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KRZYSZTOF SZUWALSKI*, ANETA USTRZYCKA**

OPTIMAL DESIGN OF CONICAL DISK WITH RESPECT TO DUCTILE CREEP RUPTURE TIME

OPTYMALIZACJA STOŻKOWEJ TARCZY PIERŚCIENIOWEJ ZE WZGLĘDU NA CZAS ZNISZCZENIA CIĄGLIWEGO

A b s t r a c t

This paper presents the problem of optimal design with respect to ductile creep rupture time for rotating disk. The material is described by the Norton–Bailey nonlinear creep law, here generalized for true stresses and logarithmic strains. For complex stress states, the law of similarity of deviators, combined with the Huber–Mises–Hencky hypothesis is applied. The set of four partial differential equations describes the creep conditions of annular disk. The optimal shape of the disk is found using parametric optimisation with one free parameter. The results are compared with disks of uniform thickness.

Keywords: *optimal design*, *annular disk*, *ductile creep rupture time*

S t r e s z c z e n i e

W niniejszym artykule przedstawiono problem optymalizacji tarczy pierścieniowej ze względu na czas zniszczenia ciągliwego. Do opisu materiału stosowano teorię nieliniowego pełzania Nortona–Baileya, uogólnioną dla naprężeń rzeczywistych i odkształceń logarytmicznych. W odniesieniu do złożonych stanów naprężeń stosowano prawo podobieństwa dewiatorów w połączeniu z hipotezą Hubera–Misesa–Hencky'ego. Proces pełzania tarczy wirującej opisuje układ czterech nieliniowych równań różniczkowych. Wyniki otrzymano przez zastosowanie optymalizacji jednoparametrycznej, w odniesieniu do płaskiej tarczy pełnej.

Słowa kluczowe: *optymalizacja*, *tarcza pierścieniowa*, *czas zniszczenia ciągliwego*, *pełzanie*

* Prof. dr hab. inż. Krzysztof Szuwalski, Instytut Mechaniki Stosowanej, Wydział Mechaniczny, Politechnika Krakowska.
** Mgr Aneta Ustrzycka, doktorantka, Wydział Mechaniczny, Politechnika Krakowska.

1. Introduction

The problems of structural optimization in creep conditions are relatively new ones and offer a wide scope of investigations. Overview of such problems with new possibilities of objective functions was given by Betten [1] and Życzkowski [5]. Among those functions of special importance, due to practical applications, are criteria connected with time of work to creep rupture. Vast majority of already known solutions took advantage of brittle creep rupture theory proposed by Kachanov because of its comparative simplicity (small strain theory). Applications of Hoff's ductile creep damage theory in optimization problems are rather scarce, as it requires finite strain theory, resulting in significant complication of problem. For the first time it was used by Szuwalski [3], who also formulated some general theorems on optimality of full disks [4]. The complexity of such problems is connected with nonlinearities both: physical and geometric and moreover existence of additional time factor. In present paper the problem of optimal shape with respect to ductile rupture time for the rotating disk clamped on the rigid shaft is investigated. Properties of material are described by the Norton creep law for complex stress state

$$\dot{\varepsilon}_e = k\sigma_e^n \tag{1}$$

where σ_e and $\dot{\varepsilon}_e$ denote: effective stress and velocity of effective strain, respectively, according to Huber–Mises–Hencky hypothesis. With regard to simplicity and cost of machinering only conical disks were investigated, under assumption of plane stress ($\sigma_z \equiv 0$).

For given dimensions (internal radius a and external b) and volume of material V, initial profile of disk $H(R)$, ensuring the longest time to ductile rupture was sought.

2. Governing equations

Ductile creep rupture is always preceded by large deformations and therefore the rigidification theorem must be neglected. The element of already deformed disk, in which all parameters connected with initial configuration (for $t = 0$) are denoted by capital letters, while their current values by the small ones, is investigated. The problem is solved in material (Lagrangian) coordinates and due to axial symmetry all quantities will be functions of two independent variables: radius R and time t. Deformation of the disk is described by dependence of spatial (Eulerian) coordinate on material one – $r(R, t)$. Arbitrarily small element of the disk, limited previously by two cylindrical surfaces of radii R and $R + dR$, and two planes forming the angle $d\vartheta$ after deformation is shown in Fig. 1.

The internal equilibrium condition, with body force caused by rotation with constant angle velocity ω taken into account, takes form

$$\frac{1}{hr'}\frac{\partial}{\partial R}(h\sigma_r) + \frac{\sigma_r - \sigma_\vartheta}{r} + \frac{\gamma}{g}\omega^2 r = 0 \tag{2}$$

where:

- σ_r – stands for current value of radial stress,
- σ_ϑ – of circumferential one,
- h – for current thickness,

γ – specific weight of material,
g – acceleration of gravity.

Assumption of incompressibility leads to

$$HRdR = hrdr \tag{3}$$

Finite strains require logarithmic strains

$$\varepsilon_r = \ln\frac{\partial r}{\partial R} = \ln r'; \ \ \varepsilon_\vartheta = \ln\frac{r}{R}; \ \ \varepsilon_z = \ln\frac{h}{H} \tag{4}$$

And their velocities

$$\dot{\varepsilon}_r = \frac{\dot{r}'}{r'}; \ \ \dot{\varepsilon}_\vartheta = \frac{\dot{r}}{r}; \ \ \dot{\varepsilon}_z = \frac{h'}{h} \tag{5}$$

Partial derivatives with respect to material coordinates are denoted by "primes", and with respect to time by dots.

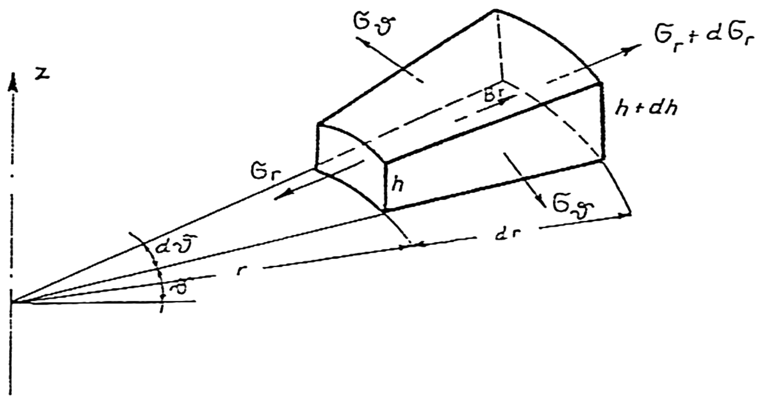

Fig. 1. Deformed element of the disk

Rys. 1. Element zdeformowanej tarczy

The shape change law, assumed in form of similarity of true stresses and velocities of logarithmic strains deviators leads to

$$\frac{\dot{r}}{r} = \frac{1}{2}k\sigma_e^{n-1}(2\sigma_\vartheta - \sigma_r) \tag{6}$$

Resulting from definitions of strains (4) compatibility condition

$$\dot{\varepsilon}_r - \dot{\varepsilon}_\vartheta = \frac{R\dfrac{\partial \dot{\varepsilon}_\vartheta}{\partial R}}{1 + R\dfrac{\partial \varepsilon_\vartheta}{\partial R}} \tag{7}$$

with help of the shape change law may be written

$$\bar{\sigma}'_\vartheta = \frac{6\bar{\sigma}_e^2(\bar{\sigma}_r - \bar{\sigma}_\vartheta)\dfrac{\bar{r}'\lambda}{\bar{r}\lambda+1} - \bar{\sigma}'_r\left[(n-1)\left(2\bar{\sigma}_r - \bar{\sigma}_\vartheta\right)(2\bar{\sigma}_\vartheta - \bar{\sigma}_r) - 2\bar{\sigma}_e^2\right]}{(n-1)(2\bar{\sigma}_\vartheta - \bar{\sigma}_r)^2 + 4\bar{\sigma}_e^2} \tag{8}$$

Given above equations make it possible to calculate four unknowns: true stresses σ_r and σ_ϑ and parameters describing deformation of the disk, spatial coordinate r and current thickness h.

For numerical calculations dimensionless quantities, denoted by overbears are introduced. Material and spatial coordinates are referred to the width of the disk

$$\bar{R} = \frac{R-a}{b-a}; \quad \bar{r} = \frac{r-a}{b-a} \tag{9}$$

Thickness of the disk is related to mean thickness h_m of annular disk of volume V

$$\bar{H} = \frac{\pi\cdot(b^2-a^2)}{V}\cdot H; \quad \bar{h} = \frac{\pi\cdot(b^2-a^2)}{V}\cdot h \tag{10}$$

Radial loading at radius b of rotating disk is resulting from uniformly distributed on the outer edge mass M. Total radial force is there equal

$$N_r(b) = M\omega^2 r(b,t) \tag{11}$$

As a result boundary condition on the radius b takes form

$$\sigma_r(b) = p = \frac{N_r(b)}{2\pi r(b)h(b)} = \frac{M\omega^2}{2\pi h(b)} \tag{12}$$

Dimensionless stresses are referred to calculated using rigidification theorem stress s in motionless full plane disk subject to tension with uniform pressure p (12)

$$s = \frac{M\omega^2}{2\pi h_m} = \frac{M\omega^2(b^2-a^2)}{2V} \tag{13}$$

Consequently

$$\bar{\sigma}_i = \frac{2V}{M\omega^2(b^2-a^2)}\cdot\sigma_i; \quad i = r,\vartheta \tag{14}$$

Dimensionless time is defined

$$\bar{t} = \frac{t}{\tau} \tag{15}$$

where: τ stands for the time of ductile rupture for full plane di,sk, rotating with the angle velocity ω, loaded by uniformly distributed at the external edge mass M, with neglected own mass.

For such a disk relation between true stresses and logarithmic strains takes form

$$\dot{\varepsilon}_z = \frac{\dot{h}}{h} = \frac{3}{2} k \sigma_e^{n-1} (\sigma_z - \sigma_m) \tag{16}$$

and effective stress is equal

$$\sigma_e = \sigma_r = \sigma_\vartheta = p \tag{17}$$

While mean stress, under assumption of plane stress ($\sigma_z \equiv 0$)

$$\sigma_m = \frac{2}{3} p \tag{18}$$

Proper substitutions to (16) lead to

$$\frac{1}{h}\frac{dh}{dt} = -k\left(\frac{M\omega^2}{2\pi h}\right)^n \tag{19}$$

With help of initial condition

$$t = 0\,;\ h = h_m \tag{20}$$

time to rupture τ was established

$$\tau = \frac{1}{nk\left(\frac{M\omega^2}{2\pi h_m}\right)^n} = \frac{1}{nks^n} \tag{21}$$

Consequently dimensionless time is calculated

$$\bar{t} = nks^n t \tag{22}$$

Finally the set of four dimensionless equations describing the process of creep for optimized rotating disk takes form

$$\bar{\sigma}_r' = \frac{\bar{\sigma}_\vartheta - \bar{\sigma}_r}{\bar{r}\lambda + 1} \cdot \bar{r}'\varepsilon - \frac{2r'\lambda(\bar{r}\lambda + 1)}{\lambda(\lambda + 2)} \cdot \mu - \frac{\bar{h}'}{\bar{h}} \bar{\sigma}_r \tag{23}$$

$$\bar{\sigma}'_\vartheta = \frac{6\bar{\sigma}_e^2(\bar{\sigma}_r - \bar{\sigma}_\vartheta)\frac{\bar{r}'\lambda}{\bar{r}\lambda + 1} - \bar{\sigma}_r'\left[(n-1)\left(2\bar{\sigma}_r - \bar{\sigma}_\vartheta\right)(2\bar{\sigma}_\vartheta - \bar{\sigma}_r) - 2\bar{\sigma}_e^2\right]}{(n-1)(2\bar{\sigma}_\vartheta - \bar{\sigma}_r)^2 + 4\bar{\sigma}_e^2} \tag{24}$$

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{r}\lambda + 1}{2\lambda \cdot n}\left(\bar{\sigma}_r^2 + \bar{\sigma}_\vartheta^2 - \bar{\sigma}_r\bar{\sigma}_\vartheta\right)^{\frac{n-1}{2}}\left(2\bar{\sigma}_\vartheta - \bar{\sigma}_r\right) \tag{25}$$

$$\bar{h} = \frac{\bar{H}(\bar{R}\lambda + 1)}{\bar{r}'\left(\bar{r}\lambda + 1\right)} \tag{26}$$

In those equations some auxiliary quantities are used

$$\mu = \frac{\gamma \cdot V}{gM}; \ \beta = \frac{b}{a}; \ \lambda = \beta - 1 \tag{27}$$

In succession: μ as ratio of own disk mass to mass distributed at the outer radius, β and λ as ratio of radii external and internal. Set written in given above form is very convenient for numerical calculations.

3. Numerical calculations

At the beginning of the creep process (for $t = 0$) disk remains undeformed, therefore the initial conditions take form

$$\bar{r}(\bar{R},0) = \bar{R}\ ; \quad \bar{h}(\bar{R},0) = \bar{H}(\bar{R}) \tag{28}$$

For disk clamped on the rigid shaft, boundary conditions at internal radius may be written

$$\bar{r}(0,\bar{t}) = 0\ ;\ \dot{\bar{r}}(0,\bar{t}) = 0 \tag{29}$$

The condition at external radius (12), where the mass M is distributed, in dimensionless form

$$\bar{\sigma}_r(1,t) = \frac{1}{\bar{h}(1,t)} \tag{30}$$

For conical disk the equation describing its initial shape has a linear form

$$\bar{H}(\bar{R};u_0,u_1) = u_0 + u_1\bar{R} \tag{31}$$

Parameters u_0 and u_1, which optimal values are being sought, are linked together by the condition of given volume V

$$u_1 = \frac{1-u_0}{\frac{2}{3}\lambda+1}(\lambda+1) \tag{32}$$

Parameter u_0, which may be interpreted as the thickness of disk at radius a $(\bar{R}=0)$ is treated as free, steering parameter. Values of this parameter are limited by condition of nonnegative thickness at the outer edge

$$\bar{H}(0) > 0 \Rightarrow u_0 > 0; \quad \bar{H}(1) > 0 \Rightarrow \left(u_0 < \frac{\lambda+2}{\frac{1}{3}\lambda} \right) \tag{33}$$

Numerical calculations were carried out according to the flow-chart given in Fig. 2.

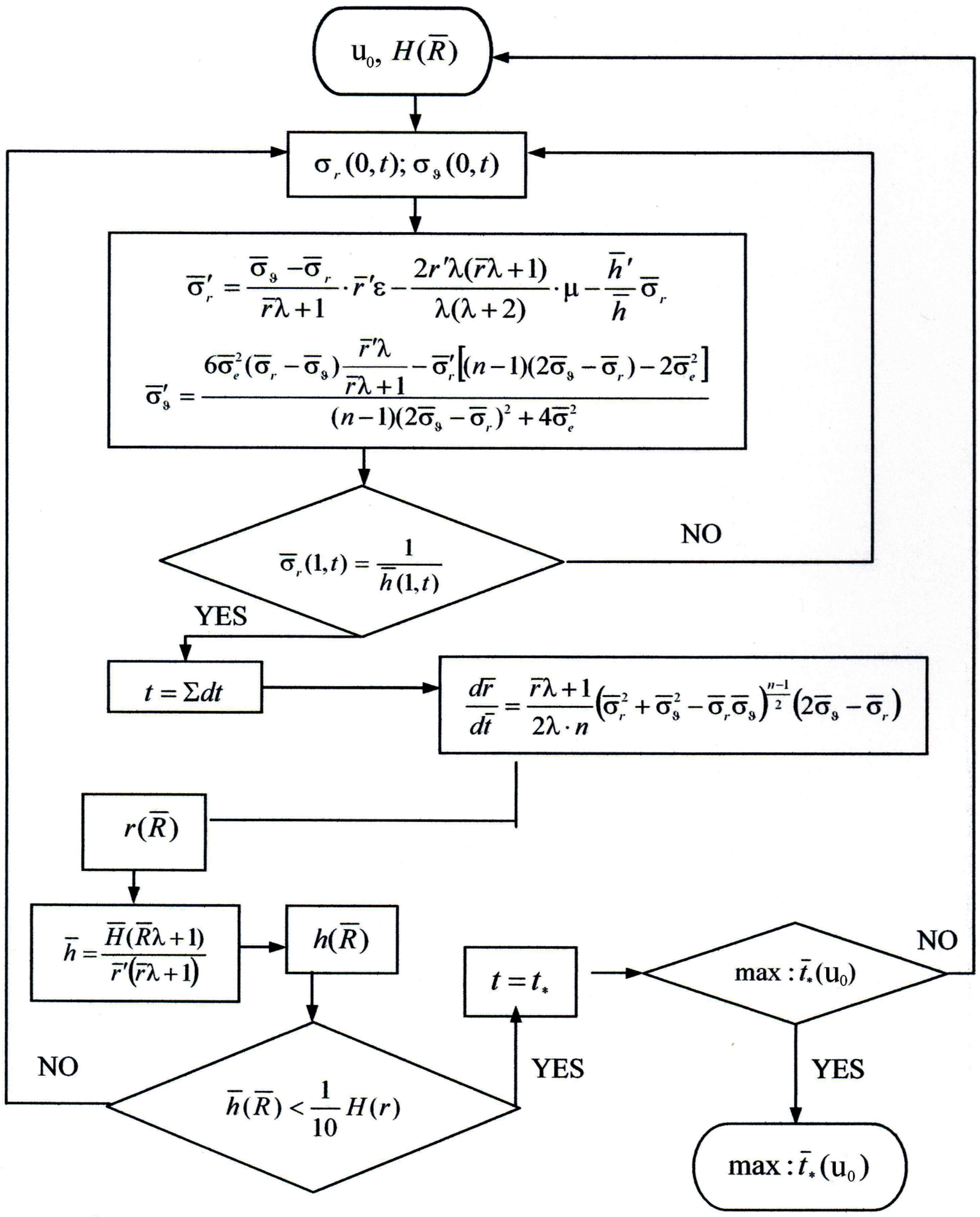

Fig. 2. Algorithm of numerical calculations – flow chart

Rys. 2. Algorytm obliczeń numerycznych – schemat blokowy

The results of numerical calculations are presented in diagrams:

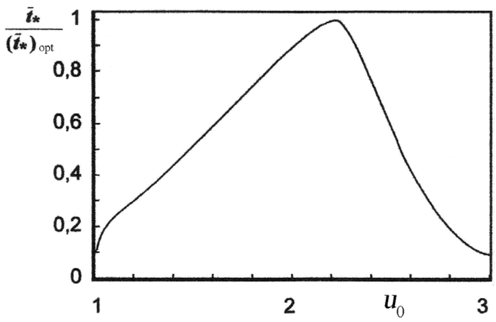

Fig. 3. The dependences of the ductile creep rupture time on the parameter u_0, $n = 3$, $\beta = 30$, $\mu = 1$

Rys. 3. Zależności czasu zniszczenia ciągliwego od parametru u_0, $n = 3$, $\beta = 30$, $\mu = 1$

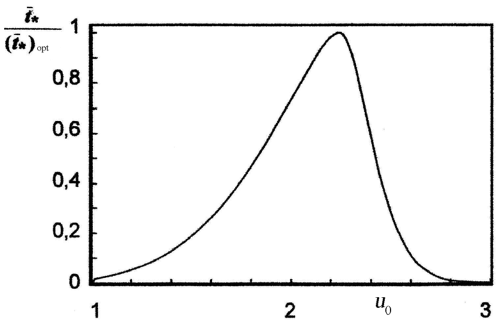

Fig. 4. The dependences of the ductile creep rupture time on the parameter u_0, $n = 9$, $\beta = 30$, $\mu = 1$

Rys. 4. Zależności czasu zniszczenia ciągliwego od parametru u_0, $n = 9$, $\beta = 30$, $\mu = 1$

As an instance, the dependence of time to rupture on free parameter u_0, for two different values of exponent in Norton law n, is shown. Maximum of curves, appointing the optimal value of parameter, becomes more "sharp" for larger values of n. Change of ratio of own and external masses μ does not introduce significant qualitative changes. However, for larger values of μ maximum moves to the right (disks more steep), while for smaller (less significant influence of own mass) to the left – optimal disk is almost flat.

This effect is even more visible in Fig. 5 where parameter μ is presented on logarithmic scale. When influence of own mass is insignificant, disks of almost constant thickness are optimal, while for large values of μ thickness of optimal disks at the outer radius tends to zero. The impact of exponent n is not very important.

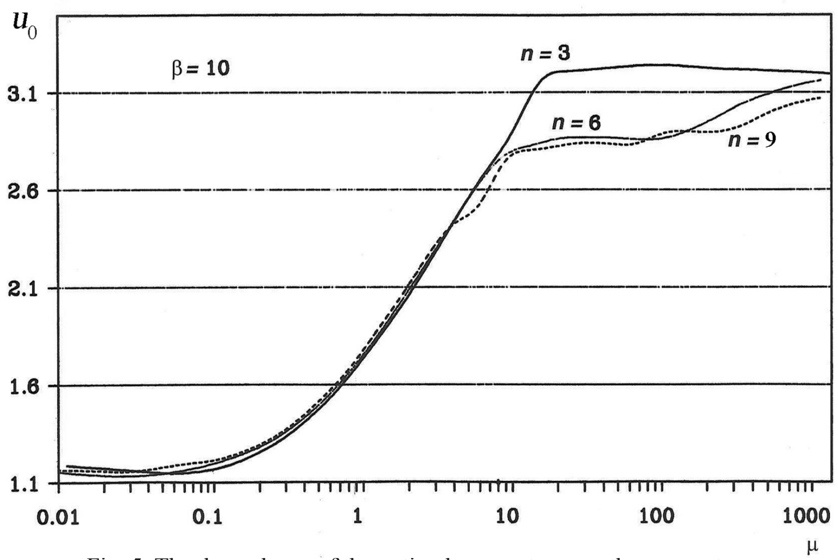

Fig. 5. The dependence of the optimal parameter u_0 on the parameter μ

Rys. 5. Zależność parametru optymalnego u_0 od parametru μ

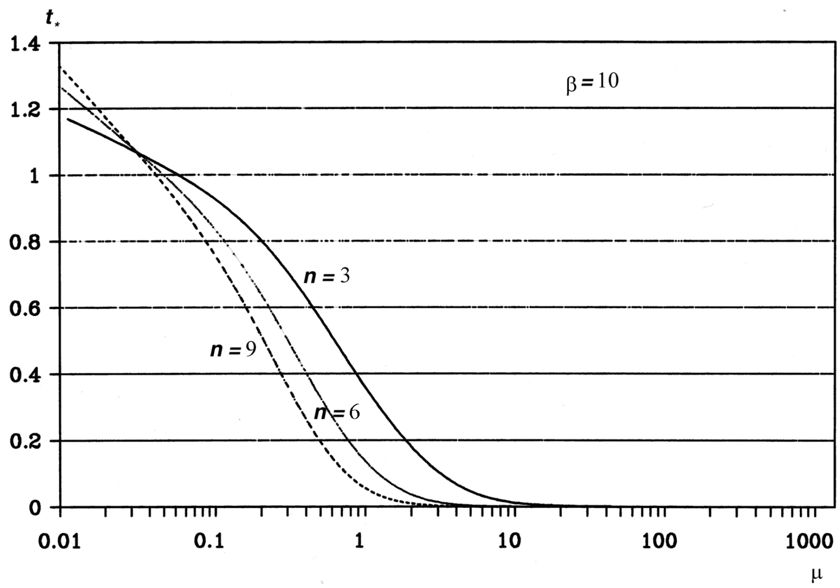

Fig. 6. The dependence of optimal ductile creep rupture time on parameter μ

Rys. 6. Zależność optymalnego czasu zniszczenia ciągliwego t_* od parametru μ

Time to ductile creep rupture for optimal disks depends on exponent *n* only for values smaller than 10. What is interesting, for very small μ (small influence of disk own mass) longer may work disks with larger values of exponent, but for $\mu > 0{,}1$ the longest time to rupture is obtained for $n = 3$ and increase of exponent results in shortening of time to rupture. The results in Figs. 5 and 6 were obtained for $\beta = 10$. For disks of different width results are very similar.

4. Conclusions

Among many new criteria of structural optimization in creep conditions of the greatest importance are criteria connected with time to rupture. An attempt of application of more difficult ductile creep rupture theory to problems of optimization of annular rotating disks mounted on the rigid shaft was made. Because of complexity of the problems (nonlinearities material and geometrical) parametric optimization was applied and special optimization procedure was introduced.

Obtained solutions strongly depend on ratio μ of own mass of the disk to mass uniformly distributed at the outer edge, causing there tensile pressure. When this mass is very large in comparison with own mass of the disk optimal disks are close to flat ones. On contrary, when influence of own mass is dominating, thickness of optimal disk at the outer edge tends to zero.

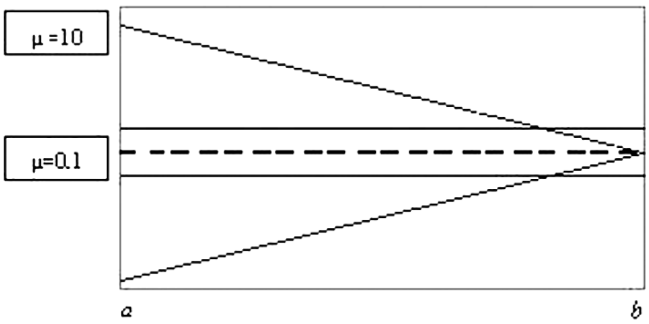

Fig. 7. Geometric profile of the disk

Rys. 7. Geometryczny profil tarczy

It turned out, that influence of exponent *n* in Norton law on the shape of optimal disk and on its time to ductile rupture depends on the ratio μ. Also dependence of optimal solution on the width of disk was investigated, but it did not introduce significant qualitative changes.

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