BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 63, No. 1, 2015
DOI: 10.1515/bpasts-2015-0047

Polynomial chaos expansion method in estimating probability distribution of rotor-shaft dynamic responses

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Abstract. The main purpose of the study is an assessment of computational efficiency of selected numerical methods for estimation of vibrational response statistics of a large multi-bearing turbo-generator rotor-shaft system. The effective estimation of the probability distribution of structural responses is essential for robust design optimization and reliability analysis of such systems. The analyzed scatter of responses is caused by random residual unbalances as well as random stiffness and damping parameters of the journal bearings. A proper representation of these uncertain parameters leads to multidimensional stochastic models. Three estimation techniques are compared: Monte Carlo sampling, Latin hypercube sampling and the sparse polynomial chaos expansion method. Based on the estimated values of the first four statistical moments the probability density function of the maximal vibration amplitude is evaluated by the maximal entropy principle method. The method is inherently suited for an accurate representation of the probability density functions with an exponential behavior, which appears to be characteristic for the investigated rotor-shaft responses. Performing multiple numerical tests for a range of sample sizes it was found that the sparse polynomial chaos method provides the best balance between the accuracy and computational effectiveness in estimating the unknown probability distribution of the maximal vibration amplitude.

Key words: stochastic moment estimation, sparse polynomial chaos expansion, maximum entropy principle, rotor, uncertainties, hybrid mechanical model, random unbalance distribution.

1. Introduction

The analysis of structures with random parameters plays an important role in the structural design, optimization and reliability assessment. In particular, heavily affected rotating machines must assure possibly a high level of reliability, durability and safety in operation. Care should be taken to eliminate effects of unavoidable dynamic excitation or to reduce them to minimum. The random rotor-shaft residual unbalances and uncertain journal bearing parameters should be classified as major sources of the observed scatter of rotor-shaft vibration responses. Therefore, focusing on the random nature of abovementioned parameters, the main objective of the present study is to examine methods, which allow for efficient probability density estimation of the rotor-shaft responses.

Efficient methods of stochastic moments estimation are a crucial component of robust design optimization (RDO) algorithms (a comprehensive survey of RDO formulations and solution techniques is given in e.g. [1–3]). The goal of the rotor-shaft RDO is to find the optimal design that is not sensitive with respect to parameter imperfections even when the rotor-shaft is subjected to considerable bending or torsional resonant vibrations. An alternative formulation of the structural design optimization problem that accounts for uncertainties is reliability-based design optimization (RBDO). Contrary to the most popular RDO formulations, in RBDO the design constraints are expressed in terms of failure probabilities corresponding to selected failure criteria. Since in most of the design cases of practical importance, the value of failure probability is very sensitive to the shape of probability density function (PDF), an accurate PDF estimation of the rotor-shaft responses is essential to perform RBDO. It must be emphasized that problems concerning the uncertainty propagation in the analysis of complex systems have already been addressed by many authors in numerous papers, see references [4–9] for a comprehensive overviews, however, these issues do not seem to have been explored for the rotor-shaft systems, where the stochastic model typically contains a big number of random variables.

The investigated methods include sampling techniques, namely, the classical Monte Carlo and Latin hypercube sampling. The first technique, referred in the text as random sampling (RS), often requires a considerable number of simulation runs to precisely estimate higher stochastic moments of a multidimensional random function and its PDF. Nevertheless, due to its popularity and the asymptotic convergence properties RS is used here for comparison and for reference purposes. The Latin hypercube sampling (LHS), belonging to the class of descriptive sampling techniques, is known for better estimation properties than RS, see e.g. [7, 10, 11]. Still it was interesting to verify the feasibility of this method for the considered random structural performances and the task of PDF estimation.

A potential remedy for high computational cost of the sampling techniques may be given by the polynomial chaos expansion method (PCE) [12, 13]. In PCE orthogonal polynomials are used as the basis functions, and this property simplifies the calculation of statistical moments. Ghanem and Spanos [12] showed that the set of multidimensional Her-

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mite polynomials forms an orthogonal basis for the probability space and that the method is convergent in the mean-square sense. The original Hermite polynomial chaos basis are proposed for modeling stochastic responses with Gaussian random variables. In recent years, the PCE method has been extended to the generalized polynomial chaos, employing the Askey scheme of orthogonal polynomials [14] to allow for a direct use of such parametric distributions as uniform, beta, gamma, etc. The method has been further modified to the so-called multi-element generalized polynomial chaos, see e.g. [15], where the random space is decomposed into local elements and the generalised polynomial chaos method is subsequently implemented within individual elements. Usage of PCE for statistics of the rotor system responses under uncertainties modeled by Gaussian random variables is considered in [16], where PCE is using at the stage of solving the equations of motion. In the current document non-Gaussian random variables are dealt with by means of the transformation technique [17–19]. Even though, in some cases this technique may be considered as non-optimal solution, the transformation-based PCE is adopted here due to its simplicity and flexibility. The stochastic model of the rotor-shaft system is considered as a black box from the applied PCE method point of view. Unfortunately, the PCE method is not immune to the curse of dimensionality. The number of required structural response evaluations substantially increases with the number of random variables. Since the stochastic model of the large multi-bearing rotor-shaft system of the steam turbo-generator considered in this paper contains nearly 60 random variables, to alleviate this difficulty, an adaptive sparse algorithm proposed by Blatman and Sudret [13] was employed. As it will be demonstrated in the rotor-shaft example, this method allows for a substantial reduction of the computational cost of the corresponding complete PCE problem.

In the present study the efficiency and estimation accuracy of above mentioned techniques is compared using the problem of PDF estimation of the rotor-shaft lateral vibration amplitude. Based on the values of the first four statistical moments calculated my means of various methods the response PDF is approximated by maximal entropy principle [20].

The paper consists of five main sections. In Sec. 2 each of the investigated scatter analysis techniques is shortly introduced. Section 3 describes the maximal entropy principle method for PDF estimation. In Sec. 4 the employed hybrid mechanical model of the rotor-shaft system is presented. It is underlined that thanks to its high computational efficiency together with sufficient technical accuracy this model is particularly convenient for stochastic analysis. Finally, in Sec. 5 the effectiveness of the selected methods is compared using the rotor-shaft vibration analysis problem and in Sec. 6 concluding remarks are contained.

2. Estimation of response statistics

It is not unusual in mechanical and civil engineering that some quantities which describe a structural system and applied loads should be modeled as random variables, $X_1, \ldots, X_n$. They are called the basic variables and constitute a random vector $\boldsymbol{X}$ whose samples $\boldsymbol{x}$ belong to the Euclidian space with the probability measure defined by the joint probability density function (PDF) $f_{\boldsymbol{X}}(\boldsymbol{x})$. Assuming a random variable Y, e.g. a rotor-shaft vibration response, is a scalar-valued function of the basic variables in the form $Y = h(\boldsymbol{X})$, in the current paper there are investigated methods for estimating statistical moments of Y with the main purpose of using them for approximation of the response PDF. The first four statistical moments, which are considered here are:

$$\text{mean value} \quad \mu_Y = \mathrm{E}(Y), \tag{1}$$

$$\text{variance} \quad \mathrm{Var}(Y) = \sigma_Y^2 = \mathrm{E}\left[(Y-\mu_Y)^2\right], \tag{2}$$

$$\text{skewness} \quad \mathcal{S}_Y = \frac{1}{\sigma_Y^3}\mathrm{E}\left[(Y-\mu_Y)^3\right], \tag{3}$$

$$\text{kurtosis} \quad \mathcal{K}_Y = \frac{1}{\sigma_Y^4}\mathrm{E}\left[(Y-\mu_Y)^4\right]. \tag{4}$$

In the following a particular emphasis is put on the Polynomial Chaos Expansion method and on comparing its accuracy and computational efficiency with sampling techniques. Below a short description of the abovementioned methods is presented.

2.1. Sampling methods. Random sampling (Monte Carlo sampling) as well as descriptive sampling methods employ samples of basic random variables $\boldsymbol{X}$ to assess the values of population statistics. The commonly used unbiased estimators of the first four statistical moments are formulated as follows:

$$\mu_Y \approx \overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y^{(i)} = \frac{1}{N}\sum_{i=1}^{N} h\left(\boldsymbol{X}^{(i)}\right), \tag{5}$$

$$\begin{aligned}\sigma_Y^2 \approx \widehat{\sigma}_Y^2 &= \frac{1}{N-1}\sum_{i=1}^{N}(Y^{(i)}-\overline{Y})^2 \\ &= \frac{1}{N-1}\sum_{i=1}^{N}\left(h(\boldsymbol{X}^{(i)})-\overline{Y}\right)^2,\end{aligned} \tag{6}$$

$$\mathcal{S}_Y \approx \widehat{\mathcal{S}}_Y = \frac{N\sqrt{N-1}}{N-2}\frac{\sum\limits_{i=1}^{N}\left(h(\boldsymbol{X}^{(i)})-\overline{Y}\right)^3}{\left[\sum\limits_{i=1}^{N}\left(h(\boldsymbol{X}^{(i)})-\overline{Y}\right)^2\right]^{3/2}}, \tag{7}$$

$$\mathcal{K}_Y \approx \widehat{\mathcal{K}}_Y = \frac{N(N+1)(N-1)}{(N-2)(N-3)}\frac{\sum\limits_{i=1}^{N}\left(h(\boldsymbol{X}^{(i)})-\overline{Y}\right)^4}{\left[\sum\limits_{i=1}^{N}\left(h(\boldsymbol{X}^{(i)})-\overline{Y}\right)^2\right]^{2}}. \tag{8}$$

Realizations $\boldsymbol{x}^{(i)}$, $i = 1, \ldots, N$, of the random vector $\boldsymbol{X}$ are drawn from the distribution of $\boldsymbol{X}$. The simulation methods differ mainly by the way the samples are obtained. One may distinguish two major sampling techniques: random sampling (RS) and descriptive sampling [21]. Under some assumptions, the so-called Latin hypercube sampling (LHS) [7, 11, 22] can be classified as a descriptive sampling technique. In the performed study the efficiency of RS as well as LHS are examined.

2.2. Polynomial chaos expansion method. The presentation of the method follows the one given in article [13] of Blatman and Sudret. Assuming variable $Y = h(\boldsymbol{X})$ has a finite variance, it can be expanded onto the so-called "polynomial chaos" (PC) basis as follows, see [12, 23]:

$$Y = h(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \mathrm{N}^n} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}), \tag{9}$$

where $a_{\boldsymbol{\alpha}}$ are the unknown deterministic coefficients and $\psi_{\boldsymbol{\alpha}}$ are the multivariate polynomials, orthogonal with respect to the joint PDF $f_{\boldsymbol{X}}(\boldsymbol{x})$, which reads

$$\mathrm{E}[\psi_{\boldsymbol{\alpha}}(\boldsymbol{X})\psi_{\boldsymbol{\beta}}(\boldsymbol{X})] = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{10}$$

where $\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$, if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and 0 otherwise.

If the random variables X_i, $i = 1, \ldots, n$, are independent, the probability density function of the random vector $\boldsymbol{X}$ can be expressed as a following product of respective PDFs:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n). \tag{11}$$

In the case of correlated variables, they should be first transformed into the space of independent standard Gaussian variables, as it is described in, e.g. [18, 24]. Taking advantage of the form of Eq. (11), the polynomials $\psi_{\boldsymbol{\alpha}}$ can be then constructed as a product of n univariate orthogonal polynomials

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) = P^{(1)}_{\alpha_1}(X_1) P^{(2)}_{\alpha_2}(X_2) \cdots P^{(n)}_{\alpha_n}(X_n). \tag{12}$$

Hence, the elements of vector indices $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of the multivariate orthogonal polynomials $\psi_{\boldsymbol{\alpha}}$ can be identified as degrees of univariate polynomials constituting the above product. Denoting by $\mathcal{D}_{X_i}$ the support of X_i, the orthogonal polynomials $\{P^{(i)}_k, k \geq 0\}$ satisfying

$$\mathrm{E}[P^{(i)}_k(X_i) P^{(i)}_l(X_i)] = \int_{\mathcal{D}_{X_i}} P^{(i)}_k(x) P^{(i)}_l(x) f_{X_i}(x) \mathrm{d}x = \delta_{k,l}, \quad \forall (k,l) \in \mathrm{N}^2, \tag{13}$$

are computed by classical algorithms, see [25]. For standard normal variables the polynomials, which are orthogonal with respect to this PDF, are the Hermite polynomials $P_k(\boldsymbol{x}) = H_k(\boldsymbol{x})$, $k \geq 0$

$$\begin{aligned} &H_0(x) = 1, \qquad H_1(x) = x, \quad \ldots \\ &H_{k+1}(x) = xH_k(x) - kH_{k-1}(x). \end{aligned} \tag{14}$$

For the sake of computational efficiency the series in Eq. (9) is truncated after a finite number of terms. Most often, the polynomials, which degree $|\boldsymbol{\alpha}| = \sum_{i=1}^{n} \alpha_i$ is higher than a given degree p, are eliminated from the series

$$Y = h(\boldsymbol{X}) \approx \sum_{|\boldsymbol{\alpha}| \leq p} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}). \tag{15}$$

The number of $a_{\boldsymbol{\alpha}}$ coefficients that have to be computed is then equal to

$$M = \binom{n+p}{p}. \tag{16}$$

It is claimed by some authors, see e.g. [13, 26], that the PC approximation with $p = 2$ is usually sufficiently accurate for estimating the first two statistical moments of random functions, but to perform the structural reliability analysis one should use higher degree polynomials, $p \geq 3$. On the other hand, such general statements seem to be only of limited value, since the usefulness of a particular approximation strongly depends on the analysed problem. As it will be shown later in the text, the stochastic model of the turbo-generator rotor-shaft under study consists of a big number of random variables and only a low degree PC expansion can be afforded due to high computational costs. Therefore, one of the main research tasks is to evaluate if the PC approximation with $p = 2$ can be used for higher moment estimation as well as for PDF assessment.

The unknown coefficients in (15) are computed either by the so-called projection approach or by the regression approach. The first approach often turns out to be computationally expensive, especially for problems with a large number of random variables. In [27] it is proved that the less expensive projection method, which is based on Smolyak quadrature, is at least 2^p times more costly than the regression method, where p is the PCE order. The minimal number of the random function evaluations for the regression method is equal to the number of unknown coefficients in PCE. It is the reason, why only the regression approach is briefly described below in the text.

The considered method is based on the concept of linear regression and consists in fitting an *a priori* assumed response surface (here it is the truncated PC expansion) to the actual functional relationship given by its values in a sample of experimental points. It is convenient to rewrite Eq. (15) in an equivalent matrix form

$$h(\boldsymbol{X}) \approx \mathcal{H}_p(\boldsymbol{X}) = \sum_{|\boldsymbol{\alpha}| \leq p} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{\psi}(\boldsymbol{X}), \tag{17}$$

where $\boldsymbol{a}$ is the vector of coefficients $\{a_{\boldsymbol{\alpha}}, 0 \leq |\boldsymbol{\alpha}| \leq p\}$ and $\boldsymbol{\psi}$ gathers the basis polynomials $\{\psi_{\boldsymbol{\alpha}}, 0 \leq |\boldsymbol{\alpha}| \leq p\}$. Based on the results of N numerical experiments $\{\boldsymbol{x}_i, y_i\}$, $i = 1, \ldots, N$, where $y_i = h(\boldsymbol{x}_i)$, the coefficients in Eq. (17) are computed by minimizing a norm of residuals $y_i - \mathcal{H}_p(\boldsymbol{x}_i)$ usually given by

$$S(\boldsymbol{a}) = \sum_{i=1}^{N} \left[h(\boldsymbol{x}_i) - \boldsymbol{a}^{\mathrm{T}} \boldsymbol{\psi}(\boldsymbol{x}_i) \right]^2. \tag{18}$$

The solution vector $\widehat{\boldsymbol{a}}$ is expressed by the well-known formula

$$\widehat{\boldsymbol{a}} = (\boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{\Psi})^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{y}, \tag{19}$$

where $\boldsymbol{y} = \{y_1, \ldots, y_N\}$ is the vector of computed function values at the experimental points and the matrix $\boldsymbol{\Psi}_{N \times M}$ has the form

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{\boldsymbol{\alpha}_1}(\boldsymbol{x}_1) & \cdots & \psi_{\boldsymbol{\alpha}_M}(\boldsymbol{x}_1) \\ \vdots & \ddots & \vdots \\ \psi_{\boldsymbol{\alpha}_1}(\boldsymbol{x}_N) & \cdots & \psi_{\boldsymbol{\alpha}_M}(\boldsymbol{x}_N) \end{bmatrix}. \tag{20}$$

In order to make this problem well-posed matrix $\boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{\Psi}$ must be well conditioned. It is then necessary that the employed design of experiments contains a sufficient number of points, preferably significantly more than M.

It is easy to verify, see Eq. (16), that the number of coefficients in expansion (15) grows rapidly with the number of variables n and the polynomial degree p. Therefore, in order to reduce the computational burden and to improve the approximation quality of the method, Blatman and Sudret proposed in [13] an adaptive sparse polynomial chaos expansion (SPCE). In their approach the iterative algorithm allows to eliminate these of the expansion coefficients which are not significant in approximating the function $h(\boldsymbol{X})$ leading to the optimal polynomial representation. A version of the sparse PC expansion algorithm, implemented for the purpose of the current study is the one, which is described as "based on a fixed experimental design" in [13]. Reduction of the number of expansion terms allows here to improve the approximation quality by increasing the ratio of the number of experimental points to the number of unknown coefficients.

Applying the assumption of the stochastic independence of random variables $\boldsymbol{X}$ and the orthogonality of base polynomials $\psi_{\boldsymbol{\alpha}}$, see Eqs. (11)–(13), it is easy to show that the mean value and variance of $h(\boldsymbol{X})$ are given by

$$\mu_Y \approx \mathrm{E}[\mathcal{H}_p(\boldsymbol{X})] = \mu_{Y,p} = a_{\mathbf{0}}, \tag{21}$$

$$\mathrm{Var}(Y) \approx \mathrm{Var}[\mathcal{H}_p(\boldsymbol{X})] = \sigma^2_{Y,p} = \sum_{0<|\boldsymbol{\alpha}|\le p} a^2_{\boldsymbol{\alpha}}, \tag{22}$$

so they are immediately available after obtaining the expansion coefficients $\boldsymbol{a_\alpha}$. More involved are calculation of skewness and kurtosis coefficients, see (3) and (4),

$$\mathcal{S}_Y \approx \mathcal{S}_{Y,p} = \frac{1}{\sigma^3_{Y,p}} \sum_{0<|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|\le p} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} a_{\boldsymbol{\gamma}}\, \mathrm{E} \cdot [\psi_{\boldsymbol{\alpha}}(\boldsymbol{X}), \psi_{\boldsymbol{\beta}}(\boldsymbol{X}), \psi_{\boldsymbol{\gamma}}(\boldsymbol{X})], \tag{23}$$

$$\mathcal{K}_Y \approx \mathcal{K}_{Y,p} = \frac{1}{\sigma^4_{Y,p}} \sum_{0<|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|,|\boldsymbol{\delta}|\le p} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} a_{\boldsymbol{\gamma}} a_{\boldsymbol{\delta}}\, \mathrm{E} \cdot [\psi_{\boldsymbol{\alpha}}(\boldsymbol{X}), \psi_{\boldsymbol{\beta}}(\boldsymbol{X}), \psi_{\boldsymbol{\gamma}}(\boldsymbol{X}), \psi_{\boldsymbol{\delta}}(\boldsymbol{X})]. \tag{24}$$

For the sake of computational efficiency they are evaluated by LHS using the PC approximation (17) instead of the actual model

$$\mathcal{S}_{Y,p} = \frac{1}{\sigma^3_{Y,p}} \mathrm{E}\left[(\mathcal{H}_p(\boldsymbol{X}) - \mu_{Y,p})^3\right], \tag{25}$$

$$\mathcal{K}_{Y,p} = \frac{1}{\sigma^4_{Y,p}} \mathrm{E}\left[(\mathcal{H}_p(\boldsymbol{X}) - \mu_{Y,p})^4\right]. \tag{26}$$

3. Probability density function estimation by maximal energy principle

In order to estimate PDF of random rotor-shaft responses, when the available information is in the form of moment constraints, a method based on maximal entropy principle (MEP) proposed by Jaynes [28] is applied. It was shown in [20] that contrary to Pearson system or the saddlepoint approximation, the MEP method is numerically stable. It is also inherently suited to accurately represent PDFs with an exponential behavior. As it will be demonstrated later in the text, such a distribution shape appears to be characteristic to vibration responses investigated in this paper.

The PDF of the response Y is approximated by maximizing the Shannon information entropy subject to the known values of statistical moments. The problem can be formulated as follows:

$$\text{maximize:} \quad Q(f_Y) = -\int_{a_1}^{a_2} f_Y(y) \ln(f_Y(y))\, \mathrm{d}y, \tag{27}$$

$$\text{subject to:} \quad f_Y(y) \ge 0, \quad \int_{a_1}^{a_2} f_Y(y)\, \mathrm{d}y = 1, \tag{28}$$

$$\int_{a_1}^{a_2} y^j f_Y(y)\, \mathrm{d}y = \mu_j, \quad j = 1, 2, \ldots, M, \tag{29}$$

where a_1 and a_2 are the lower and upper bounds of the response and μ_j, $j = 1, \ldots, M$, are the known raw moments of Y. By using the Lagrange multiplier method an analytical solution of the above problem is obtained as

$$f_Y(y) = \exp\left(-\lambda_0 - \sum_{j=1}^{M} \lambda_j y^j\right), \tag{30}$$

where $\lambda_0, \ldots, \lambda_M$, are the Lagrange multipliers corresponding to the $M+1$ constraints. To determine the unknown multipliers the solution (30) is substituted into (28) and (29) to get

$$\lambda_0 = \ln\left(\int_{a_1}^{a_2} \exp\Big(-\sum_{j=1}^{M} \lambda_j y^j\Big) \mathrm{d}y\right), \tag{31}$$

$$\mu'_r = \frac{\displaystyle\int_{a_1}^{a_2} y^r \exp\left(-\sum_{j=1}^{M} \lambda_j y^j\right) \mathrm{d}y}{\displaystyle\int_{a_1}^{a_2} \exp\left(-\sum_{j=1}^{M} \lambda_j y^j\right) \mathrm{d}y}, \quad j = 1, \ldots, M. \tag{32}$$

Next, the multipliers $\lambda_1, \ldots, \lambda_M$ are computed by the least-squared method minimizing the following sum of squares of residuals, see [29]:

$$R = \sum_{r=1}^{M} R_r^2 = \sum_{r=1}^{M} \left(1 - \frac{\mu'_r}{\mu_r}\right)^2. \tag{33}$$

In the current study the differential evolution algorithm proposed by Storn and Price [30] has been applied to solve this problem. The method is suitable for global optimization and proves to be robust and efficient.

4. Description of the hybrid mechanical model of the rotor-shaft system

In order to obtain sufficiently reliable results of theoretical calculations for the rotor-shaft system, one-dimensional beam-type finite element models are usually applied. Such models are often characterized by relatively large numbers of degrees of freedom, which makes them computationally troublesome, particularly when very many simulation cases have to be considered and the commonly applied degree of freedom reduction methods can lead to essential inaccuracies. To avoid the abovementioned drawback and to maintain the obvious advantages of the finite element approach, in this paper, similarly as in [31–34], the dynamic analysis of the entire rotating system is performed by means of the one-dimensional hybrid structural model consisting of continuous visco-elastic macro-elements and discrete oscillators. Using such a model the rotor-shaft geometry as well as its material properties can be described in an identical way as in the analogous mentioned above finite element model of the same structure. The hybrid model is employed here for eigenvalue analyses as well as for numerical simulations of lateral vibrations of the rotor-shaft. In this model successive cylindrical segments of the stepped rotor-shaft are represented by flexurally and torsionally deformable cylindrical macro-elements of continuously distributed inertial-visco-elastic properties. With an accuracy that is sufficient for practical purposes, in the proposed hybrid model of the rotor-shaft system, some heavy rotors or coupling disks can be substituted by rigid bodies attached to the macro-element extreme cross-sections. Each journal bearing is represented by a dynamic oscillator of two degrees of freedom, where apart from the oil-film interaction also the visco-elastic properties of the bearing housing and foundation are taken into consideration. This bearing model makes possible to represent with a relatively high accuracy kinetostatic and dynamic anisotropic and anti-symmetric properties of the oil-film in the form of constant or variable stiffness and damping coefficients. The complete mathematical formulation and solutions for such hybrid models of the rotor-shaft systems can be found e.g. in [31–33].

For relatively small magnitude of the rotor-shaft system unbalance, e.g. due to residual static and dynamic unbalance of the shaft segments and of the rotor-disks, the coupling effect between the torsional and bending vibrations is usually negligible, which has been demonstrated in [31] and in other publications written by numerous authors. Moreover, since in majority of fluid-flow rotating machinery operating in steady-state conditions the fluctuating components of dynamic torques transmitted by their rotor-shaft systems are very small, in the carried out considerations only flexural excitation due to unbalances causing bending vibrations is going to be taken into account. Thus, simulations of torsional forced vibrations will not be performed.

In the hybrid model flexural motion of cross-sections of each visco-elastic macro-element is governed by the partial differential equations derived using the Timoshenko and Rayleigh rotating beam theory. In these equations the gyroscopic forces mutually coupling rotor-shaft bending vibrations in the vertical and horizontal plane are contained. The analogous coupling effect caused by the system rotational speed dependent shaft material damping is also taken into consideration.

Similarly as in [31–33], mutual connections of the successive macro-elements creating the stepped shaft as well as their interactions with the supports and rigid bodies representing the heavy rotors are described by equations of boundary conditions. These equations contain geometrical conditions of conformity for translational and rotational displacements of extreme macroelement cross-sections. The second group of boundary conditions are dynamic ones, which in general contain linear, nonlinear and parametric equations of equilibrium for concentrated external forces, static and dynamic unbalance forces and moments, inertial, elastic and external damping forces, support reactions and gyroscopic moments. Shaft interactions with discrete oscillators representing the shaft supports in journal bearings are also described by means of the dynamic boundary conditions. Here, similarly as in [32, 33], such boundary conditions contain anti-symmetrical terms with cross-coupling oil-film stiffness components, which couple shaft bending vibrations in two mutually perpendicular planes. In these equations the stiffness and damping coefficients can be constant or variable, when non-linear properties of the oil-film are taken into consideration.

The solution for the forced bending vibration analysis has been obtained using the analytical – computational approach demonstrated in detail in [31–33]. Solving the differential eigenvalue problem for the linearized orthogonal system and an application of the Fourier solutions in the form of series in orthogonal eigenfunctions leads to the set of modal equations in the Lagrange coordinates

$$\begin{aligned} \boldsymbol{M}_0\ddot{\boldsymbol{r}}(t) + \boldsymbol{D}(\Omega(t))\dot{\boldsymbol{r}}(t) + \boldsymbol{K}(\Omega(t))\boldsymbol{r}(t) \\ = \boldsymbol{F}(\Omega^2(t), \Theta(t)), \end{aligned} \tag{34}$$

where

$$\boldsymbol{D}(\Omega(t)) = \boldsymbol{D}_0 + \boldsymbol{D}_g(\Omega(t)),$$

$$\boldsymbol{K}(\Omega(t)) = \boldsymbol{K}_0 + \boldsymbol{K}_b + \boldsymbol{K}_d(\Omega(t)), \quad \Theta(t) = \int_0^t \Omega(\tau) d\tau.$$

The symbols $\boldsymbol{M}_0$, $\boldsymbol{K}_0$ denote, respectively, the constant diagonal modal mass and stiffness matrices, $\boldsymbol{D}_0$ is the constant symmetrical damping matrix and $\boldsymbol{D}_g(\Omega(t))$ denotes the skew-symmetrical matrix of gyroscopic effects. Anti-symmetric elastic properties of the journal bearings are described by the skew-symmetrical matrix $\boldsymbol{K}_b$. Anti-symmetric effects due to Kelvin-Voigt material damping model of the rotating shaft are expressed by the skew-symmetrical matrix $\boldsymbol{K}_d(\Omega(t))$ and the symbol $\boldsymbol{F}(\Omega^2(t), \Theta(t))$ denotes the external excitation vector due to the unbalance and gravitational forces. The Lagrange coordinate vector $\boldsymbol{r}(t)$ consists of the unknown time functions $\xi_m(t)$ in the Fourier solutions, $m = 1, 2, \ldots$. The number of Eqs. (34) corresponds to the number of bending eigenmodes taken into consideration in the range of frequency of interest. These equations are mutually coupled by the out-of-diagonal terms in matrices $\boldsymbol{D}$ and $\boldsymbol{K}$ regarded as the response-dependent external excitations expanded in series in the base of orthogonal analytical eigenfunctions. A fast convergence of the applied Fourier solution enables us to reduce the appropriate number of the modal equations to solve in order to obtain a sufficient accuracy of results in the given range of frequency. Here, it is necessary to solve only 10–30 coupled modal Eqs. (34), even in cases of complex mechanical systems, contrary to the classical one-dimensional beam finite element formulation usually leading to large numbers of motion equations corresponding each to more than one hundred or many hundreds degrees of freedom (if the artificial and often error-prone model reduction algorithms are not applied). Thus, the proposed approach is much more convenient for a stable and efficient numerical simulations. Moreover, due to the natural, continuous distribution of inertial-visco-elastic properties of the beam macro-elements the hybrid modeling assures at least the same or even better representation of real objects.

In a general case, i.e. for the variable in time shaft average rotational speed $\Omega(t)$ during system start-ups or run-downs, in order to obtain the system's dynamic response Eqs. (34) can be solved by means of a direct integration. However, for the constant shaft rotational speed Ω and for constant stiffness and damping coefficients of the bearing supports Eqs. (34) become a system of linear ordinary differential equations with constant coefficients and harmonic external excitation due to the residual unbalances. Then, in order to obtain the system's steady-state dynamic response, an analytical solution of Eqs. (34) is very convenient. For the mentioned above harmonic excitation the induced steady-state vibrations are also harmonic with the same synchronous circular frequency Ω. Thus, the analytical solutions for the successive modal functions $\xi_m(t)$ contained in vector $\boldsymbol{r}(t)$ can be assumed in the following form:

$$\boldsymbol{r}(t) = \boldsymbol{G} + \boldsymbol{C}\cos(\Omega t) + \boldsymbol{S}\sin(\Omega t), \tag{35}$$

where vectors $\boldsymbol{C}$, $\boldsymbol{S}$ contain respectively the modal cosine- and sine-components of forced vibration amplitudes and vector $\boldsymbol{G}$ contains the modal components of the rotor-shaft static deflection due to the gravitational load. Then, by introducing (35) into (34) simplified for $\Omega = \text{const.}$ one obtains the following systems of linear algebraic equations:

$$\begin{gathered} \boldsymbol{K}(\Omega)\boldsymbol{G} = \boldsymbol{Q}, \\ \big(\boldsymbol{K}(\Omega) - \Omega^2\boldsymbol{M}_0\big)\boldsymbol{C} + \Omega\boldsymbol{D}(\Omega)\boldsymbol{S} = \boldsymbol{P}(\Omega^2), \\ \big(\boldsymbol{K}(\Omega) - \Omega^2\boldsymbol{M}_0\big)\boldsymbol{S} - \Omega\boldsymbol{D}(\Omega)\boldsymbol{C} = \boldsymbol{R}(\Omega^2), \end{gathered} \tag{36}$$

where vectors $\boldsymbol{P}(\Omega^2), \boldsymbol{R}(\Omega^2)$ contain the modal components of unbalance amplitudes and vector $\boldsymbol{Q}$ contains the modal components of the rotor-shaft static gravitational load. These equations are very easy to solve with respect of the unknown components of vectors $\boldsymbol{C}$, $\boldsymbol{S}$ and $\boldsymbol{G}$.

5. Numerical example: the steam turbo-generator rotor-shaft

5.1. Model description. The methodology of vibration analysis presented in Sec. 4 is applied here by constructing numerical example of a rotor-shaft system of the typical 200 MW steam turbo-generator consisting of the single high- (HP), intermediate- (IP) and low-pressure (LP) turbines as well as of the generator-rotor (GEN). The rotor-shaft system is supported by seven journal bearings, as shown in Fig. 1. For the purpose of this study it seems to be sufficient to model the considered stepped-rotor shaft of the total length 25.9 m by means of $n_e = 49$ continuous macro-elements, as an initial approximation of its geometry. All geometrical parameters of the successive real rotor-shaft segments as well as their material constants have been determined using the detailed technical documentation of this turbo-generator. The average stiffness and damping coefficients of the oil film in the bearings as well as the equivalent masses and stiffness and damping coefficients of the bearing housings are obtained by means of measurements and identification performed on the real object.

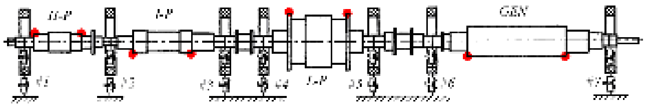

Fig. 1. Hybrid mechanical model of the steam turbo-generator rotor-shaft system

By comparison of eigenvibration analyzes performed using Timoshenko's and Rayleigh's rotating beam theories for the nominal rotational speed 3000 rpm, the shear effect taken into consideration in the case of Timoshenko's beam results in a little bit smaller natural frequency values than these determined by means of Rayleigh's beam model. In the frequency range 0–150 Hz, which is the most important from the engineering viewpoint, the respective differences slightly exceed 3%. The eigenfunctions corresponding to these natural frequencies and determined using both beam theories respectively overlay each other. Therefore, one can conclude that in this frequency range an application of Rayleigh's rotating beam theory for simulations of forced vibrations seems to be sufficiently accurate.

Since typical steam turbo-generators are the devices operating almost permanently in steady-state, out-of-resonance working conditions during a majority of their life, their start-ups and run-downs are rather rare exploitation phases. Thus, in the considered case simulations of passages through lateral vibration resonance zones are not necessary. Therefore, the dynamic and stochastic analyses of the steam turbo-generator rotor-shaft system are going to be carried out only for the steady-state, out-of-resonance operation with the constant nominal rotational speed 3000 rpm corresponding to the excitation of bending vibrations by means of residual unbalances with the synchronous frequency equal to 50 Hz. According to the above, computer simulations of forced bending vibrations of the turbo-generator rotor-shaft system reduce to solving the algebraic Eqs. (36). Here, for the assumed hybrid model of this object in the frequency range of a practical interest 0–500 Hz, 22 bending eigenmodes have been considered in computing forced vibration amplitudes to solve Eqs. (36) with a sufficiently high computational accuracy of the obtained results.

5.2. PDF estimation. The uncertain parameters of the rotor-shaft system are represented by 59 random variables. The stiffness and damping coefficients of 7 journal bearings are modeled by normal random variables. It is assumed that their standard deviations are related to the corresponding mean (nominal) values by the coefficient of variation equal to 5%. Therefore, there are 56 random variables that correspond to the journal bearings, i.e. $7\times$ (4 stiffness coefficients $+4$ damping coefficients). The remaining 3 variables account for random values of the residual unbalances.

The rotor-shaft system of the considered turbo-generator consists of the 3 units described in previous subsection, which are independently manufactured and then mutually connected during on-site assembly process of the entire device. Each of them is characterized by a combined cross-sectional structure consisting of the load carrying shaft core and of the strip created by the turbine blade rims or generator windings, respectively, attached along this core by means of a shrink-fit connection. Thus, the residual unbalance distributions of the HP-IP and LP turbines as well as of the generator rotor are in principle not related to the machining process. Taking this into account, it seems to be reasonable to assume that the unbalance of each rotor-shaft unit is proportional to the successive shaft segment diameters with the common proportionality factor for all segments in the entire unit. For the 3 rotor-shaft units, this assumption results in 3 variables that model the uncertainty of residual unbalances. The 3 proportionality factors are given by realizations of log-normally distributed random variables. Based on the technical data for the considered turbo-generator rotor-shaft system, the mean values of the 3 uncertain factors were estimated as: $5.6\cdot10^{-5}$ for the HP and IP turbines, $2.0\cdot10^{-5}$ for the LP turbine and $3.2\cdot10^{-6}$ for the generator rotor. The coefficient of variation of these variables was taken equal to 10%. According to this, each rotor-shaft unit is characterized by the common phase shift angle for all unbalance amplitudes corresponding to successive shaft cylindrical segments. The obtained in this way 3 phase shift angles for each abovementioned rotor-shaft units are not random, but they are determined from respective identification measurements performed for the real object and assumed equal to zero for the HP-IP turbine, 2.79 rad for the LP turbine and zero for the generator rotor unit.

To get baseline reference values for the first four statistical moments of maximal rotor-shaft lateral displacement a thorough random sampling with the sample size $N=1\,000\,000$ was performed. The obtained estimations, see (5)–(8), are respectively: $\overline{Y}=0.09721$ mm, $\widehat{\sigma}_Y=0.00910$ mm, $\widehat{\mathcal{S}}_Y=0.42673$ and $\widehat{\mathcal{K}}_Y=3.42533$. With these reference values, it is possible to examine the accuracy of MEP in order to verify, whether such an approximation is sufficient for further comparative studies. In Fig. 2 there is shown that the MEP-based PDF curve is very close to the empirical histogram. In addition, the largest relative difference between values of MEP-based cumulative density function (CDF) and the corresponding empirical CDF is only 0.63%. This seem to by a good justification of using the MEP approximation as a reference PDF. Below, in Figs. 3–6 there are shown mean percentage estimation errors for the moments computed by RS, LHS and SPCE methods. The mean percentage estimation error is given as

$$MEE=\mathrm{E}\left(\left|\frac{m_{\mathrm{ref}}-m}{m_{\mathrm{ref}}}\right|\right)\cdot100\%,\tag{37}$$

where m_{ref} and m are respectively the reference value and the computed value of the considered statistical moment.

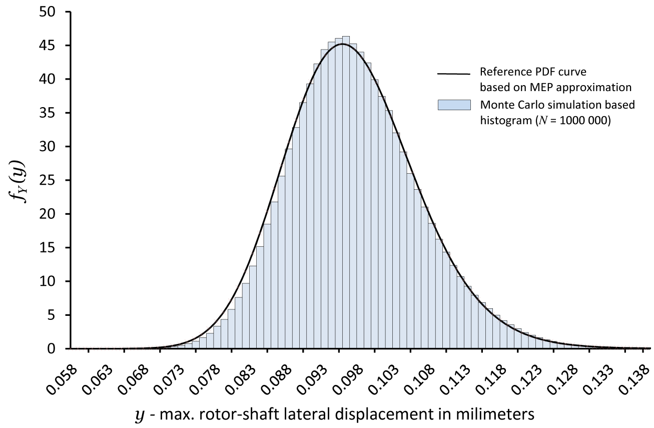

Fig. 2. The maximal lateral displacement histogram obtained with random sampling (sample size $N=1\,000\,000$) and the corresponding MEP-based PDF

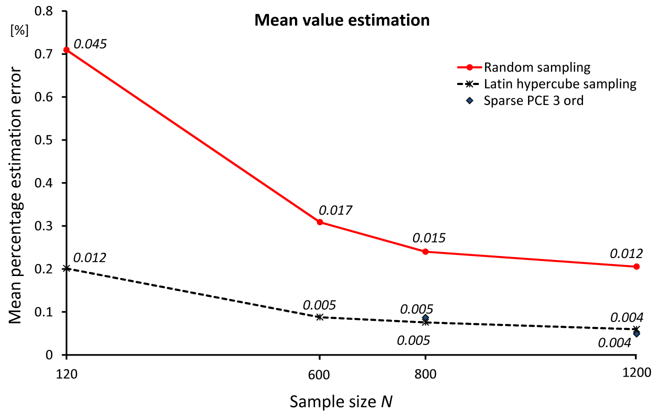

Fig. 3. The turbo-generator rotor shaft example. Mean relative percentage error of the mean value estimation of the maximal rotor-shaft vibration amplitude obtained using RS, LHS and sparse PCE

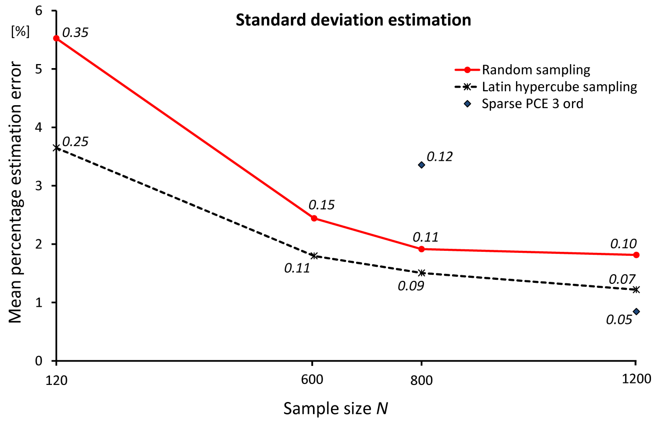

Fig. 4. The turbo-generator rotor shaft example. Mean relative percentage error of the standard deviation estimation of the maximal rotor-shaft vibration amplitude obtained using RS, LHS and sparse PCE

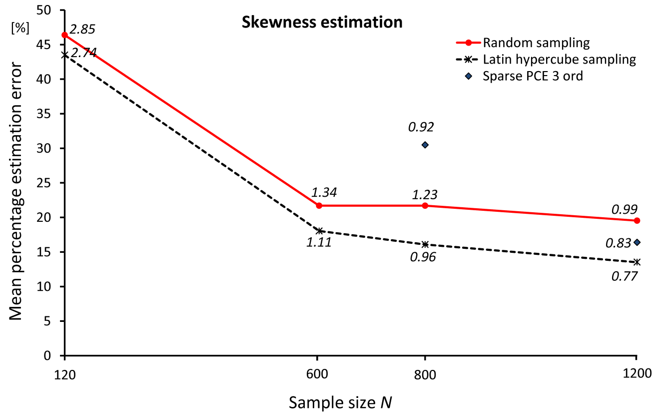

Fig. 5. The turbo-generator rotor shaft example. Mean relative percentage error of the skewness estimation of the maximal rotor-shaft vibration amplitude obtained using RS, LHS and sparse PCE

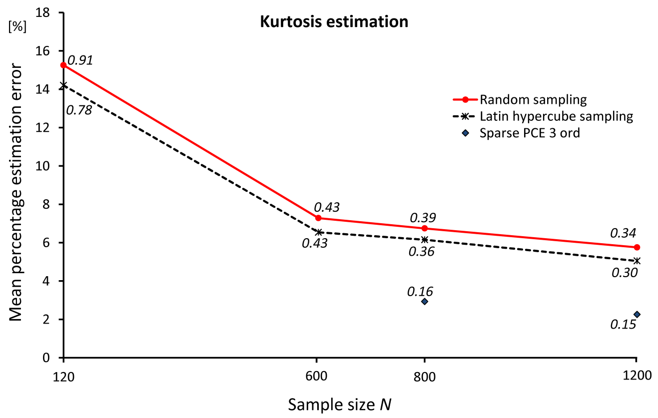

Fig. 6. The turbo-generator rotor shaft example. Mean relative percentage error of the standard deviation estimation of the maximal rotor-shaft vibration amplitude obtained using RS, LHS and sparse PCE

The error statistics are based on 150 repetitions of a given method for each value of the sample size N. The numbers next to the graph points are the standard errors of the mean error estimates. As it can be seen, the estimation obtained with Latin hypercube sampling is always superior to the random sampling irrespective of the sample size. It is interesting to analyze the results corresponding to the SPCE method. In the study the third order expansion was employed. According to Eq. (16), the full PCE representation for $n = 59$ variables consists of 37820 terms. Using the SPCE approach it is possible to greatly reduce this number by identifying about 350 significant terms (depending on the sample this number varies from 200 to 500). It was therefore decided to evaluate the SPCE estimation results for two sample sizes: $N = 800$ and $N = 1200$, which are, respectively, 47 and 31 times lower than the number of terms in the complete third order PC representation. SPCE based on the large samples ($N = 1200$) yields the smallest mean percentage estimation errors for all considered statistical moments (only in the case of skewness LH sampling leads to approximately the same estimation quality as SPCE). On the other hand, in the case of standard deviation and skewness the estimation error obtained for $N = 800$ is significantly greater than the corresponding errors of the sampling methods with the equal sample size. Hence, it can be concluded that such relatively small samples (800 numerical experiments compared to 37820 unknown coefficients) may lead to substantial errors in PDF estimation.

Accounting for the poor estimation quality of all the considered methods in the case of a skewness estimation, see Fig. 5, it was interesting to find out how these errors influence final shape of the PDF curves. In Figs. 7 and 8 there are shown min-max estimation corridors corresponding to RS, LHS and SPCE methods. For each value of the investigated rotor-shaft response the corresponding sample minimal and maximal PDF values were registered. They form the upper and lower bounding lines presented in the figures. However, contrary to the reference line, these curves are not actual PDFs. They establish envelopes containing all the PDF realizations for considered estimation techniques. As it was supposed, due to substantial estimation errors of some statistical moments for $N = 800$ sample points, see Fig. 7, the scatter around the reference PDF is quite significant. This may cause important errors in estimating, in particular, “tail-based statistics” of the response. On the other hand, for the sample size $N = 1200$, Fig. 8, the PDF scatter is considerably reduced. A very narrow envelope corresponding to the SPCE method tightly encloses the reference PDF. In addition, a greater sample size leads to a decrease of the PDF tail approximation error, which is shown in Fig. 9. The presented box plots illustrate a small variance of SPCE results with respect to the sampling methods, which despite a little bias of this technique, leads to very good overall performance in estimating high quantiles. It allows to conclude that the probabilistic design constraints defined as, e.g., $\mathrm{P}[Y < Y_a] \geq 0.99$, where Y_a is an admissible displacement, can be evaluated with an acceptable accuracy. Therefore, the SPCE method is well suited for the reliability-based design optimization as well as for the robust design optimization of rotor-shaft structures, with the design constraints imposed on maximal lateral displacements.

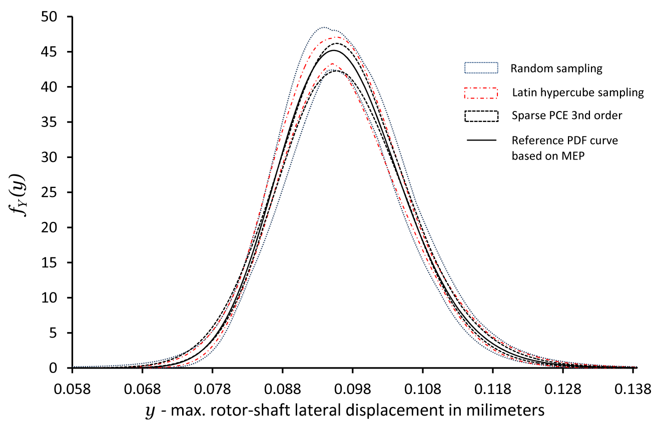

Fig. 7. The MEP-based PDF envelopes for various methods of statistical moments estimation. Sample size $N = 800$

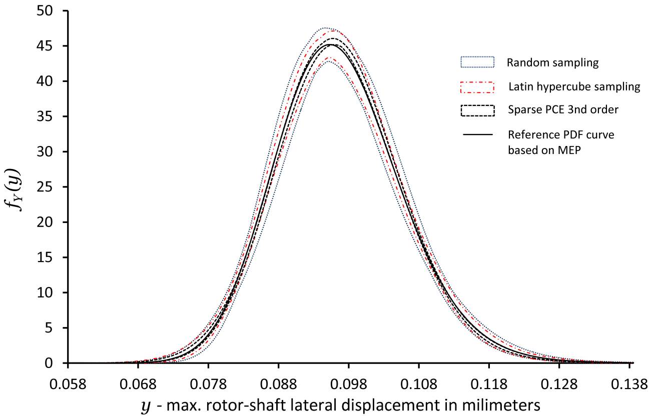

Fig. 8. The MEP-based PDF envelopes for various methods of statistical moments estimation. Sample size $N = 1200$

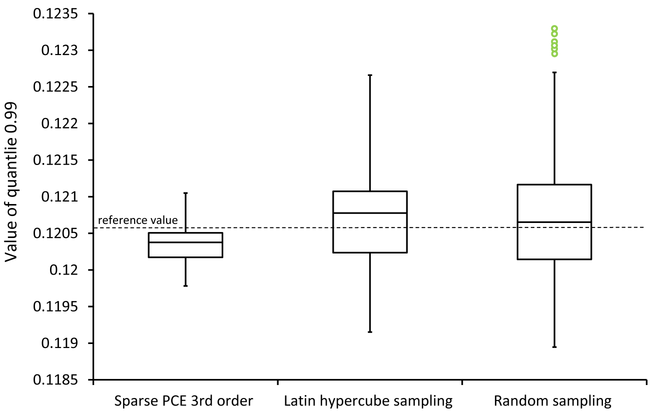

Fig. 9. Box plots for MEP-based 0.99 quantile computation for various methods of statistical moments estimation (sample size $N = 1200$). The reference value is established by an extensive ($N = 1\,000\,000$) random sampling

The question that remains to be answered concerns the sample size that would guarantee satisfactory PDF estimation of random rotor-shaft responses. Here, using third order SPCE and a relatively big number of random variables to describe bearing parameters and residual unbalances, it was shown that samples of the size $N \approx 20n$ provide sufficient accuracy. This, however, may turn out to be too conservative assumption when smaller stochastic models are employed.

6. Conclusions

The objective of this study was to examine the feasibility of sparse polynomial chaos expansion method (SPCE) in estimating probability density function of rotor shaft vibration responses. The observed scatter of, e.g. the lateral vibration amplitude, is mainly due to the uncertainty of residual unbalances as well as random characteristics of stiffness and damping coefficients of the journal bearings. An accurate estimation of the response probability distribution is essential for the effectiveness of reliability-based design optimization as well as robust design optimization, which are considered as important elements of modern design practice.

The first four stochastic moments of the response estimated by SPCE and two sampling techniques (classical Monte Carlo method and Latin hypercube sampling) were used as inputs to the maximal entropy principle method (MEP) in order to approximate the response's probability density function (PDF). MEP method turned out to be numerically stable and well suited to accurately represent PDFs with an exponential behavior, i.e. a distribution shape characteristic to the investigated vibration responses.

Accounting for a big number of random variables constituting the stochastic model of the rotor-shaft system, the complete third order polynomial chaos expansion is computationally too expensive and provides no practical alternative to the simulation techniques. On the other hand, the SPCE version of this method yields very accurate PDF estimation with about 1% of this computational effort allowing for the optimal selection of significant polynomial chaos expansion terms. In the presented numerical example, which included 59 random variables, the full PCE method required more than 32500 third order terms. With the SPCE method this huge number can be greatly reduced. Such an approach leads to considerably higher computational accuracy in comparison with methods based on the second order expansion only. This method can be recommended for scatter analysis as well as reliability-based design optimization and robust design optimization of rotor-shaft systems.

The vibration analysis was carried out by means of the hybrid structural model consisting of one-dimensional beam-like continuous visco-elastic macro-elements and discrete oscillators. Such a hybrid model proved to be very computationally efficient and reliable, which is of a major importance in the context of stochastic analysis.

Acknowledgements. The authors would like to gratefully acknowledge the full support of European Regional Development Fund within the framework of the Innovative Economy Program, project number POIG.01.03.01-14-209/09, acronym – NUMPRESS.

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