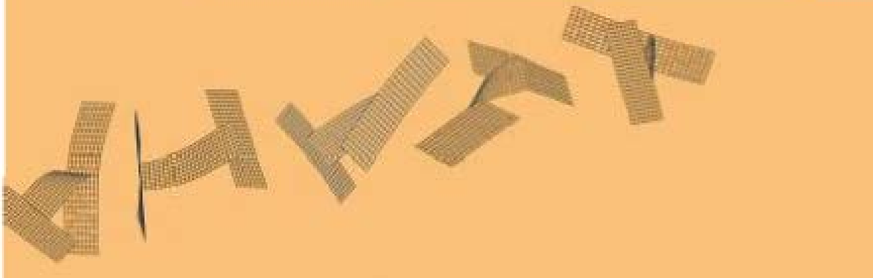
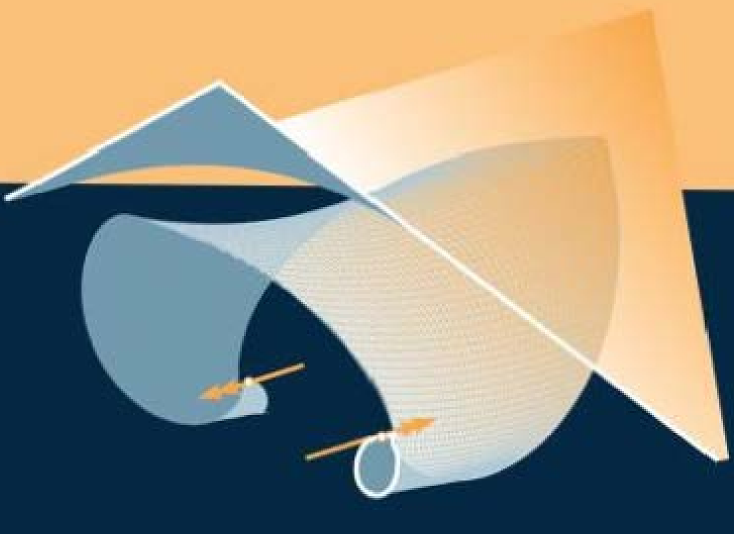


Shell Structures

Theory and Applications

Volume 3

Editors: W. Pietraszkiewicz & J. Górski



CRC Press
Taylor & Francis Group
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PROCEEDINGS OF THE 10TH SSTA CONFERENCE, GDAŃSK, POLAND, 16–18 OCTOBER 2013

Shell Structures: Theory and Applications

VOLUME 3

Editors

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CRC Press
Taylor & Francis Group
Boca Raton London New York Leiden

CRC Press is an imprint of the Taylor & Francis Group, an **informa** business

A BALKEMA BOOK

Shell Structures: Theory and Applications, Vol. 3 – Pietraszkiewicz & Górski (Eds)
© 2014 Taylor & Francis Group, London, ISBN 978-1-138-00082-7

On mixed/enhanced Hu-Washizu shell elements with drilling rotation

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ABSTRACT: Mixed/enhanced four-node shell elements with six dofs/node based on the Hu-Washizu (HW) functional are developed for Green strain. The shell HW functional is derived from the shell potential energy functional instead of from the three-dimensional HW functional. Partial HW functionals, differing in the bending/twisting part and the transverse shear part, are obtained. For the membrane part of HW shell elements, a 7-parameter stress, a 9-parameter strain and a 2-parameter EADG enhancement are selected as performing best. The assumed representations of stress and strain are defined in skew coordinates in the natural basis at the element's center. The drilling rotation is included through the drilling Rotation Constraint (RC) equation and the Perturbed Lagrange method. The spurious mode is stabilized using the gamma method. Several versions of shell HW elements are tested using several benchmark examples and the optimally performing element is selected (HW29) in (Wisniewski & Turska 2012): Additional examples are presented here.

1 INTRODUCTION

Currently, the most promising are the shell elements based on the Hu-Washizu (HW) functional. The four-node HW shell element *without* the drilling rotation clearly has better convergence properties than the EAS shell element, see (Wagner & Gruttmann 2005) and (Gruttmann & Wagner 2006). Besides, HW elements show better accuracy and robustness than the enhanced EADG elements, as shown for 2D HW elements in (Wisniewski & Turska 2009) and (Wisniewski, Wagner, Turska, & Gruttmann 2010). The methodology developed in these papers provided solid ground for the HW shell elements *with* the drilling rotation described in (Wisniewski & Turska 2012).

Note that in the class of mixed elements *with* the drilling rotation, several additional questions must be addressed:

- The implementation of the drilling RC involves difficulties comparable to these encountered for the in-plane shear strain. The equal-order bi-linear interpolations of displacements and the drilling rotation render that the drilling RC is incorrectly approximated, which must be corrected. Then we obtain a spurious zero eigenvalue and a stabilization is needed.
- The next question is to select suitable representations of the assumed stress and strain for the membrane part. Note that an enhancement of the deformation gradient **F** (not of the strain) is needed to obtain good performance because it affects also the drilling RC. In consequence, the best performing assumed representations for the elements *with* the drilling rotation are different from the ones for the elements *without* the drilling rotation.
- The form of the shell functional strongly affects the properties of a shell element. The most versatile is the approach which enables the derivation of the so-called partial (or incomplete) HW functionals. Only these parts of the shell strain energy are converted to the HW form which yield improved convergence properties. We do not have doubts about the membrane part and scrutinize the bending/twisting and the transverse shear part.

2 FORMULATIONS INCLUDING DRILLING ROTATION

Extended configuration space.
The classical configuration space of the non-polar Cauchy continuum is defined as: $\mathcal{C} \doteq \{\boldsymbol{\chi} : B \to R^3\}$, where $\boldsymbol{\chi}$ is the deformation function defined on the reference configuration of the body B. In the present work, we consider an *extended configuration* space, defined as follows:

$$\mathcal{C}_{ext} \doteq \{(\boldsymbol{\chi}, \mathbf{Q}) : B \to R^3 \times SO(3) \quad | \quad \boldsymbol{\chi} \in \mathcal{C}\}, \quad (1)$$

where the rotations $\mathbf{Q} \in SO(3)$ are constrained by the Rotation Constraint (RC) equation

$$\text{skew}(\mathbf{Q}^T\mathbf{F}) = \mathbf{0}, \quad (2)$$

where $\mathbf{F} \doteq \nabla\boldsymbol{\chi}$. Generally, in the Cauchy continuum, the rotations can be obtained by polar decomposition of $\mathbf{F}$ but this requires the calculation of $\mathbf{U}^{-1}$, where $\mathbf{U} \doteq (\mathbf{F}^T\mathbf{F})^{1/2}$, see (Pietraszkiewicz 1979). Alternatively, we can find $\mathbf{Q}$ from eq. (2), which is equivalent to $\mathbf{Q}^T\mathbf{F} = \mathbf{U}$. This approach was used in (Badur & Pietraszkiewicz 1986), (Simo, Fox, & Hughes 1992) and is applied in the present work as well. For the Cosserat-type kinematics of shells, in which the rotations are not constrained, see (Chroscielewski, Makowski, & Stumpf 1992).

Reissner shell kinematics.
The initial configuration of a shell is parameterized by the normal coordinates $\{\xi^\alpha, \zeta\}$ $(\alpha = 1, 2)$, where $\xi^\alpha \in [-1, +1]$ are the natural coordinates parameterizing the reference (middle) surface, and $\zeta \in [-h/2, +h/2]$ is the coordinate in the direction normal to this surface. h denotes the initial shell thickness.

The position vector of an arbitrary point of a shell in the initial configuration is expressed as $\mathbf{y}(\xi^\alpha, \zeta) = \mathbf{y}_0(\xi^\alpha) + \zeta\, \mathbf{t}_3(\xi^\alpha)$, where $\mathbf{y}_0$ is a position of the reference surface, and $\mathbf{t}_3$ is the shell director, normal to the reference surface. In the deformed configuration, the position vector is expressed by the Reissner hypothesis,

$$\mathbf{x}(\xi^\alpha, \zeta) = \mathbf{x}_0(\xi^\alpha) + \zeta\, \mathbf{Q}_0(\xi^\alpha)\, \mathbf{t}_3(\xi^\alpha), \tag{3}$$

where $\mathbf{x}_0$ is a position of the reference surface, and $\mathbf{Q}_0 \in SO(3)$ is a rotation tensor, which is parameterized by the canonical rotation vector $\boldsymbol{\psi}$ as follows:

$$\mathbf{Q}_0(\psi) \doteq \mathbf{I} + \frac{\sin\omega}{\omega}\tilde{\psi} + \frac{1-\cos\omega}{\omega^2}\tilde{\psi}^2, \tag{4}$$

where $\omega \doteq \|\boldsymbol{\psi}\| = \sqrt{\boldsymbol{\psi}\cdot\boldsymbol{\psi}} \geq 0$ and $\tilde{\boldsymbol{\psi}} \doteq \boldsymbol{\psi} \times \mathbf{I}$. As a result of linearization in ζ, the Green strain $\mathbf{E}(\zeta) \approx \boldsymbol{\varepsilon} + \zeta\boldsymbol{\kappa}$. Besides, the transverse components $\kappa_{\alpha 3}$ are neglected.

Drilling Rotation Constraint.
For shells, we can neglect in eq. (2) the terms which depend on the tangent components of a rotation vector $(\boldsymbol{\psi} \cdot \mathbf{t}_\alpha)$, and then it is reduced to the scalar drilling Rotation Constraint,

$$[\mathrm{skew}(\mathbf{Q}^T\mathbf{F})]_{12} = 0. \tag{5}$$

The drilling rotation is defined as a normal component of rotation vector, $\omega \doteq \boldsymbol{\psi} \cdot \mathbf{t}_3$, and its physical interpretation implied by eq. (5) is given in (Wisniewski 2010), p. 26. Assuming small stretches, we obtain

$$\omega \approx \frac{1}{2}(\beta_1 + \beta_2) + k\pi, \qquad k = 0, \ldots, K, \tag{6}$$

i.e. the drilling angle ω is an average of rotations β_α of the initial tangent (unit) vectors $\mathbf{t}_\alpha$.

Note that in eq. (1) the deformation $\boldsymbol{\chi}$ is required to belong to the classical C despite the presence of the rotation $\mathbf{Q}$. Likewise, we expect the solution displacements $\mathbf{u}$ be unaffected by the presence of ω for the cases which can be solved for $\mathbf{u}$ solely.

3 SHELL HU-WASHIZU FUNCTIONALS WITH ROTATIONS

3D HW functional with rotations.
Our formulation is based on the 2nd Piola-Kirchhoff stress $\mathbf{S}$ and the Green strain $\mathbf{E} \doteq \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. Independent fields of stress and strain of the HW functional are designated as $\mathbf{S}^*$ and $\mathbf{E}^*$.

To incorporate the rotations into a 3D formulation, we constrain the governing HW functional by the weak form of the RC of eq. (2). Consider the classical form of the three-field HW functional,

$$F_{HW}(\mathbf{u}, \mathbf{S}^*, \mathbf{E}^*) \doteq \int_V \{\mathcal{W}(\mathbf{E}^*) + \mathbf{S}^* \cdot [\mathbf{E}(\nabla\mathbf{u}) - \mathbf{E}^*]\}\; dV - F_{ext}, \tag{7}$$

where $\mathcal{W}(\mathbf{E}^*)$ is the strain energy expressed by the independent strain $\mathbf{E}^*$, and the independent stress $\mathbf{S}^*$ plays the role of the Lagrange multiplier of the relation between the independent strain $\mathbf{E}^*$ and the Green strain $\mathbf{E}(\nabla\mathbf{u})$. Besides, F_{ext} is the potential of the external loads, the body force, and the displacement boundary conditions, and V is the volume of the 3D body.

To obtain the HW functional with rotations, the Lagrange multiplier method is applied to append the RC of eq. (2) to the functional of eq. (7). Then we obtain the five-field functional,

$$F_5(\mathbf{u}, \mathbf{Q}, \mathbf{S}^*, \mathbf{E}^*, \mathbf{T}^*) \doteq \int_V \{\mathcal{W}(\mathbf{E}^*) + \mathbf{S}^* \cdot [\mathbf{E}(\nabla\mathbf{u}) - \mathbf{E}^*] + \underline{\mathbf{T}^* \cdot \mathrm{skew}(\mathbf{Q}^T\mathbf{F})}\Big\}\, dV - F_{ext}, \tag{8}$$

where $\mathbf{T}^*$ is the skew-symmetric Lagrange multiplier for the RC equation.

Pure HW functional for shells.
To derive the HW functional for shells, we can use as a starting point eq. (8) and the strain $\mathbf{E}(\zeta) \approx \boldsymbol{\varepsilon} + \zeta\boldsymbol{\kappa}$. Let us define the shell strain energy as

$$\mathcal{W}^{sh}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \doteq \int_{-h/2}^{+h/2} \mathcal{W}(\mathbf{E})\, \mu d\zeta, \tag{9}$$

where $\mu \doteq \det\mathbf{Z}$ and $\mathbf{Z}$ is the shifter tensor. For the linear material, we obtain the well-known $\mathcal{W}^{sh}(h, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = h\mathcal{W}(\boldsymbol{\varepsilon}) + (h^3/12)\mathcal{W}(\boldsymbol{\kappa})$.

Let us assume that the independent strain $\mathbf{E}^*(\zeta) \doteq \boldsymbol{\varepsilon}^* + \zeta\boldsymbol{\kappa}^*$. By integration of the HW functional of eq. (7) over the shell thickness, we obtain its shell counterpart,

$$F^{sh}_{HW} \;\doteq\; \int_A \Big\{\mathcal{W}^{sh}(\boldsymbol{\varepsilon}^*, \boldsymbol{\kappa}^*) \;+\; \mathbf{N}^* \cdot [\boldsymbol{\varepsilon}(\mathbf{u}, \mathbf{Q}) - \boldsymbol{\varepsilon}^*] \;+\; \mathbf{M}^* \cdot [\boldsymbol{\kappa}(\mathbf{u}, \mathbf{Q}) - \boldsymbol{\kappa}^*]\} \, dA - F^{sh}_{ext}, \tag{10}$$

where $\mathbf{N}^* \doteq \int_{-h/2}^{+h/2} \mathbf{S}^* \mu d\zeta$ and $\mathbf{M}^* \doteq \int_{-h/2}^{+h/2} \zeta\mathbf{S}^* \mu d\zeta$, and A is the area of the shell reference surface. Note that 6 fields $(\mathbf{u}, \mathbf{Q}, \mathbf{N}^*, \mathbf{M}^*, \boldsymbol{\varepsilon}^*, \boldsymbol{\kappa}^*)$ are involved.

To obtain the functional with the drilling rotation, the Lagrange multiplier method is used to append the drilling RC of eq. (5) to the functional of eq. (10). Then we obtain the seven-field functional (with the additional field T^*)

$$F_7^{sh} \doteq F_{HW}^{sh} + F_{RC}^{drill}, \tag{11}$$

where the drilling rotation term F_{RC}^{drill} has the Perturbed Lagrange (PL) form,

$$F_{RC}^{drill} \doteq \int_A \left\{ T^* \, [\text{skew}(\mathbf{Q}^T\mathbf{F})]_{12} - \frac{1}{2\gamma}(T^*)^2 \right\} dA, \tag{12}$$

where T^* is the Lagrange multiplier. This functional was additionally regularized in T^* by a small perturbation term, where the regularization parameter $\gamma \in (0, \infty)$. The above PL form is better than the penalty form of the drilling RC; the resulting element is less sensitive to distortions and has a larger radius of convergence in non-linear problems.

Note that the pure HW functional for shells is constructed for all strain components, which implies a large number of parameters and is not efficient. More efficient is the method described below, within which the HW functional can be constructed for selected strain components; such functionals are termed 'partial' (or 'incomplete').

Partial HW functionals for shells.
We start the derivation from the potential energy functional, $F_{PE}(\mathbf{u}) \doteq \int_V \mathcal{W}(\mathbf{E}(\nabla\mathbf{u}))dV - F_{ext}$, which, by integration over the thickness, yields the shell potential energy functional

$$F_{PE}^{sh}(\mathbf{u}, \mathbf{Q}) = \int_A \mathcal{W}^{sh}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})dA - F_{ext}^{sh}, \tag{13}$$

where $\mathcal{W}^{sh}$ is defined in eq. (9). Using this shell functional, we can construct the shell HW functional for a selected strain type while still using the potential energy functional for the other type. The so-derived functionals can be used to select the formulation with a minimum number of additional parameters. To obtain a partial functional with the drilling rotation, we proceed in the same way as in deriving eq. (11) from eq. (10).

Using the above methodology, we derived in (Wisniewski & Turska 2012) several HW functionals for shells. These functionals and the identifiers (with the number of additional parameters) of the corresponding elements are as follows:

- The pure HW functional is used in the HW47 and the HW39 elements,
- The partial HW functional is used for all shell strain components except $\varepsilon_{\alpha 3}$ in the HW31 element,
- The partial HW functional used in the HW29 element is

$$\begin{aligned} \tilde{F}_{HW}^{sh} \doteq \int_A \Big\{ & \mathcal{W}^{sh}(\varepsilon_{\alpha\beta}^*, \varepsilon_{\alpha 3}^*, \kappa_{\alpha\beta}) \\ & + N_{\alpha\beta}^* \left[\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^*\right] + N_{\alpha 3}^* \left[\varepsilon_{\alpha 3} - \varepsilon_{\alpha 3}^*\right] \Big\} \, dA \\ & - F_{ext}^{sh} + F_{RC}^{drill}, \qquad \alpha, \beta = 1, 2, \end{aligned} \tag{14}$$

where the strain energy functional is used for $\kappa_{\alpha\beta}$, while the HW functional is used for other shell strain components.

4 NUMERICAL EXAMPLE

All the above specified four-node HW shell elements with 6 dofs/node have a correct rank and pass the membrane and bending patch tests; their performance is presented and compared with the enhanced EADG elements in (Wisniewski & Turska 2012). Here we present two additional examples and only the mixed/enhanced HW29 shell element *with* drilling rotations is tested.

4.1 *Example 1. Non-zero drilling rotation*

The two tests presented below check correctness of *nonzero* drilling rotations. (Note that in the membrane patch test, the zero drilling rotation is tested.)

We model the rectangular membrane of Fig. 1 by a single four-node shell element. The in-plane displacements and the drilling rotation are considered only (the normal displacement and tangent rotations are constrained to zero). Linear strains are applied.

(1) To obtain in-plane rigid rotation, the displacements can be prescribed as follows:

$$u(x, y) = -y, \qquad v(x, y) = x. \tag{15}$$

The drilling rotation is not constrained, and, as a solution, we should obtain $\omega \doteq \frac{1}{2}(v_{,x} - u_{,y}) = 1$ at all nodes. Besides, $\varepsilon_{\alpha\beta} = 0$ should be obtained.

(2) To obtain pure in-plane bending, the displacements can be prescribed as follows:

$$u(x, y) = -xy, \qquad v(x, y) = -\frac{1}{2}(1 - x^2). \tag{16}$$

The drilling rotation is not constrained, and, as a solution, we should obtain $\omega \doteq \frac{1}{2}(v_{,x} - u_{,y}) = x$. Besides, $\varepsilon_{11} = -y$, $\varepsilon_{22} = 0$ and $\varepsilon_{12} = 0$; the last value indicates the lack of in-plane shear locking, see (MacNeal 1994), p. 213.

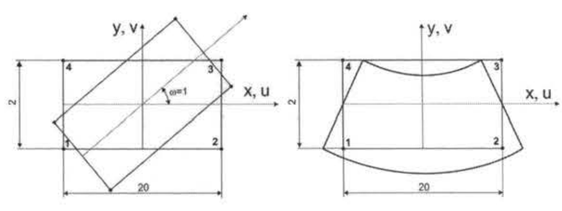

Figure 1. Rigid rotation test and pure bending test.

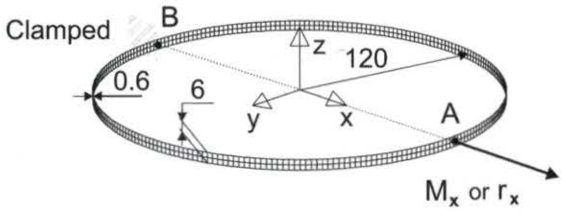

Figure 2. Twisted ring. $E = 2 \times 10^5$, $\nu = 0.3$, $h = 0.6$, width $w = 6$, $R = 120$. Mesh 2×248 elements.

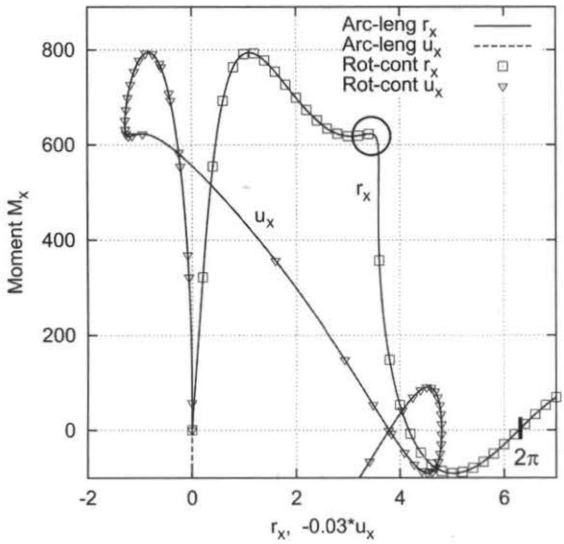

Figure 3. Twisted ring. Non-linear solutions by two methods.

All our shell elements pass these two tests for nonzero drilling rotations.

4.2 *Ring twisted by drilling rotations*

This test of (Goto, Watanabe, Kasugai, & Obata 1992) is difficult because finite rotations are involved. The ring is twisted at point A and is clamped at the opposite point B, see Fig. 2. The problem is solved using two methods: (1) the arc-length method for the initial twisting moment $M_x = 50$, and (2) the rotation-control method for the increment of rotation $r_x = 0.2$.

Two shell elements are tested: the enhanced element (EADG5A) and the mixed/enhanced element (HW29); the latter was selected as optimal in (Wisniewski & Turska 2012). The obtained drilling rotation r_x and the radial displacement u_x at point A are shown in Fig. 3b.

Using the arc-length method, we obtained solutions for both tested elements and they are identical. But using the rotation-control method, a solution was obtained only by the HW29 element while the EADG5A element failed (in the region marked by the circle) even for the four times smaller increments of r_x.

We conclude that in this test the HW shell element is much more robust than the EADG shell element.

REFERENCES

Badur, J. & W. Pietraszkiewicz (1986). On geometrically non-linear theory of elastic shells derived from pseudo-cosserat continuum with constrained micro-rotations. In W. Pietraszkiewicz (Ed.), *Finite Rotations in Structural Mechanics.*, pp. 19–32. Springer.

Chroscielewski, J., J. Makowski, & H. Stumpf (1992). Genuinely resultant shell finite elements accounting for geometric and material nonlinearity. *Int. J. Num. Meth. Engng. 35*, 63–94.

Goto, Y., Y. Watanabe, T. Kasugai, & M. Obata (1992). Elastic buckling phenomenon applicable to deployable rings. *Int. J. Solids Structures 29*, 893–909.

Gruttmann, F. & W. Wagner (2006). Structural analysis of composite laminates using a mixed hybrid shell element. *Comput. Mech. 37*, 479.

MacNeal, R. (1994). *Finite Elements: Their Design and Performance.* Marcel Dekker Inc., New York.

Pietraszkiewicz, W. (1979). *Finite rotations and Lagrangean description in the non-linear theory of shells.* Warszawa – Poznan: Polish Scientific Publisher.

Simo, J., D. Fox, & T. Hughes (1992). Formulations of finite elasticity with independent rotations. *Comput.Methods Appl. Mech. Engng. 95*, 227–2886.

Wagner, W. & F. Gruttmann (2005). A robust nonlinear mixed hybrid quadrilateral shell element. *Int. J. Num. Meth. Engng. 64*(5), 635–666.

Wisniewski, K. (2010). *Finite Rotation Shells. Basic Equations and Finite Elements for Reissner Kinematics.* Springer.

Wisniewski, K. & E. Turska (2009). Improved four-node hu-washizu elements based on skew coordinates. *Computers & Structures 87*, 407–424.

Wisniewski, K. & E. Turska (2012). Four-node mixed hu-washizu shell element with drilling rotation. *Int. J. Num. Meth. Engng. 90*, 506–536.

Wisniewski, K., W. Wagner, E. Turska, & F. Gruttmann (2010). Four-node hu-washizu elements based on skew coordinates and contravariant assumed strain. *Computers & Structures 88*, 1278–1284.