MODE APPROACH TO RATIONAL SYNTHESIS OF STRUCTURES UNDER IMPULSIVE AND DYNAMIC PRESSURE LOADING

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SUMMARY

An important problem in structural dynamics is concerned with the analysis of structure behaviour under dynamic pressure or impulsive loading in the inelastic range. To simplify this analysis, models of rigid-plastic, non-linear elastic, or non-linear viscous materials have been used. For such simplified models, the permanent mode motions exist, $u(x) = w(x) \cdot F(t)$, and correspond to solutions of non-linear eigenvalue problems. Such mode motions were investigated by Martin and Symonds Proc-ASCE 92, EMS, 43 (1966), Lee and Martin, ZAMP 21 (1970), 1011, and others. It was shown that mode approximations provide reasonable prediction of finite deflections even in cases when initial transient behavior precedes the mode motion.

The aim of this work is to study optimal synthesis of beams and plates which are subjected to initial impulses of given kinetic energy or dynamic pressure loading. Such form of a structure is to be determined for which mean or local final deflections are minimized for given material volume and prescribed kinetic energy of the initial impulse. Starting from mode approximation, it was shown that maximization of the eigenvalue of mode solution leads to minimization of the mean deflection and the problem of maximizing this eigenvalue can be used as a major criterion in generating optimal design solutions. On the other hand, for minimization of a local final deflection, the numerical search technique must be applied. In particular, it was shown that for perfectly plastic structures the static criteria of optimal design based on a concept of a simultaneous failure mode are not directly applicable in the case of dynamic loading and the significance of one-degree-of-freedom modes was demonstrated. In the case of dynamic pressure loading, the piecewise mode solutions were applied in the range of moderate pressures in order to determine optimal designs. A direct variational approach or optimal control theory are useful tools in deriving optimality conditions and relative numerical methods.

Examples of beams and circular plates are considered in detail in order to illustrate general results reached in the first part of work. Designs with continuously varying and piecewise constant cross sections were considered and their effectiveness with respect to uniform designs was studied. Non-unique mode solutions were found to exist in rigid-plastic structures and coexistence of several modes was theoretically observed for some ranges of design parameters. The present work constitutes the first step towards rational synthesis of inelastic structures and the presented approach may prove too simplified since no constraint was imposed on the acceleration of motion which is of essential importance in damping of impact in vehicle structures. However, there is a large class of problems (for instance, in nuclear technology), where plastic damping capacity of a structure should be utilized without necessary constraints on accelerations. For such cases the present formulation may be useful in determining rational designs of flexural structural elements.

1. Introduction

One of essential problems in structural dynamics is rational synthesis of structural elements subjected to dynamic loads and in particular pressure or impulsive loading. Due to ability to plastic deformation, the structure may absorb imposed kinetic energy through plastic dissipation and the rational synthesis should be aimed at utilization of maximum damping capacity of the structure. The present paper is aimed at discussing such optimal design conditions.

A problem of optimal synthesis of structures subjected to dynamic loads has not so far been widely investigated. Due to complexity of mechanical behaviour under different loading conditions, the general optimality criteria which could be applied in any particular case are not likely to be derived. It is therefore natural to consider first some simpler cases of loading in order to obtain better insight into the problem. If the structure executes the steady state vibrations induced by cyclically varying loads, the optimal design may be aimed at minimizing material volume or structure cost for prescribed maximum amplitudes of vibrations /1, 2 /. For free vibrations of a linear elastic structure, the optimal design is usually constrained by specifying one or several free frequencies,/3/.

In this paper, we shall discuss the optimal synthesis of nonlinearly-viscous and rigid-plastic structures subjected to dynamic pressure loading /pulse loading/ or impulsive loads when some kinetic energy is introduced during short initial instant and the structures executes a free motion afterwards. For viscous or plastic structures the subsequent motion will be monotonic until final rest, resulting in permanent deflections. We shall consider such cases when motion can be presented in a mode form

$$u_i/x,\ t/ = w_i/x/\Phi/t/ \qquad /1/$$

during the whole period of motion $0 \le t \le t_f$, or as a sequence of different mode motions,

$$u_i/x,\ t/ = w_{1i}/x/\ \Phi_1/t/\Big|_0^{t_1} + w_{2i}/x/\ \Phi_2/t/\Big|_{t_1}^{t_2} + \dots + w_{ki}/x/\ \Phi_k/t/\Big|_{t_{k-1}}^{t} \qquad /2/$$

lasting over finite time intervals. The second representation is useful in considering the case of pressure loading, whereas the permanent mode representation /1/ will be applied in analysing the case of impulsive loads. It will be shown that in the latter case the problem of free motion can be reduced to a non-linear eigenvalue problem and optimization will be aimed at determining such design for which the mean or local def-

lectionic minimized for given initial kinetic energy K_o.

2. Impulsive loading: formulation of the problem

Consider a nonlinearly-viscous material for which the viscous potential is a homogeneous function of order m+1 of strain rates, that is

$$U(\dot{\varepsilon}_{ij}) = \int \sigma_{ij}\, d\dot{\varepsilon}_{ij} \,, \qquad \sigma_{ij} = \frac{\partial U(\dot{\varepsilon}_{ij})}{\partial \dot{\varepsilon}_{ij}} \qquad /3/$$

where $\dot{\varepsilon}_{ij}$ denotes the viscous strain rate and $0 \leq m \leq 1$. The dissipation function is expressed as follows

$$D(\dot{\varepsilon}_{ij}) = \sigma_{ij}\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}\frac{\partial U(\dot{\varepsilon}_{ij})}{\partial \dot{\varepsilon}_{ij}} = (m+1)\, U(\dot{\varepsilon}_{ij}) \qquad /4/$$

Let us note that for m=0, the dissipation function is homogeneous of order one of strain rates, $D(\dot{\varepsilon}_{ij}) = U(\dot{\varepsilon}_{ij})$. In this limiting case, the viscous material becomes a rigid, perfectly plastic material, Fig 1.

Assume the velocity field to be presented in the form /1/, that is

$$\dot{u}_i(x,t) = v_i(x)\,\phi(t) \qquad /5/$$

and the corresponding strain rate, stress σ_{ij} , and acceleration fields $\ddot{u}_i(x,t)$ are

$$\dot{\varepsilon}_{ij}(x,t) = e_{ij}(x)\,\phi(t) \,, \qquad \ddot{u}_i(x,t) = v_i(x)\,\dot{\phi}(t) \,,$$
$$\sigma_{ij}(x,t) = s_{ij}(x)\,\phi^m(t) = \phi^m(t)\frac{\partial U(\dot{e}_{ij})}{\partial \dot{e}_{ij}} \qquad /6/$$

The equations of motion now takes the form

$$\phi^m \frac{\partial s_{ij}}{\partial x_j} - \rho v_i \dot{\phi} = 0 \qquad /7/$$

and the function $\phi(t)$ should satisfy the condition

$$\dot{\phi} + \lambda^2 \phi^m = 0 \qquad /8/$$

where λ^2 is an eigenvalue of the modal motion and the mode $v_i(x)$ is an eigenfunction. Integrating /8/, we obtain

$$\phi^{1-m} = \phi_o^{1-m} - (1-m)\lambda^2 t \qquad /9/$$

where ϕ_o denotes the initial value of $\phi(t)$ at t = 0. The impulse induced motion terminates at $t = t_f$ when $\dot{\phi} = 0$, thus

$$t_f = \frac{\phi_o^{1-m}}{(1-m)\lambda^2} \qquad /10/$$

Assume that at the initial instant t = 0 the initial velocity fiels is imparted to the structure $\dot{u}_i(x,0) = v_i(x)\,\phi_o$ such that the kinetic energy K_o is given. Thus

$$K_0 = \int \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i \, dV = \phi_0^2 \int \tfrac{1}{2}\rho v_i v_i \, dV \qquad /11/$$

The mean displacement at the end of motion can be expressed as follows

$$\bar{u} = \left(\int \rho u_i u_i \, dV\right)^{\frac{1}{2}} = \left(\int \rho v_i v_i \, dV\right)^{\frac{1}{2}} \int_0^{t_f} \phi(t)\,dt = \frac{(2K_0)^{\frac{1}{2}}}{\phi_0} \int_0^{t_f} \phi(t)\,dt =$$

$$= \frac{(2K_0)^{\frac{1}{2}}}{\phi_0} \frac{\phi_0^{2-m}}{(2-m)\lambda^2} = \frac{(2K_0)^{\frac{1}{2}}}{\phi_0} I_f \qquad /12/$$

Let us note that we can set $\phi_0 = 1$ and then /12/ becomes

$$\bar{u} = (2K_0)^{\frac{1}{2}} \frac{1}{(2-m)\lambda^2} \qquad /13/$$

It is seen that the mean deflection is minimized when the eigenvalue λ is maximized for the class of structure designs preserving constant volume of material. Alternatively, the value of $\bar{u}$ may be specified and a design minimizing material volume is to be determined. Such formulation of the problem will be discussed in the next section.

Let us note that for a linearly viscous material, m = 1, from /8/ we obtain

$$\phi = \phi_0 - \exp(-\lambda^2 t) \qquad /14/$$

and $t_f \to \infty$, whereas $I_f = \phi_0/\lambda^2$. On the other hand, when m = 0, the nonlinearly-viscous material coincides with a rigid, perfectly plastic material and the structure undergoes a motion with constant negative acceleration, Fig. 2

$$\phi = \phi_0 - \lambda^2 t. \qquad /15/$$

The equations of motion /7/, after using /8/ take the form

$$\frac{\partial s_{ij}}{\partial x_j} + \rho v_i \lambda^2 = 0 \qquad /16/$$

and the principle of virtual work gives

$$\int \sigma_{ij} e^k_{ij} \, dV = \int \rho \lambda^2 v_i v_i^k \, dV, \qquad /17/$$

where $e^k_{ij}(x), v_i^k(x)$ denote the kinematically admissible velocity field and the corresponding strain rate field. When $\dot{e}_{ij} = e^k_{ij}$, from /16/ we obtain

$$\int D(\dot{e}_{ij})\,dV = \lambda^2 \int \rho v_i v_i \, dV \qquad /18/$$

The eigenvalue λ^2 can thus be determined from /18/

$$\lambda^2 = \frac{\int D(\dot{e}_{ij})\,dV}{\int \rho v_i v_i \, dV} \qquad /19/$$

and the following functional

$$\Pi = \int U(\dot{e}_{ij})\,dV - \tfrac{1}{2}\int \rho v_i v_i \, dV \qquad /20/$$

attains an extremum in the class of kinematically admissible velocity fields.

Extremum of Π was first shown by Lee and Martin / 5 /.

3. Optimization problem for beams and plates

Let us restrict our discussion to such structures as beams and plates for which generalized stresses Q_i and strains q_i can be used instead of σ_{ij} and ε_{ij}. The dissipation function and the constitutive relations now have the form

$$D(\dot{q}_i, h) = d(h) D^{\circ}(\dot{q}_i), \qquad Q_i = \frac{\partial D^{\circ}}{\partial \dot{q}_i} d(h) \qquad /21/$$

for instance, for the case of beams of width B and height h, equations /21/ take the form

$$D = \frac{cB\, h^{n+2}}{(n+2)\, 2^{n+1}} \dot{k}^{n+1}, \qquad M = \frac{cB}{(n+2)\, 2^{n+1}} h^{n+2} \dot{k}^{n}, \qquad /22/$$

where M and k denote the bending moment and the curvature and c is a material constant. Using mode representation

$$\dot{w}(x,t) = v(x)\, \phi(t), \qquad /23/$$

instead of /19/, we have

$$\lambda^2 = \frac{\int D^{\circ}(\dot{q}_i)\, d(h)\, dA}{\int \rho v_i v_i h\, dA} \qquad /24/$$

and the mean final deflection equals

$$\bar{w} = \left[\int w^2(x,t)\, h\, dA \right]^{\frac{1}{2}} = \sqrt{\frac{2K_0}{\rho}}\, \frac{1}{(2-m)\lambda^2} \qquad /25/$$

The optimization problem will thus be formulated as follows: for specified kinetic energy K_0 imposed by the impulse, minimize the mean or local deflection for a structure of fixed volume or material cost, thus

$$\begin{aligned} &\text{minimize } \bar{w} \text{ or } w_a \\ &\text{subject to } K = \int \tfrac{1}{2} \rho v_i v_i h\, dA = K_0, \quad V = \int h\, dA \leq V_0 \end{aligned} \qquad /26/$$

In view of /24/, it is seen that instead of minimizing $\bar{w}$, we may maximize λ^2 or maximize the total rate of dissipation, thus

$$\begin{aligned} &\text{maximize } \int D^{\circ}(\dot{q}_i)\, d(h)\, dA = (m+1) \int U(\dot{q}_i, h)\, dA \\ &\text{subject to } K = \tfrac{1}{2} \int h \rho v_i v_i\, dA = K_0, \qquad \int h\, dA \leq V_0 \end{aligned} \qquad /27/$$

The formulation /27/ is aimed at finding the structure of maximal damping capacity. In order to derive the optimality conditions, let us introduce Lagrangian multipliers η and μ and consider the following functional

$$J(\dot{q}_i, h) = \int U(\dot{q}_i, h)\, dA - \eta \left[\int \tfrac{1}{2} \rho h v_i v_i\, dA - K_0 \right] - \mu \left[\int h\, dA - V_0 \right] \qquad /28/$$

whose variation with respect to $\dot{q}_i$ and v_i equals

$$\delta J = \int \frac{\partial L}{\partial q_i} \delta q_i \, dA + \int \frac{\partial L}{\partial h} \delta h \, dA - \eta \int h \rho v_i \delta v_i \, dA - \mu \int \delta h \, dA - \eta \int \tfrac{1}{2} \rho v_i v_i \delta h \, dA = 0 \qquad /28/$$

Identifying the Lagrangian multiplier η with λ^2 and using the virtual work principle

$$\int \frac{\partial L}{\partial q_i} \delta q_i \, dA - \lambda^2 \int h \rho v_i \delta v_i \, dA = 0 , \qquad /30/$$

we arrive at the stationarity conditions

$$\frac{\partial L}{\partial h} - \lambda^2 \tfrac{1}{2} \rho v_i v_i - \mu = 0 \qquad /31/$$

These stationarity criteria apply when infinitesimal variation of the design variable involves infinitesimal variation of the form of mode. Such smooth variation of modes can be expected for non-linear viscous structures. However, for a rigid-plastic material the abrupt change from one to the other mode form may occur and the conditions /31/ do not apply to such cases.

4. Rigid-plastic structures. Examples

Consider a stepped beam or plate of thicknesses h_1, h_2, h_3, ... and steps of lengths shown in Fig. 3. Considering the case of impulsive loading, we require that optimal design corresponds to minimum of local deflection at beam or plate center w_g or mean deflection $\bar{w}$, subject to the condition

$$V = 2B \left[a h_1 + (b - a) h_2 + (l - b) h_3 \right] = V_0$$

$$K = \tfrac{1}{2} \int_0^l \rho v^2 dx = K_0$$

Considering mode motion, let us assume that plastic hinges may occur at ends of segments and at the beam center. A detailed analysis of equations of motion is presented in / 5 /. Here, we quote only some results. Introducing the non-dimensional quantities

$$\alpha = \frac{a}{l} , \quad \beta = \frac{b}{l} , \quad \xi = \frac{x}{l} , \quad \gamma = \frac{h_1}{h_3} , \quad \delta = \frac{h_2}{h_3}$$

the final deflection at the beam center can be presented in the form

$$w(0, t_f) = \frac{16 l^3 K_0}{\sigma_0 V^2} F(\alpha, \beta, \gamma, \delta) \qquad /29/$$

and similarly the mean deflection is proportional to the function $G = G(\alpha, \beta, \gamma, \delta)$.

By varying the design parameters, we may study the effectiveness of particular designs. Figs. 4 and 5show the variation of F and G in function of α and γ for a two segment beam. It is seen that for $\gamma_* \leq \gamma \leq \gamma^*$ the coexistence of all three modes may occur and optimal design corresponds to a one-degree of freedom mode with plastic hinges at segment ends. Similar results were obtained for stepped plates. They indicate that static concepts of design may not be applicable in the dynamic case.

The case of pressure loading has been treated by using the representation /2/, that is assuming mode forms to occur at consecutive time intervals. Particular solutions obtained indicate that one degree of freedom modes are essential in design since the final deflections depend mainly on these modes.

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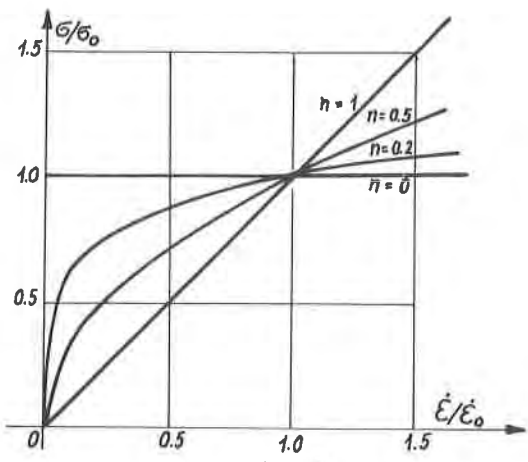


Fig. 1

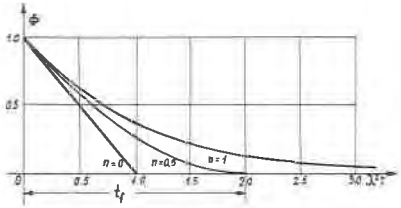


Fig. 2

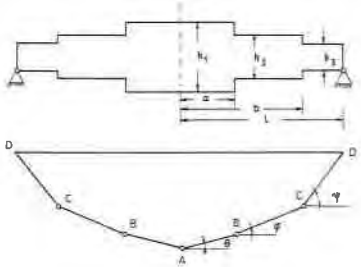
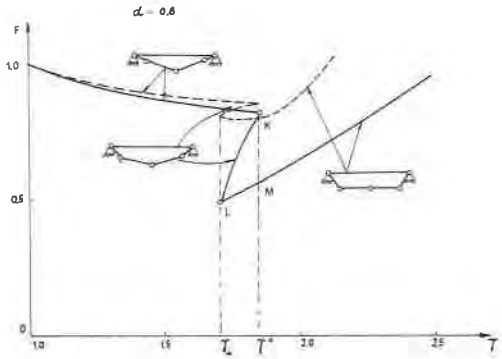


Fig. 3



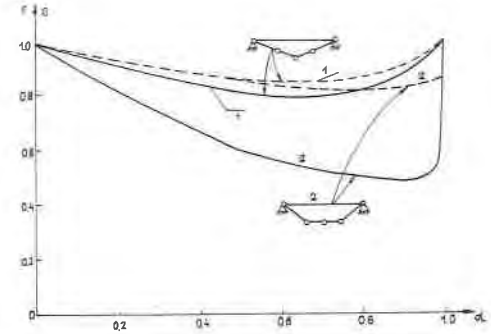


Fig. 4