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THE RELATIVISTIC BOLTZMANN EQUATION – MATHEMATICAL AND PHYSICAL ASPECTS

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Mathematical structure of the linearized Boltzmann equation is outlined. Physical interpretation of the assumed relativistic cross-sections is discussed.

1. Introduction

The relativistic Boltzmann equation, one of the basic equations used by statistical physicists, is currently a subject of interest also for mathematicians. In particular, asymptotic stability of the relativistic Maxwellian [1] and existence of global solutions to the relativistic Boltzmann equation with near-vacuum data [2] have been recently proved. Therefore in this paper some basic properties of the linearized relativistic Boltzmann equation are outlined, valid for a wide class of physically meaningful interactions between the relativistic particles.

Some time ago, in the theory of relativistic hydrodynamics a deep confusion appeared, due to various formulations of the basic postulates and, consequently, various sets of both the hydrodynamical variables and equations. Among the best known approaches, one should mention ECKART [3], LANDAU [4], ISRAEL [5], ISRAEL and STEWART [6–8], MÜLLER [9] and VAN KAMPEN [10] versions of relativistic hydrodynamics.

There is still no definite experimental evidence in favour of any of these hydrodynamical theories. On the other hand, certain relativistic hydrodynamical equations are needed to be applied both for a description of the quark plasma formed in heavy ion collisions [11–15] and in many models used in astrophysics [16].

To choose the adequate equations in a mathematically rigorous justification of relativistic hydrodynamics on the ground of microscopic theory is of great importance. However, microscopic theory of relativistic gases and fluids has not been formulated in a satisfactory way because of the well-known conceptual difficulties [17]. Although the relativistic Boltzmann equation has been extensively investigated and applied [16, 19–21], but even for this well-known microscopic equation the complete mathematical theory has not been constructed.

Thus in this paper we shall give an outline of our mathematical analysis of the linearized relativistic Boltzmann equation [22, 23], based on the well-known non-relativistic procedure [24–27]. If the connection between the long time asymptotics of the Boltzmann equation and the hydrodynamical description of the system is assumed to hold in the relativistic theory as well as it holds in the non-relativistic one, then our approach gives a possibility to choose one of the existing formulations of the relativistic hydrodynamics.

Apart from this physical motivation, mathematical structure of the relativistic Boltzmann equation is interesting by itself. As we shall show, the linearized relativistic Boltzmann equation has a unique, global in time, non-increasing in norm, causal solution in $L^2(\mathbf{r},\mathbf{p})$ for a wide class of collisional cross-sections.

2. Assumptions

We consider one-component classical relativistic gas of massive, uncharged particles in the flat space-time. We assume that before collisions, there are no particle correlations and thus we use a one-particle distribution function and the Boltzmann equation to determine the state of the system. Moreover, the gas is close to the global equilibrium, thus its evolution is described by the linearized relativistic Boltzmann equation, which for an arbitrary inertial observer has the form:

$$(2.1)_1 \qquad \frac{\partial f}{\partial t} + \frac{c\mathbf{p}}{p_0} \cdot \frac{\partial f}{\partial \mathbf{r}} = L[f],$$

with the collision operator L given by:

$$(2.1)_2 \quad L[f] = \frac{\pi c f_0^{1/2}}{p_0} \int d^3\mathbf{p}_1 \int d\theta \, \sin\theta \; \sigma\left(g,\theta\right) \frac{g s^{1/2}}{p_{10}} \; f_{10} \left[\frac{f_1'}{{f_{10}'}^{1/2}} + \frac{f'}{{f_0'}^{1/2}} - \frac{f_1}{f_{10}^{1/2}} - \frac{f}{f_0^{1/2}} \right],$$

where

$$(2.2) \qquad f_0 = \frac{n \, \exp\left(-p_\mu U^\mu / k_B T\right)}{4\pi m^2 c k_B T K_2\left(mc^2/k_B T\right)}$$

is the Jüttner relativistic equilibrium distribution function and $f_0^{1/2} f$ is the linearized distribution function.

In Eq. (2.1), the collision between particles with momenta: $\mathbf{p}$ and $\mathbf{p}_1$ before the collision, and $\mathbf{p}'$, $\mathbf{p}_1'$ after the collision is described in c.m. frame by:
the total energy:

$$(2.3) \qquad c s^{1/2} := c \left| p_1^\mu + p^\mu \right|,$$

the relative momentum:

$$(2.4) \qquad 2g := \left| p_1^\mu - p^\mu \right|,$$

the angle of scattering:

$$\cos\theta := 1 - 2\left(p_\mu - p_{1\mu}\right)\left(p^{\mu} - p'^{\mu}\right)\left(4m^2c^2 - s\right)^{-1}, \tag{2.5}$$

and the differential scattering cross-section $\sigma(g,\theta)$.

In Eq. (2.2) K_2 is the Bessel function of the second kind of index two; n, T and U^{μ} are respectively: density, temperature and velocity of the system in the global equilibrium state.

In the following we perform the analysis of the linearized relativistic Boltzmann equation (2.1), called LRBE, in the Hilbert space $L^2(\mathbf{r},\mathbf{p})$ of real-valued functions with the scalar product given by:

$$(f|g) := \int d^3\mathbf{r}\, d^3\mathbf{p}\;\; fg. \tag{2.6}$$

Functions from this space have a simple physical interpretation as describing systems with finite entropy[1)].

We assume that there exist constants α, β, γ, B and B' such that the scattering cross-section $\sigma(g,\theta)$ obeys the condition:

$$\sigma(g,\theta) \leqslant \left(Bg^{\beta} + B'g^{-\alpha}\right)\sin^{\gamma}\theta, \tag{2.7}$$

with $\gamma > -2$, $0 < \alpha < \min(4, 4+\gamma)$ and $0 \leqslant \beta < \gamma + 2$.

As it will be discussed below, the class of functions $\sigma(g,\theta)$ defined by (2.7) seems to be broad enough to contain all those physically reasonable cross-sections, for which the Boltzmann equation may be applicable.

For small momentum g, the scattering cross-section $\sigma(g,\theta)$ should of course describe non-relativistic interactions. It is easy to notice that $\sigma(g,\theta)$ satisfying condition (2.7), fulfills also the non-relativistic restriction on the scattering cross-section $\sigma(g,\theta)$ given by Drange [25]:

$$\sigma(V,\theta) \leqslant V^{\beta}\Psi(\theta), \tag{2.8}$$

where Ψ is a non-negative function defined on $[0,\pi]$, bounded in $[\delta_1,\delta_2]$ $\forall$ $\delta_1,\delta_2 \in (0,\pi)$ and satisfying the condition:

$$\begin{aligned} &\Psi(\theta) < B\theta^{\gamma}, && \text{when} && \theta \to 0,\\ &\Psi(\theta) < B(\pi-\theta)^{\gamma}, && \text{when} && \theta \to \pi, \end{aligned}$$

with $-4 < \beta < \gamma + 2$ and $\gamma > -2$.

In the non-relativistic physics, the condition (2.7) (or 2.8) can be interpreted in terms of a simplified intermolecular potential [24]:

$$\Phi = \frac{\kappa}{r^n}. \tag{2.9}$$

The classical cross-section corresponding to (2.9) has the form:

$$\sigma(V,\theta) = V^{\beta}\Psi(\theta), \tag{2.10}$$

1) Entropy for the linearised Boltzmann equation is to be understood as the linearised entropy [28], i.e. $\int d^3r\, d^3p\; f^2$.

with $\beta = -4/n$ and $\Psi(\theta)$ having a non-integrable singularity for grazing collisions, i.e. for $\theta \to 0$. However, the singularity is absent from the quantum treatment [29]. It can be easily checked [29] that the condition (2.7) is satisfied by all interactions via the quantum inverse power potential (2.9) with $n > 2$. However, for $n \leqslant 2$ collective effects in the system, making the Boltzmann equation inadequate, cannot be excluded [30, 31]. Thus for small momenta g, the class of adopted functions $\sigma(g,\theta)$ (2.7) seems to be wide enough.

Equation (2.7) is a simple, but nontrivial generalization of (2.8) for large momenta[2)] g. Physical interpretation of (2.7) in terms of any potential of course fails in the relativistic regime. But according to the experimental data on the hadron-hadron scattering, the elastic cross-section is slowly increasing for momenta g large enough [32].

Field theoretical analysis puts an upper bound (so-called Froissart's restriction) [33] on the large momentum behaviour of σ:

$$\int d\theta \ \sin\theta \ \sigma(g,\theta) < C[\log(g/g_0)]^2 \qquad \text{for} \qquad g \to \infty, \tag{2.11}$$

where C and g_0 are constants.

Of course all cross-sections satisfying (2.11) fulfill also condition (2.7).

With all the above listed assumptions we are ready to analyse the LRBE (2.1).

3. Properties of the Equation

Under the assumptions made in the previous section, the LRBE (2.1) may be rewritten in the form [22]:

$$\frac{\partial f}{\partial t} + \frac{c\mathbf{p}}{p_0} \cdot \frac{\partial f}{\partial \mathbf{r}} = -\nu(\mathbf{p}) + K[f], \tag{3.1}$$

where $\nu(\mathbf{p})$ is the collision frequency:

$$\nu(\mathbf{p}) = \frac{\pi c}{p_0} \int d^3\mathbf{p}_1 \int_0^{\pi} d\theta \ \sin\theta \ \frac{g s^{1/2}}{p_{10}} \ \sigma(g,\theta) \ f_{10}, \tag{3.2}$$

and the operator K acts as:

$$K\left[f(\mathbf{r},\mathbf{p},t)\right] = \frac{\pi c f_0^{1/2}}{p_0} \int d^3\mathbf{p}_1 \int_0^{\pi} d\theta \sin\theta \ \frac{g s^{1/2}}{p_{10}} \ \sigma(g,\theta) \ f_{10} \left[\frac{f_1'}{{f_{10}'}^{1/2}} + \frac{f'}{{f_0'}^{1/2}} - \frac{f_1}{f_{10}^{1/2}} \right]. \tag{3.3}$$

Similarly as in the non-relativistic theory [24, 25], to analyse the properties of the LBRE (3.1) first we must examine the structure of the K operator and the behaviour of the collision frequency ν. The results are given in the following theorems:

2) Note that using the relativistic relative velocity $4cgs^{-1/2}$ instead of the relativistic relative momentum $2g$ to generalize (2.8), we would get a condition different than (2.7).

Theorem 1

The K operator is compact in $L^2(\mathbf{p})$.

Proof is given by us in Ref. [22].

Theorem 2A

Let us assume that $\exists\, \gamma > -2$, $0 \leqslant \beta < \gamma + 2$, $B > 0$ and $c_0 > 0$ so that

$$\sigma(g,\theta) > B\frac{g^{\beta+1}}{c_0+g}\sin^{\gamma}\theta. \tag{3.4}$$

Then the collision frequency ν obeys:

$$\nu(\mathbf{p}) > \nu_0[p_0/mc]^{\beta/2}. \tag{3.5}$$

Theorem 2B

Let us assume that $\exists\, \gamma > -2$, $0 \leqslant \alpha < 4$ and $B' > 0$ so that

$$\sigma(g,\theta) < B' g^{-\alpha}\sin^{\gamma}\theta. \tag{3.6}$$

Then:

$$\nu(\mathbf{p}) < \nu_0[p_0/mc]^{-\epsilon/2} \leqslant \nu_0, \tag{3.7}$$

where

$$\epsilon = \begin{cases} \alpha, & \text{for} \quad 0 < \alpha < 3, \\ \alpha - 2, & \text{for} \quad 3 < \alpha < 4, \\ \delta + 1, & \text{for} \quad \alpha = 3, \qquad \text{and} \qquad 0 < \delta < 1. \end{cases} \tag{3.8}$$

Proofs of theorems 2A and 2B are given in Ref. [22].

The analysis of the collision frequency $\nu(\mathbf{p})$ performed in Theorems 2A and 2B allows to discriminate between relativistic soft and hard interactions, defined as satisfying (3.6) or (3.4), respectively. The basic properties of the LRBE (such as existence and uniqueness of solutions, provided by Theorem 3) can be obtained for the whole class of adopted interactions (2.7). But from the physical point of view one expects that for arbitrary initial values $f(\mathbf{r},\mathbf{p}) \in L^2(\mathbf{r},\mathbf{p})$, the solution should decay in time in such a way that asymptotically the hydrodynamical approximation holds. This property can be proved (see Theorem 4) for hard interactions only [23].

Theorem 3

Assume that $\sigma(g,\theta)$ satisfies (2.7).

Then the operator $B = -\nabla\mathbf{p}/p_0 + L$ generates a strongly continuous contraction semigroup on $L^2(\mathbf{r},\mathbf{p})$ given explicitly as:

$$\exp(tB)f(\mathbf{r},\mathbf{p}) = \frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\ \exp(i\mathbf{k}\cdot\mathbf{r})\ \exp(B_k t)\ f(\mathbf{k}\cdot\mathbf{p}), \tag{3.9}$$

where $B_k = i\mathbf{k}\cdot\mathbf{p}/p_0 + L$ is in general an unbounded operator on $L^2(\mathbf{p})$ with the domain:

$$D(B_k) = \{f(\mathbf{p}) \in L^2(\mathbf{p}) :\ \nu(\mathbf{p})f(\mathbf{p}) \in L(\mathbf{p})\}. \tag{3.10}$$

Proof of this theorem is given in Ref. [22].

The Cauchy problem for the LBRE is then solved by the linear Boltzmann semigroup as:

$$f(t) = \exp(tB)\ f(t{=}0). \tag{3.11}$$

We see immediately from Theorem 3 that $f(t)$ is unique and that:

$$||f(t)||_{L^2(\mathbf{r},\mathbf{p})} \ \leqslant\ ||f(0)||_{L^2(\mathbf{r},\mathbf{p})}. \tag{3.12}$$

As we have shown in [34, 35], the solution is causal.

Theorem 4

Assume that $\sigma(g,\theta)$ *satisfies* (2.7) *and* (3.4). *Then there exist* $\delta, \beta_1, \beta_2 > 0$ *such that for any* $f \in D(B_k)$, *the following properties are satisfied*:

i) *For any* $\mathbf{k}, |\mathbf{k}| \leqslant \delta$,

$$\exp[tB_k]f = \sum_{j=1}^{5} \exp[t\lambda_j(\mathbf{k})]\langle e_j(-\mathbf{k})|f\rangle_{L^2(\mathbf{p})}\ \mathbf{e}_j(k) + \ \exp\left[tA_k\right] f \ + \ \exp\left[-t\beta_1\right] Z_1(\mathbf{k},t)f, \tag{3.13}$$

where λ_j *and* e_j *are eigenvalues and eigenvectors of* B_k *and for* $|\mathbf{k}| < \delta$ *they have analytical expansion in* k *as:*

$$\lambda_j(\mathbf{k}) = \sum_{n=1}^{3} a_{j,n}(i|\mathbf{k}|)^n + o\left(|\mathbf{k}|^9\right), \tag{3.14}$$

$$e_j(\mathbf{k}) = \sum_{n=0}^{3} e_{j,n}(\mathbf{k}/|\mathbf{k}|)\ (i|\mathbf{k}|)^n + o\left(|\mathbf{k}|^9\right), \tag{3.15}$$

$a_{j,n}$ *are constants with* $a_{j,2} > 0$, $e_{j,n}$ *are functions of* $(\mathbf{k}/|\mathbf{k}|)$ *and*

$$\langle e_i(-\mathbf{k})|e_j(\mathbf{k})\rangle_{L^2(\mathbf{p})} = \delta_{i,j} \qquad i,j = 1,\ldots,5. \tag{3.16}$$

ii) *For any* $\mathbf{k}$, $|\mathbf{k}| > \delta$,

$$\exp[tB_k]f = \exp[tA_k]f \ + \ \exp[-t\beta_2]Z_2(\mathbf{k},t)f, \tag{3.17}$$

where

$$A_k = -\nu(\mathbf{p}) + i\mathbf{kp}/p_0, \tag{3.18}$$

$$Z_j(\mathbf{k},t)f = \lim_{\gamma\to\infty} \frac{1}{2\pi} \int_{-i\gamma}^{i\gamma} \exp(it\gamma)\ Z(-\beta_j + i\gamma;\mathbf{k})\ f\ d\gamma, \tag{3.19}$$

$$Z(\lambda, \mathbf{k}) = (\lambda - A_k)^{-1} \left(I - K(\lambda - A_k)^{-1}\right)^{-1} K(\lambda - A_k)^{-1} \tag{3.20}$$

and

$$|Z_j(\mathbf{k}, t) f|_{L^2(\mathbf{p})} \leqslant C |f|_{L^2(\mathbf{p})}, \quad \text{with } C \text{ independent of } \mathbf{k} \text{ and } t. \tag{3.21}$$

Proof is given in Ref. [23].

Theorem 4 has a very important physical interpretation. From (3.13) and (3.17) it is clear that for long times, the solution of the Boltzmann equation is well approximated by its "hydrodynamical part" given as:

$$f_H = \sum_{j=1}^{5} \exp\left[t\lambda_j(\mathbf{k})\right] \langle e_j(-\mathbf{k}) | f \rangle_{L^2(\mathbf{p})} e_j(\mathbf{k}). \tag{3.22}$$

For small k one can perform the perturbation expansion of (3.22) and then use it to obtain a set of hydrodynamical equations [23].

4. Conclusions

After tedious estimations [22] the basic mathematical structure of the linearised relativistic Boltzmann equation turned out to be the same as that of the non-relativistic one [24]. Namely, for a physically reasonable class of scattering cross-sections, the collision operator was proved to have a form of the multiplication operator $\nu(\mathbf{p})$ plus the compact in $L^2(\mathbf{p})$ perturbation K. However, the analysis of the collision frequency $\nu(\mathbf{p})$ led to a discrimination between the relativistic soft and hard intermolecular interactions different than in the non-relativistic case. Namely, there is a large class of functions σ, which corresponds to scattering cross-sections for relativistic soft, but non-relativistic hard interactions [22].

The basic structure of the collision operator was used to prove existence, uniqueness and causality of the solution to the linearized relativistic Boltzmann equation in $L^2(\mathbf{r}, \mathbf{p})$ [22, 34, 35].

For the relativistic hard interactions the solution was proved to converge asymptotically in time to its hydrodynamical part [23], similarly as it had been done in the non-relativistic theory [26, 27]. This gives a possibility to derive rigorously a set of relativistic hydrodynamical equations from the microscopic equation, using the perturbation expansion in small wave number k. Note that relativistic perturbation expansion in small wave number k is analytical, unlike the non-relativistic one. This difference can be easily understood, because the relativistic perturbation operator $i\mathbf{k}\mathbf{p}/p_0$ is bounded in p, contrary to its non-relativistic counterpart $i\mathbf{k}\mathbf{p}$.

References

1. L. Hsiao, H.J. Yu, *Asymptotic stability of the relativistic Maxwellian*, Math. Method Appl. Sci., **29**, 1481–1499, 2006.
2. R.T. Glassey, *Global Solutions to the Cauchy Problem for the Relativistic Boltzmann Equation with Near-Vacuum Data*, Commun. Math. Phys., **264**, 705–724, 2006.
3. C. Eckart, *The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid*, Phys. Rev., **58**, 919–924, 1940.

4. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Oxford 1959.
5. W. Israel, *Relativistic Kinetic Theory of a Simple Gas*, J. Math. Phys., **4**, 1163, 1963.
6. W. Israel, J.M. Stewart, *Thermodynamics of nonstationary and transient effects in a relativistic gas*, Phys. Letters, **58A**, 213–215, 1976.
7. W. Israel, J. M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Ann. Phys. (N.Y.), **118**, 341–372, 1979.
8. W. Israel, J. M. Stewart, *On Transient relativistic thermodynamics and kinetic theory. II*, Proc. R. Soc. London, **A 365**, 43–52, 1979.
9. I. Shih Liu, I. Müller and T. Ruggeri, *Relativistic thermodynamics of gases*, Ann. Phys., **169**, 191–219, 1986.
10. N. G. van Kampen, *Chapman-Enskog as an application of the method for eliminating fast variables*, J. Stat. Phys., **46**, 709–727, 1987.
11. U. Heinz, *Kinetic Theory for Plasmas with Non-Abelian Interactions*, Phys. Rev. Lett., **51**, 351–354, 1983.
12. D. D. Holm, B. A. Kupershmidt, *Relativistic chromohydrodynamics and Yang-Mills Vlasov plasma*, Phys. Letters A, **105**, 225–228, 1984.
13. K. Kajantie, *Hydrodynamics and approach to equilibrium in quark-gluon plasma*, Nuclear Physics A, **418**, 41–55, 1984.
14. G. Baym, [in:] *Quark Matter '84*, K. Kajantie [Ed.], Lecture Notes in Physics, Vol. 221, Springer, Berlin 1985.
15. O. K. Kalashnikov, [in:] *Quantum Field Theory and Quantum Statistics*, [Ed.] I.A. Batalin *et al.*, Vol. 1, p. 499, Adam Hilger, Bristol 1987.
16. D. Mihalas, B. W. Mihalas, *Foundations of radiation hydrodynamics*, Oxford University Press, 1984.
17. P. G. Bergmann, *Generalized statistical mechanics*, Phys. Rev., **84**, 1026–1033, 1951.
18. P. Havas, [in:] *Statistical mechanics of equilibrium and non-equilibrium*, J. Meixner [Ed.], North Holland, Amsterdam 1965.
19. J. L. Synge, *The relativistic gas*, North Holland, Amsterdam 1957.
20. J. M. Stewart, *Non-equilibrium relativistic kinetic theory.* Lecture Notes in Physics, Vol. 10. Springer, Berlin 1971.
21. S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, *Non-Equilibrium relativistic kinetic theory-principles and applications.* North Holland, Amsterdam 1980.
22. M. Dudyński, M. L. Ekiel-Jeżewska, *On the linearized relativistic Boltzmann equation*, Commun. Math. Phys., **115**, 607, 1988.
23. M. Dudyński, *On the linearized relativistic Boltzmann equation. II. Existence of hydrodynamics*, J. Stat. Phys., **57**, 199, 1989.
24. H. Grad, *Asymptotic theory of the Boltzmann equation, II*, [in:] *Third international rarefied gas symposium*, J.A. Lauremann [Ed.], Academic Press, **1,** p. 26, New York 1963.
25. H. B. Drange, *The linearized boltzmann collision operator for cut-off potentials*, SIAM J. Appl. Math., **29**, 665–676, 1975.
26. R. S. Ellis, M. A. Pinsky, *The first and second fluid approximations to the linearized Boltzmann equation*, J. Math. Pures et Appl., **54**, 125–156, 1975, and *The projection of the Navier-Stokes equations upon the Euler equations*, J. Math. Pures et Appl., **54**, 157–182, 1975.
27. T. Nishida, *Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation*, Commun. Math. Phys., **61**, 119–148, 1978.
28. H. Grad, [in:] *Rarefied Gas Dynamics*, F. M. Devienne [Ed.], Pergamon Press, London 1960.

29. L. D. Landau, E. M. Lifshitz, *Quantum Mechanics–Non-Relativistic Theory*, Part 1, Sec. 106 and 108, Pergamon Press, Oxford 1965.

30. M. Klaus, *The linear Boltzmann operator-spectral properties and short-wavelength limit*, Helv. Phys. Acta, **48**, 99–129, 1975 and *Boltzmann collision operator without cut-off*, Helv. Phys. Acta, **50**, 893–903, 1977.

31. Young-Ping Pao, *Boltzmann collision operator with inverse-power intermolecular potentials, I*, Comm. Pure Appl. Math., **27**, 407–428, and *Boltzmann collision operator with inverse-power intermolecular potentials, II* Comm. Pure Appl. Math., 559–581, 1974.

32. G. Siacomelli, *Total cross sections and elastic scattering at high energies*, Phys. Reports, **23**, 123–235, 1976.

33. B.H. Bransden, D. Evans, J.V. Major, *The fundamental particles*, Van Nostrand Reinhold Company, London 1973.

34. M. Dudyński, M.L. Ekiel-Jeżewska, *Causality of the linearized relativistic Boltzmann equation*, Phys. Rev. Lett., **55**, 2831–2834, 1985.

35. M. Dudyński, M.L. Ekiel-Jeżewska, *Causality problem in the relativistic kinetic theory*, [in:] *Recent developments in nonequilibrium thermodynamics: Fluids and related topics*, J. Casas–Vázquez *et. al.* [Ed.], Lecture Notes in Physics, Vol. 253. Springer, Berlin 1986.

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