_j\, \hat{\Delta} u_j\, d\xi + \int_{S_T} \left\{ T_j^0 + \frac{1}{2} f_j[\hat{\Delta}\underline{u}] \right\} \hat{\Delta} u_j\, da + \dots \;. \tag{14}$$

All quantities in (14), and likewise their differences denoted by $\hat{\Delta}$, are calculated at an instant following the instant of branching. Hence, at the instant of branching when $\Delta\tilde{u} = 0$ we have

$$-\,\Delta\ddot{\Omega} = \int_V (2\,\dot{b}_j\, \Delta v_j + b_j\, \Delta\dot{v}_j)\, d\xi + \int_{S_T} \left\{ (2\,\dot{T}_j^0 + f_j[\Delta\underline{v}])\,\Delta v_j + T_j^0\, \Delta\dot{v}_j \right\} da. \tag{15}$$

In calculating the second time derivative of ΔW, we take into account the possibility of a velocity-gradient discontinuity across a moving surface whose image S_D in the reference configuration moves with a normal speed v_n. By using the transport theorem, we obtain

$$\ddot{W} = \int_V (\dot{s}_{ij}\, v_{j,i} + s_{ij}\, \dot{v}_{j,i})\, d\xi - \int_{S_D} s_{ij} [\![v_{j,i}]\!] v_n\, da \ , \tag{16}$$

where $[\![\cdot]\!]$ denotes a jump across S_D with the usual sign convention. Since $\tilde{v}$ is and remains continuous, the standard compatibility conditions require that there is a jump in accelerations $\dot{v}$ across S_D such that

$$[\![v_{j,i}]\!] v_n = -[\![\dot{v}_j]\!] n_i \ , \tag{17}$$

where $\underline{n}$ is the (appropriately directed) unit normal to S_D. By substituting (17) to (16) and using the equilibrium conditions for $\underline{s}^0$ and the divergence theorem, we obtain that at the instant of branching there is

$$\Delta\ddot{W} = \Delta\Big\{\int_V \dot{s}_{ij}\, v_{j,i}\, d\xi\Big\} + \int_V b_j\, \Delta\dot{v}_j\, d\xi + \int_{S_T} T^0_j\, \Delta\dot{v}_j\, da \ . \tag{18}$$

Finally, by combining (15) and (18) and using (2), (12) and (13), we arrive at the result

$$\frac{1}{2}\, \Delta\ddot{E} = \Delta J[\tilde{v}] \ , \tag{19}$$

where

$$J[\tilde{v}] = \int_V \{U[\underline{v}] - \dot{b}_j\, v_j\}\, d\xi - \int_{S_T} \Big\{ \frac{\partial T_j}{\partial \lambda}\, \dot{\lambda} + \frac{1}{2}\, f_j[\underline{v}]\Big\}\, v_j\, da \tag{20}$$

is a functional defined on the class of continuous and piecewise-continuously differentiable velocity fields $\tilde{v}$ taking the values prescribed over S_u at the instant of branching. For convenience, from now on $\tilde{w}$ will denote an arbitrary continuous and piecewise-continuously differentiable vector field on V which vanishes over S_u. Each velocity field admissible in (20) has now the form $\tilde{v}^0 + \tilde{w}$. By substituting (19) in the Criterion 2, we obtain the following minimum principle for velocities.

Criterion 3. The fundamental process is stable only if

$$J[\tilde{v}] \geqq J[\tilde{v}^0] \qquad \text{for all} \quad \tilde{v} = \tilde{v}^0 + \tilde{w} \ . \tag{21}$$

This is an extension of the criterion derived in [21] to a wider class of loadings and with velocity gradient discontinuities taken into account. The Criterion 3 reduces the problem of "detecting" instabilities to establishing the circumstances under which the fundamental velocity field $\tilde{v}^0$ fails to minimize the value of the functional (20).

The functional (20) coincides with that considered by Hill [3,4]. As shown by Hill, its first weak (Gateaux) variation vanishes at $\tilde{v} = \tilde{v}^0$ if and only if $\tilde{v}^0$ is a solution to the actual first-order rate boundary value problem. This variational principle appears here as a necessary condition for (21) and thus for stability of the fundamental deformation process.

The minimum principle (21) (with strict inequality) was proved originally by Hill [3] under the additional assumption that a condition sufficient for uniqueness of the solution is satisfied. On the contrary, (21) has been derived here from the postulated stability Criterion 1 and holds for any solution $\tilde{v}^0$, unique or not, which corresponds to a stable deformation process.

4. DISCUSSION

Along a typical path of deformation, the condition (21) (or (11), equivalently) will be satisfied with strict inequality for all non-zero $\tilde{w}$ up to a certain critical stage beyond which the functional (20) becomes indefinite. If instability did not take place earlier for another reason ((11) is only necessary for stability) then the critical stage marks the onset of instability of the fundamental deformation path. Suppose that this is the case. Now, it is esential in what way the functional (20) becomes indefinite and what properties it has at the critical stage since this strongly influences the post-critical behaviour of the body. Three typical cases are discussed below which are related to the distinct observable forms of plastic instability mentioned in the Section 1.

4.1. "Geometric" instabilities

Suppose that at the critical stage (21) still holds but with equality for some $\tilde{v}^* \neq \tilde{v}^0$. Then $\tilde{v}^*$, likewise $\tilde{v}^0$, renders the functional (20) a minimum and thus also a stationary value. From the variational principle mentioned above it follows that $\tilde{v}^*$ is another solution to the first-order rate boundary value problem, that is, at the critical stage we have bifurcation in velocities. Hence, in this case the search for the onset of instability in the sense of the energy criterion is reduced to a search for the corresponding bifurcation point. If the secondary post-bifurcation path is stable then it may replace the fundamental path beyond the critical point, leading in that way to an observable "geometric" instability.

In general, a bifurcation point may precede the considered critical stage with no consequence for stability of the fundamental path. We specify below the circumstances under which such cases are excluded so that the primary bifurcation takes place exactly at the critical stage. Suppose first that along the fundamental deformation path the potential U has continuous second derivatives with respect to $\dot{F}$ at $\dot{F}^0$, except possibly on certain surfaces, e.g. on the elastic-plastic interface. In other words, let the incremental moduli tensor C^0 relating $\dot{s}^0$ to $\dot{F}^0$ (cf. (3) and (4)) be well defined almost everywhere in V. In that case the second weak variation of the functional (20) at $\tilde{v} = \tilde{v}^0$ exists and is equal to

$$\delta^2 J[\tilde{v}^0 ; \tilde{w}] \equiv \frac{1}{2} \frac{d^2}{d\gamma^2} J[\tilde{v}^0 + \gamma \tilde{w}] \Big|_{\gamma=0} = I^0 [\tilde{w}] \quad , \tag{22}$$

where γ is a scalar and

$$I^0 [\tilde{w}] = \frac{1}{2} \int_V C^0_{ijkl} \, w_{j,i} \, w_{l,k} \, d\xi - \frac{1}{2} \int_{S_T} f_j[w] \, w_j \, da \tag{23}$$

is a quadratic functional defined on the class of $\tilde{w}$ as above.

The proof follows at once from the standard theorem of the calculus of variations applied separately to each subdomain in V in which the needed regularity conditions are satisfied, provided that in each subdomain the moduli tensor $C[v^0 + \gamma w]$ depends continuously on γ in an interval of γ containing 0 and independent of place. If this is not the case then we can separate a region $R(\hat{\gamma})$ in V such that $C[v^0 + \gamma w]$ is a continuous function of γ in $V \backslash R(\hat{\gamma})$ for $|\gamma| < \hat{\gamma}$ and apply the theorem in $V \backslash R(\hat{\gamma})$. By the assumption introduced above and the assumed regularity of U, the volume of $R(\hat{\gamma})$

can be taken to tend to zero as $\hat{\gamma} \to 0$ and in the limit we obtain (22).

Non-negativeness of the second variation is necessary for a mimimum. Hence, when the moduli $\underline{C}^0$ are well defined then the Criterion 3 yields the Criterion 4. The fundamental process is stable only if

$$I^0[\tilde{\underline{w}}] \geq 0 \qquad \text{for all } \tilde{\underline{w}} \ . \tag{24}$$

Suppose now that the moduli $\underline{C}^0$ define the incrementally linear "comparison solid" [1,2,4] such that

$$\Delta\dot{s}_{ij}\ \Delta\dot{F}_{ji} \geq C^0_{ijkl}\ \Delta\dot{F}_{ji}\ \Delta\dot{F}_{lk} \qquad \text{for all } \Delta\dot{\underline{F}} \tag{25}$$

where $\Delta\dot{\underline{s}} = \dot{\underline{s}} - \dot{\underline{s}}^0$, $\Delta\dot{\underline{F}} = \dot{\underline{F}} - \dot{\underline{F}}^0$ and $\dot{\underline{s}}$ is related to $\dot{\underline{F}}$ by the constitutive equation (2). The condition (25) is a weakened form of the "relative convexity" property [2,4] which requires (25) to hold also if $\dot{\underline{F}}^0$ is replaced by an arbitrary velocity gradient. For instance, (25) is satisfied for the elastic-plastic solids with piecewise-linear incremental response subject to the normality flow rule, either at a regular point or at a vertex on the yield surface, provided that the moduli $\underline{C}^0$ in the plastic zone correspond to the "fully" plastic branch [1,5,4].

From (25) by the same argument as in [4] we obtain that a bifurcation in velocities is excluded and (21) is satisfied as long as the functional I^0 is positive definite (i.e. $I^0[\tilde{\underline{w}}] > 0$ if $\tilde{\underline{w}} \neq \underline{0}$). In usual circumctances examined so far in many papers (cf. [5÷8]) this is so along the fundamental deformation path up to the stage when the primary bifurcation takes place. The corresponding eigenmode $\tilde{\underline{w}}^* = \tilde{\underline{v}}^* - \tilde{\underline{v}}^0$ renders the functional I^0 the zero stationary value. In general, beyond this stage the functional I^0 becomes indefinite; the Criterion 4 now shows that the instant of the primary bifurcation coincides with the onset of instability of the fundamental path in the energy sense. In those circumstances, the onset of buckling or necking found by using Hill's bifurcation theory results also from the Criteria 3 or 4 being now equivalent to each other. This is a generalization of the previous result [22] obtained for the conventional elastic--plastic solids with a smooth yield surface and for less general loading conditions. The Criterion 2 gives additionally an energy interpretation of the primary bifurcation as well as a reason for rejecting the fundamental post-bifurcation path as being unstable in the energy sense. The secondary post-bifurcation path may be stable; if there are more post-bifurcation paths then the stable path must minimize the increment δE of the energy functional (7), developed into a series up to the order needed.

4.2. "Dynamic" instabilities

To show that the Criterion 2 (or 3) can also be used to find the onset of uncontrolled dynamic deformations, we derive from it the following Criterion 5. The fundamental process is stable only if

$$I[\tilde{\underline{w}}] \equiv \int_V U[\underline{w}]\, d\xi - \frac{1}{2}\int_{S_T} f_j[\underline{w}]\, w_j\, da \geq 0 \qquad \text{for all } \tilde{\underline{w}} \ . \tag{26}$$

In proof, consider the quantity $(J[\tilde{\underline{v}}^0 + \gamma\tilde{\underline{w}}] - J[\tilde{\underline{v}}^0])/\gamma^2$ and put $\gamma \to \infty$ while $\tilde{\underline{w}}$ is held fixed. By using homogeneity and continuity of U and $\underline{f}$, we obtain in the limit $I[\tilde{\underline{w}}]$. Thus, if $I[\tilde{\underline{w}}]$ is negative then $J[\tilde{\underline{v}}^0 + \gamma\tilde{\underline{w}}]$ does not satisfy (21) for γ large enough. Hence, $I[\tilde{w}] \geq 0$ is necessary for stability, and the Criterion 5 has been proved.

Alternatively, the condition (26) can be obtained as a necessary condition for the stability of equilibrium, defined as the stability of the degenerate deformation process in which an equilibrium configuration of the body does not vary in time. The corresponding loading conditions cannot depend on time so that we must have $\dot{\lambda} = 0$ what implies $\dot{b} = 0$ in V and $\dot{u} = 0$ on S_u. Then (21) reduces simply to (26) (cf. [21,22]). Under these conditions, $I[\delta\tilde{u}]$ is the second-order expression for the amount of energy which has to be supplied to the system (body and loading device) along a direct virtual quasi-static deformation path leading from the equilibrium state to a neighbouring configuration defined by the displacement increment $\delta\tilde{u}$ (cf. (8), (10), (19) and the remark following (7)). Relation to the familiar energy criteria of stability of equilibrium (cf. [1,9]) is apparent, however, (26) is here necessary for stability rather than sufficient.

If (26) is not satisfied then there are certain modes of departure from the equilibrium state along which the energy is released from the system and may, at least in principle, be inverted into the kinetic energy since the modes require no quasi-statically applied increment of the loading parameter λ. Then, the system may start spontaneously to move dynamically; in that sense the Criterion 5 is a criterion for "dynamic" instabilities.

Consider now the stage on the deformation path at which there is an eigenmode $\tilde{w}^* \neq 0$ which renders the functional I in (26) the stationary value necessarily equal to zero. Evidently, $\tilde{w}^*$ is a solution to the first-order rate boundary value problem obtained for $\dot{\lambda} = 0$. If the functional I is positive definite before this stage then it becomes usually indefinite beyond this stage. Now, observe that that stage is necessarily reached when the loading parameter λ attains its analytic extremum value versus a typical displacement on the fundamental deformation path. For, $\dot{\lambda} = 0$ and $\tilde{v}^0 \neq 0$ at the extremum point so that the fundamental mode $\tilde{v}^0$ itself constitutes the eigenmode. This is a generalization of the classical maximum load criterion which is obtained here in the special case when the given loads vary monotonically with λ.

4.3. "Local" instabilities

All the criteria discussed above involve an integral over the body volume. For such global criteria to be satisfied, a prerequisite is that certain local conditions hold. The local conditions discussed below involve the material mechanical properties at a single point only and are independent of the boundary conditions corresponding to the fundamental deformation process.

Our starting point is again the Criterion 3 as equivalent to the Criterion 2. Observe first that in a stable process the volume integral in (20) must be minimized by $\tilde{v}^0$ within the class of all continuous and piecewise-continuously differentiable velocity fields which coincide with $\tilde{v}^0$ over the whole body surface S. Now, the theorem due to Graves [25] gives a local necessary condition for this (the condition (27) below) so that we obtain the following criterion for "local" instabilities.

Criterion 6. The fundamental process is stable only if at every point in V

$$U(v^0_{j,i} + g_j n_i) - U(v^0_{j,i}) - \dot{s}^0_{ij} g_j n_i \geq 0 \qquad \text{for all } g, n \qquad (27)$$

This criterion was derived in [22] (under less general assumptions) by considering a kinematically admissible mode of localization of deformation within a disc-like region whose thickness grew from zero at the initial instant of localization. It has been shown that if (27) does not hold then a localization mode of this type can be found which corresponds to the increment δE of the energy functional less that δE^0 by a third-order term.

Suppose now that (27) is satisfied in the body along the fundamental deformation path up to the critical stage and then ceases to hold, implying instability of the fundamental deformation process. Suppose also that (27) still holds at the critical stage, but with equality for some non--zero vectors $\underline{g}^*$, $\underline{n}^*$. Then the left hand side expression in (27) attains the minimum (zero) value for these $\underline{g}^*$, $\underline{n}^*$ and, consequently, its partial derivatives with respect to g_j vanish. This yields

$$(\dot{s}^*_{ij} - \dot{s}^0_{ij})\, n^*_i = 0 \tag{28}$$

at the critical stage, where $\dot{s}^*_{ij}$ is related by the constitutive equation (2) to the velocity gradient $(v^0_{j,i} + g^*_j n^*_i)$. $\underline{\dot{s}}^*$ can be envisaged as the stress-rate within a vanishingly thin band of orientation $\underline{n}^*$ at the initial instant of localization since the velocity gradient within such bands is just of such a form. Now, (28) is the condition of continuing equilibrium across the band, that is, the condition for <u>bifurcation</u> within the band [12]. Note that the term "bifurcation" is in this context used in a local sense only and not necessarily in the sense of an exact bifurcation in a finite body. The condition (28), re-derived here from the general stability criterion, has been widely used in the literature (cf. [12, 14÷17]) as a bifurcation criterion for the onset of the shear band localization. It has been usually used for piecewise-linear constitutive rate equations which not necessarily admit a potential but under the assumption that the moduli inside and outside the band are the same.

The above considerations allow to interpret the Criterion 6 as the energy criterion for localization of deformation within narrow regions inside the body.

Consider a material point at which the moduli $\underline{C}^0$ are well defined. By substituting $\gamma\underline{g}$ in place of $\underline{g}$ in (27) and putting $\gamma \to 0$ or $\gamma \to \infty$, we obtain that the inequalities

$$C^0_{ijkl}\, g_j\, n_i\, g_l\, n_k \geqq 0 \qquad \text{for all } \underline{g}, \underline{n}\,, \tag{29}$$

$$U(g_j n_i) \geqq 0 \qquad \text{for all } \underline{g}, \underline{n}\,, \tag{30}$$

respectively, are necessary for the local stability. (29) is the known ellipticity condition, associated with the criterion $\det(C^0_{ijkl} n^*_i n^*_k) = 0$ which follows from (28) if the moduli within the band are equal to $\underline{C}^0$ [12].

The condition (30) can alternatively be obtained directly by applying Graves' theorem to (26) and is thus necessary also for stability of equilibrium. Hence, violation of (30) may lead to a <u>dynamic localization process</u> associated with an "internal snap-through" within the body. In this context it is perhaps worthwile to mention that (30) is necessary and sufficient for the speed of all dynamic acceleration waves propagating into the material <u>being at rest</u> to be real. For, from the wave propagation condition (cf. [26]) and $(2)_2$ it follows that the expression in (30) with

$|\underline{g}| = |\underline{n}| = 1$ is proportional to the square of wave speed.

Now, consider as a special case the conventional elastic-plastic solid for which the moduli $\underline{C}$ take constant values in each of two half--spaces obtained by dissection of the strain-rate space by the hyperplane tangent to the smooth yield surface in strain space. Then (29) is implied by (30) since each velocity gradient corresponds to the moduli $\underline{C}^0$ provided its sign is appropriately chosen. Consequently, in the conventional elastic-plasic solids the loss of ellipticity by the plastic moduli results in loss of the stability of equilibrium in the energy sense. In that case, a quasi-static study of post-bifurcation behaviour may be of no physical meaning.

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