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Boussinesq equation for flow in an aquifer with time dependent porosity

R. WOJNAR*

Institute of Fundamental Technological Research, Polish Academy of Sciences, 5B Pawińskiego St., 02-106 Warszawa, Poland

Abstract. A problem of the Boussinesq type for flow of incompressible fluid through a medium with the porosity ϕ being a given function of time is studied. At first, from the continuity equation the Dupuit relation for the fluid velocity in the considered medium is found. Next the Boussinesq equation is derived for the medium. This equation, if written for quantity $H = h/\phi$ and not for the hydraulic head h, has a form of the classical Boussinesq equation, with the coefficient not constant but dependent on time, however.

An example of the solution of the modified Boussinesq equation is also given.

Key words: seepage, porosity, horizontal flow.

1. Introduction

A seepage flow in porous medium with a constant porosity is an idealised process. The porosity of aquifer depends sometimes strongly on external conditions and charges. For example, the peat deposits pose major difficulties to builders of structures, roads and railways, as they are highly compressible under even small loads [1].

In the present paper we consider a seepage flow in a medium with time dependent both coefficients, the porosity ϕ and the seepage flow coefficient C. The last coefficient is proportional to the quotient κ/η, of the porous medium permeability κ and the fluid viscosity η, and it is coefficient κ which depends on time t if the porosity depends on it.

Used in geology, hydrogeology, soil science, building science, material sciences, botany and zoology, and measured as a percent of volume, the porosity ϕ of a porous medium (such as rock, sediment, timber or bone) describes the fraction of void space in the material, where the void may contain, for example, air or water.

The hydraulic head, denoted h, is a measure of water pressure above a geodetic datum. It is usually given as a water surface elevation, expressed in units of length, from the bottom of water in the aquifer.

At first we derive the continuity of mass equation for flow of incompressible fluid through the porous medium with a varying in time porosity ϕ.

Next we deal with the Depuit conjecture. The conjecture was designed to simplify the groundwater flow equation. It states that the fluid velocity $\mathbf{v}$ is proportional to the hydraulic head gradient ∇h. The conjecture holds if the water table is relatively flat, and if the pressure in the groundwater could be regarded as hydrostatic, despite of the fluid flow [2, 3].

We show that the form of Dupuit conjecture is preserved despite of the fact that porosity of our medium is not constant in time, if other assumptions of Dupuit hold.

The Boussinesq equation permits to trace the time evolution of the hydraulic head h [4–7].

The Boussinesq type equation is derived below for the aquifer whose porosity varies with time according to a given function. This new equation, if written for quantity $H = h/\phi$ instead for the hydraulic head h itself, has a form of the classical Boussinesq equation, with the seepage coefficient not constant but dependent on time.

An example of the analytical solution of the modified Boussinesq equation is given in last section of the paper. Here method of change of independent variables was adopted from the article of Zeldovich and Kompaneets [8].

1.1. Time dependent porosity

Darcy's law. Darcy's law is a constitutive equation that describes the flow of a fluid through a porous medium, cf. below Eq. (7). The law was formulated by Henry Darcy in 1856 based on the results of experiments on the flow of water through beds of sand [9]. Recently, this law has been derived from the Stokes and elasticity equations *via* homogenization, e.g. [10].

Flow in frozen soil. During year, with succession of seasons the porosity of soil changes. The earth is getting warm in summer and cool in winter, when often rivers use to freeze over [11]. Infiltration rates in frozen soil may be extremely low due to pore blockage by ice [12]. But, even at the temperature -3°C, about 10% of water in soil is not frozen [13], also [14].

Seepage coefficient C for the water as a function of the temperature $\mathcal{T}$ is given by

$$C = C_0(0.70 + 0.03\mathcal{T}),$$

where C_0 is a constant and $\mathcal{T}$ is expressed in the Celsius scale. For $\mathcal{T} = 10^o$C we have $C = C_0$ [13].

*e-mail: rwojnar@ippt.gov.pl

Compaction and consolidation. Soil compaction occurs when the weight of livestock or heavy machinery compresses the soil, causing it to lose pore space. Soil compaction may also occur due to a lack of water in the soil. Affected soils become less able to absorb rainfall, thus increasing runoff and erosion. Plants have difficulty in compacted soil because the mineral grains are pressed together, leaving little space for air and water, which are essential for root growth. Burrowing animals also find a hostile environment, because the denser soil is more difficult to penetrate. The ability of a soil to recover from compaction depends on climate, mineralogy, and fauna.

Compaction in geology refers to the process by which a sediment progressively loses its porosity due to the effects of loading. This forms a part of the process of lithification. When a layer of sediment is originally deposited, it contains an open framework of particles with the pore space being usually filled with water. The initial porosity of a sediment depends on its lithology. Sandstones start with porosities of order 40%, mudstones greater than 60%, and carbonates sometimes as high as 70%. Results from hydrocarbon exploration wells show clear porosity reduction trends with depth [15].

Consolidation is a process by which soils decrease in volume. It occurs when stress is applied to a soil that causes the soil particles to pack together more tightly, therefore reducing its bulk volume. When this occurs in a soil that is saturated with water, water will be squeezed out of the soil [16–18].

Variation of ground water level under forest and under clearings. In the heart of a forest (in the Sumy region, northern Ukraine) the ground waters (GWs) occur at 3–5 metres or lower, and have a narrower range of fluctuations than at the edge of the forest, in glades and in the field. Clearing the forest in separate sections in the heart of the forest range calls forth a rise in the water level in the course of a number of years, even under conditions when a general fall in level is noted. The range of the annual fluctuations of the GW level in the field, the glades and the forest border is considerably greater than in the forest itself. The greatest range, as much as 1.7 m, was noted in the centre of the glade Bolshaia Poliana [19]. Similar observations were known to a forester in the Roztocze ridge forests (southern Poland), at time of my childhood. The forester remarked that in spring, a glade rises and takes the shape of a convex lens. The greatest fall in the GW level on passing from field to forest occurs in the border belt of forest. Here the inclination of the GW table is considerable and amounts to 0.01. In the heart of the forest the inclinations are 5–10 times smaller. The presence of inclination in the GW table gives rise to a GW flow, from the unwooded sections into the heart of the forest [19].

Athy's law. The permeability coefficient, κ, of a porous matrix in Darcy's equation of hydraulic conductivity depends mainly on the properties of the solid matrix, e.g., porosity, tortuosity, particle size composition, and shape of grains. As more sediment is deposited above the layer, the effect of the increased loading is to increase the particle-to-particle stresses resulting in porosity reduction primarily through a more efficient packing of the particles and to a lesser extent through elastic compression and pressure solution. In sediments compacted under self-weight, especially in sedimentary basins,the porosity profiles often show an exponential decrease, called Athy's law as first shown by Lawrence Ferdinand Athy in 1930. The estimated porosity is given by relation $\phi = \phi_0 \exp(-a\ z)$, where ϕ_0 and a are constants that vary with sediment type and history, z denotes the sub-bottom depth [20].

Flow of blood. Blood flow in the cardiovascular system is described approximately by the Darcy's law. Blood is a heterogeneous medium consisting mainly of plasma and a suspension of red blood cells. Red cells tend to coagulate when the flow shear rates are low, while increasing shear rates break these formations apart, thus reducing blood viscosity. This results in non-Newtonian blood properties, shear thinning and yield stress. In healthy large arteries the blood can be approximated as a homogeneous, Newtonian fluid since the vessel size is much greater than the size of particles and shear rates are sufficiently high that particle interactions may have a negligible effect on the flow. In smaller vessels, however, non-Newtonian blood behaviour should be taken into account.

The flow in healthy vessels is laminar, however in diseased (e.g. atherosclerotic) arteries the flow may be transitional or turbulent. Disturbed blood flow may cause ischemia and even infarction of the dependent tissue supplied by the struck vessels. Even prolonged bedrest or immobilization may disturb the blood flow [21].

The average resting blood flow in the young adult male foot ranges from 0.2 ml/100 ml tissue/min at 15°C to 16.5 ml/100 ml/min at 44°C [22].

2. Flow in an aquifer with time dependent porosity

Porosity (from Gk. poros "a pore", lit. "passage, way") is a measure of the total pore space in the soil. Porosity of a rigid porous medium creating an aquifer is characterized by the porosity coefficient

$$\phi = \frac{V_p}{V_0}. \tag{1}$$

Here V_0 is the considered bulk volume of material (representative volume element), including the solid and pore components, and V_p is the volume of pore-space. As the volume V_0 gets shrinking to zero (being still much larger than the pore size), the local porosity is approached. Below, we admit possibility of variation of porosity in a given manner with time t only, $\phi = \phi(t)$.

2.1. Equation of continuity.

We consider some volume V of an aquifer bounded by the surface ∂V. The distribution of the fluid velocity is given in the rectangular Cartesian coordinates by function $\mathbf{v} = \mathbf{v}(x, y, z, t)$ of position (x, y, z) and time t. The mass of fluid flowing in unit time through an element $\mathbf{n}dA$ of the surface is $\rho\mathbf{v}\,\mathbf{n}dA$ and $\mathbf{n}$ is the outward normal to the surface element. The total mass of fluid flowing out of the volume V in unit time is $\int_{\partial V} \rho \mathbf{v}\,\mathbf{n} \mathrm{d}A$ and is equal to the decrease per unit time in mass of the fluid in the volume V. The mass of fluid in this volume is $\int_V \phi\rho \mathrm{d}V$, where ρ is the fluid density. Thus, the conservation of mass is expressed by the following equation

$$-\frac{\partial}{\partial t}\int\limits_V \phi\rho \mathrm{d}V = \int\limits_{\partial V} \rho \mathbf{v}\,\mathbf{n} \mathrm{d}A.$$

If the fluid is incompressible then $\rho =$ constant and we get

$$-\frac{\partial}{\partial t}\int\limits_V \phi \mathrm{d}V = \int\limits_{\partial V} \mathbf{v}\,\mathbf{n} \mathrm{d}A,$$

or by the divergence theorem

$$-\frac{\partial}{\partial t}\int\limits_V \phi \mathrm{d}V = \int\limits_V \nabla\mathbf{v} \mathrm{d}V. \tag{2}$$

This is the equation of continuity for our case [7, 23, 24].

2.2. Unconfined flow. Let the plane Oxy be situated at the horizontal bottom of an aquiferous reservoir and the axis z be vertical up. In an unconfined flow the fluid, in general, does not raise to the top of the reservoir and is limited from the above by a mathematical surface, variable in time,

$$z = h(x, y, t). \tag{3}$$

Now we consider an unconfined flow for which the pressure p_0 at the free surface $z = h(x, y, t)$ of the fluid is constant.

Let the volume V be of the shape of irregular (chamfered) cylinder. Its generatrices are vertical (parallel to z axis), the lower base S rests on the impermeable bottom ($z = 0$), while the upper surface is given by Eq. (3). The continuity equation has the form

$$-\frac{\partial}{\partial t}\int\limits_S \mathrm{d}S \int\limits_0^h \phi \mathrm{d}z = \int\limits_S \mathrm{d}S \int\limits_0^h \nabla\mathbf{v} \mathrm{d}z,$$

or

$$\int\limits_S \left(\frac{\partial}{\partial t}\int\limits_0^h \phi \mathrm{d}z + \int\limits_0^h \nabla\mathbf{v} \mathrm{d}z \right) \mathrm{d}S = 0.$$

Since this equation must hold for any surface S, the integrand under $\int_S \mathrm{d}S$ must vanish and

$$\frac{\partial}{\partial t}\int\limits_0^h \phi \mathrm{d}z = -\int\limits_0^h \nabla\mathbf{v} \mathrm{d}z. \tag{4}$$

If the porosity ϕ is function of time only

$$\frac{\partial}{\partial t}(\phi h) = -\int\limits_0^h \nabla\mathbf{v} \mathrm{d}z. \tag{5}$$

As it was stated above, the quantity h depends on time, $h = h(t)$.

2.3. Darcy's and Dupuit's laws. For a fluid at rest in a uniform gravitational field, the Euler's equation takes the form $\nabla p = \rho\mathbf{g}$, where $\mathbf{g} = (0, 0, -g)$ is the acceleration due to gravity, and we get

$$p = p_0 + \rho g(h - z). \tag{6}$$

This relation, derived originally for the motionless fluid, is used farther to the fluid seepage. It is called the gently sloping flow approximation, as it means that the vertical velocity component is neglected and the flow is plane.

Seepage or the fluid flow through a porous medium is described by the Darcy law which relates the flow velocity $\mathbf{v}$ with the gradient of pressure p

$$\mathbf{v} = -\frac{\kappa}{\eta}\nabla p. \tag{7}$$

Here κ is the permeability of the porous medium and η is the viscosity of the fluid. The permeability κ as dependent on porosity ϕ, is also time dependent. After inserting (5) into (6) we get

$$v_x = -C\frac{\partial h}{\partial x}, \qquad v_y = -C\frac{\partial h}{\partial y} \quad \text{and} \quad v_z = 0, \tag{8}$$

where

$$C = \rho g \frac{\kappa}{\eta} \tag{9}$$

is the seepage coefficient. It has the velocity dimesion, $[C] =$ m/s, as dimension $[\rho] =$ kg/m^3, $[g]$=m/s^2, $[\kappa]$=m^2 and η=kg/(ms). Therefore we obtain (the Dupuit assumption)

$$\mathbf{v} = -C\,\nabla h, \tag{10}$$

where from now on $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the two dimensional gradient operator. According to our assumptions $\kappa = \kappa(t)$, thus $C = C(t)$, cf. Eq. (9).

2.4. Boussinesq equation. If $\mathbf{v}$ does not depend on z one can perform integration at right hand side of (4) and get

$$\frac{\partial}{\partial t}\int\limits_0^h \phi \mathrm{d}z = -h\nabla\mathbf{v}. \tag{11}$$

Substitution (9) into (10) gives

$$\frac{\partial}{\partial t}\int\limits_0^h \phi \mathrm{d}z = h\nabla(C\nabla h). \tag{12}$$

If ϕ and C do not depend on spatial coordinates (but they can depend on time) we have

$$\frac{\partial}{\partial t}(\phi h) = Ch\frac{\partial^2 h}{\partial x_\alpha \partial x_\alpha}. \tag{13}$$

We have put for brevity $x = x_1$ and $y = x_2$. The position vector is denoted x_α where $\alpha = 1, 2$ and the summation convention over repeated indices α is used in (13) and below in this section. We find

$$h\frac{\partial^2 h}{\partial x_\alpha \partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(h\frac{\partial h}{\partial x_\alpha}\right) - \frac{\partial h}{\partial x_\alpha}\frac{\partial h}{\partial x_\alpha} \approx \frac{\partial}{\partial x_\alpha}\left(h\frac{\partial h}{\partial x_\alpha}\right).$$

In the last approximation we exploited the gentle sloping assumption and rejected squares of the slope angles, and we write (12) as

$$\frac{\partial}{\partial t}\,(\phi h) = C\,\frac{\partial}{\partial x_\alpha}\left(h\frac{\partial h}{\partial x_\alpha}\right). \tag{14}$$

If porosity ϕ does not depend on time we regain the classical Boussinesq equation

$$\phi\,\frac{\partial h}{\partial t} = \frac{1}{2}C\,\Delta h^2, \tag{15}$$

where Δ is the two dimensional Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x_\alpha \partial x_\alpha} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

From the derivation it is seen that the classical Boussinesq equation holds also for the seepage coefficient C dependent on time, $C = C(t)$.

3. An example

Consider a one dimensional case, in which h depends on t and only on one spatial variable x, $h = h(x,t)$ (but is idependent of y). Equation (14) takes the form

$$\frac{\partial}{\partial t}\,(\phi h) = C\,\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right). \tag{16}$$

Introduce a new rescaled height H by relation

$$H = \phi h. \tag{17}$$

Hence

$$h(x,t) = \frac{1}{\phi(t)}H(x,t). \tag{18}$$

With notation

$$a = a(t) = \frac{C(t)}{[\phi(t)]^2} \tag{19}$$

Eq. (16) takes the form

$$\frac{\partial H}{\partial t} = a(t)\,\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right). \tag{20}$$

This equation, for a =constant, belongs to the class of partial differential equations discussed in the context of nonlinear heat flow by Zeldovich and Kompaneets in [8], and in the context of nonlinear diffusion in [25].

3.1. A solution. Introduce a new dimensionless independent variable

$$\xi = \frac{x}{(QA)^{1/3}}, \tag{21}$$

with

$$A = A(t) = \int_0^t a(\tau)d\tau. \tag{22}$$

The quantity Q denotes a total amount of considered fluid (found below curve $z = h(x,y,t)$ on plane Oxz) and is measured in m^2 units. The dimension of the quantity A is that of the length, [m].

We look for the solution in the form

$$H = \left(\frac{Q^2}{A}\right)^{1/3}\varphi(\xi), \tag{23}$$

where the dimensionless function $\varphi = \varphi(\xi)$ is multiplied by a coefficient having the dimension of length.

As $\partial\xi/\partial t = -a\,\xi/(3A)$ and $\partial\xi/\partial x = 1/(QA)^{1/3}$ we get

$$\frac{\partial H}{\partial t} = -\left(\frac{Q^2}{A}\right)^{1/3}\frac{a}{3A}\left(\varphi + \xi\,\frac{\partial\varphi}{\partial\xi}\right)$$

and

$$\frac{\partial H}{\partial x} = \left(\frac{Q}{A^2}\right)^{1/3}\frac{\partial\varphi}{\partial\xi}.$$

Also

$$\frac{\partial}{\partial x}\left(H\frac{\partial H}{\partial x}\right) = \frac{Q}{A}\frac{\partial}{\partial x}\left(\varphi\frac{\partial\varphi}{\partial x}\right) = \left(\frac{Q}{A^2}\right)^{2/3}\frac{\partial}{\partial\xi}\left(\varphi\frac{\partial\varphi}{\partial\xi}\right).$$

These relations permit to reduce Eq. (20) to the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi\varphi + 3\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right] = 0. \tag{24}$$

This ordinary differential equation has a solution

$$\varphi = \frac{1}{6}\left(\xi_0^2 - \xi^2\right), \tag{25}$$

where ξ_0 is a constant of integration. After substituting this expression into (23) we obtain

$$H = \left(\frac{Q^2}{A}\right)^{1/3}\frac{1}{6}\left(\xi_0^2 - \xi^2\right), \tag{26}$$

This formula describes the behaviour of rescaled height H in the interval between the points $x = \pm x_0$ corrresponding at every instant to the equations $\xi = \pm\xi_0$

$$x_0 = (QA)^{1/3}\,\xi_0, \tag{27}$$

cf. (21). Outside the interval we have $H = 0$. If coefficient a in (22) is a constant, then $A = a\ t$ and the interval of nonvanishing H expands with time as $x_0 \propto t^{1/3}$.

Substituting (26) into (18) we get

$$h(x,t) = \frac{1}{6\phi(t)}\left(\frac{Q^2}{A(t)}\right)^{1/3}\left(\xi_0^2 - \xi^2\right) \tag{28}$$

or, as ξ is given by (21),

$$h(x,t) = \frac{1}{6\phi}\left(\frac{Q^2}{A}\right)^{1/3}\left(\xi_0^2 - \frac{x^2}{(QA)^{2/3}}\right). \tag{29}$$

This is a solution of our Eq. (16). Here $\phi = \phi(t)$ and $A = A(t)$ are given functions of time, cf. (22), Q denotes the given constant amount of the fluid, while ξ_0 is a constant dependent on an initial condition.

3.2. Initial value problem. Let $t = t_1$ at the initial instant the height h has the parabolic shape

$$h(x, t_1) = B(x_{01}^2 - x^2), \tag{30}$$

with B being an unknown and x_{01} known constants. From solution (28) we have

$$h(x, t_1) = \frac{1}{6\phi(t_1)} \left(\frac{Q^2}{A(t_1)}\right)^{1/3} \left(\xi_0^2 - \xi_1^2\right). \tag{31}$$

After (27) and (21) for $t = t_1$ we get $B = 1/\left(6\phi(t_1)A(t_1)\right)$. Thus

$$h(x, t_1) = \frac{1}{6\phi(t_1)A(t_1)} \left(x_{01}^2 - x^2\right). \tag{32}$$

All amount Q of the fluid be gathered at interval $[-x_{01}, x_{01}]$ while $h(x, t1) = 0$ elsewhere. The constant x_{01} is linked with the amount Q by the relation

$$Q = \int_{-x_{01}}^{x_{01}} h(x, t_1)\mathrm{d}x_1 = \frac{2x_{01}^3}{9\phi(t_1)A(t_1)} \tag{33}$$

or

$$x_{01}^2 = \left(\frac{9}{2}\,\phi(t_1)A(t_1)Q\right). \tag{34}$$

Here, cf. (22),

$$A(t_1) = \int_0^{t_1} a(\tau)d\tau. \tag{35}$$

At the subsequent instants the height spreads according to the law

$$h(x, t) = \frac{1}{6\phi(t)A(t)^{1/3}} \left(\frac{\xi_{01}^2}{A(t_1)^{2/3}} - \frac{x^2}{(A(t))^{2/3}}\right). \tag{36}$$

The dependence on x is explicit, while dependence on t appears through the given functions $\phi(t)$ and $A(t)$.

4. Conclusions

It was shown that with classical assumptions needed to derive the Boussinesq equation only, one can derive the modified Boussinesq equation for that case when porosity varies with time in a given manner $\phi = \phi(t)$.

1. The mass continuity equation for $\phi = \phi(t)$ was obtained.

2. The utility of the Dupuit fluid velocity assumption for the time dependent porosity was indicated. Dupuit considered the groundwater flow equation, which governs the flow of groundwater. He assumed that the equation could be simplified for analytical solutions by assuming that groundwater is hydrostatic and flows horizontally. This assumption is regularly used today, and is known by hydrogeologists as the Dupuit assumption.

3. The Boussinesq equation was extended for the case of porous medium with the porosity $\phi = \phi(t)$ and the seepage coefficient $C = C(t)$ being given functions of time.

4. We gave an example of analytical solution of the new equation.

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REFERENCES

[1] http://en.wikipedia.org/wiki/Rannoch_Moor, 2009.

[2] J. Depuit, *Études Théoretiques et Pratiques sur le Mouvement des Eaux dans les Canaux Découverts et à Travers les Terrains Perméables*, Dunod, Paris, 1863.

[3] W.H. Hager, "Jules Dupuit – eminent hydraulic engineer", *J. Hydraulic Engineering* 130 (9), 843–848 (2004).

[4] J. Boussinesq, "Recherches théoriques sur écoulement des nappes d'eau infiltrées dans le sol", *J. de mathématiques pures et appliquées* 5 (X(1) 5–78), 363–394 (1904).

[5] V.V. Vedernikov, "A physical picture of free seepage", *Comptes Rendus de l'Académie de Sciences de l'URSS* 55 (3), 199–202 (1947).

[6] P.Y. Polubarinova-Kochina, *Theory of Ground Water Movement*, Princeton University Press, Princeton, New Jersey, 1962.

[7] G.I. Barenblatt, V.M. Entov and V.M. Ryzhik, *Theory of Fluid Flows Through Natural Rocks*, Kluver Academic Publishers, London, 1984.

[8] Y.B. Zeldovich and A.S. Kompaneets, "The theory of heat propagation in the case where conductivity depends on temperature", in: *Collection of Papers Celebrating the Seventieth Birthday of Academician A.F. Ioffe*, pp. 61–71, ed. P.I. Lukirskii, Izdat. Akad. Nauk SSSR, Moskva, 1950, (in Russian).

[9] H. Darcy, *Les Fontaines Publiques de la Ville de Dijon*, Dalmont, Paris, 1856.

[10] J.J. Telega and R. Wojnar, "Electrokinetics in random piezoelectric porous media", *Bull. Pol. Ac.: Tech.* 55 (1), 125–128 (2007).

[11] P. Vergili Maronis, *Bucolica Georgica Aeneis*, Otto Güthling, B.G. Teubner, Lipsiae, 1894-1896.

[12] M.S. Seyfried and M.D. Murdock, "Use of air permeability to estimate infiltrability of frozen soil", *J. Hydrology* 202 (1–4), 95–107 (1997).

[13] W. Derski, R. Izbicki, I. Kisiel, and Z. Mróz, *Mechanics of Rocks and Soils*, PWN, Warszawa, 1982, (in Polish).

[14] S.T. Willatt, "Changes in water content in and under frozen soil in Iowa", *Geoderma* 22 (4), 323–331 (1979).

[15] Soil compaction: en.wikipedia.org/wiki/Soil_compaction, 2009.

[16] Consolidation (soil): en.wikipedia.org/wiki/Consolidation_(soil), 2010.

[17] W. Derski, "A method of solving of the system of equations of consolidation theory", *Bull. Pol. Ac.: Tech.* XII (10), 489–493 (1964).

[18] W. Derski, "Equations of the consolidation theory for the case of a source of fluid", *Bull. Pol. Ac.: Tech.* XIII (1), 37–43 (1965).

[19] I.T. Grudinskaya and I.S. Shpak, "Effect of forest and forest strips on the moisture conditions of soils and ground water levels in the forest-steppe and steppe districts of the Ukraine", *Symposium of Hannoversch-Münden* 1, 252–270 (1959).

[20] L.F. Athy, "Density, porosity, and compaction of sedimentary rocks", *AAPG Bulletin – American Association of Petroleum Geologists* 14 (1), 1–24 (1930).

[21] Blood flow: en.wikipedia.org/wiki/Blood_flow, 2009.

[22] M.J. Allwood and H.S. Burry, "The effect of local temperature on blood flow in the human foot", *J. Physiology* 124 (2), 345—357 (1954).

[23] L. Landau and E. Lifszyc, *Mechanics of Continuous Media*, PWN, Warszawa, 1958.

[24] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, Oxford, 1963.

[25] R. Wojnar, "Subdiffusion with external time modulation", *Acta Physica Polonica* A 114 (3), 607–611 (2008).