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YIELD CRITERION ACCOUNTING FOR THE THIRD INVARIANT OF STRESS TENSOR DEVIATOR. PART I. PROPOSITION OF THE YIELD CRITERION BASED ON THE CONCEPT OF INFLUENCE FUNCTIONS

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A proposition of an energy-based hypothesis of material effort for isotropic materials exhibiting strength-differential (SD) effect, pressure-sensitivity and Lode angle dependence is discussed. It is a special case of a general hypothesis proposed by the authors in [11] for anisotropic bodies, based on Burzyński's concept of influence functions [2] and Rychlewski's concept of elastic energy decomposition [16]. General condition of the convexity of the yield surface is introduced, and its derivation is given in the second part of the paper. Limit condition is specified for Inconel 718 alloy, referring to the experimental results published by Iyer and Lissenden [7].

1. Introduction

1.1. Motivation

In recent years, the number of new materials (e.g. composites, modern alloys) exhibiting certain uncommon properties – such as low elastic symmetry, pressure sensitivity, Lode angle dependence, strength differential effect – still increases and they become more and more commonly used. Furthermore, the precision of the measurement tools and accuracy of mathematical or numerical models used for the description of industrial processes is still improved, so some of the mentioned phenomena, which for decades have been considered negligible, now seem to be necessary to be involved in the mechanical analysis of the considered processes. Classical yield criteria, which are still commonly used both in elastic and plastic analysis (as limit conditions or plastic potentials in case of associated flow rule), cannot deal with those specific features of modern materials in a satisfactory way.

Many propositions of the yield criteria for anisotropic bodies were already stated (e.g. Mises [10], Burzyński [2], Hill [4], Hoffman [5], Tsai-Wu [20], Rychlewski [15], Theocaris [19] etc.), however some of them were poorly motivated physically being just of purely mathematical nature [5] or having only empirical character [19, 20]. Such approach enables one-to-one correlation between the final values of the parameters of the criterion and the limit quantities obtained from the experiment. Despite its great practical meaning, such an approach makes no contribution to the research on the nature of material effort. Furthermore, mathematical form of the criterion (arbitrary chosen by author) often constrains it in such a way that it is not possible to account for some of the phenomena mentioned above. In case of physically motivated limit criteria by Burzyński [2] and Rychlewski [15], other problems occur. Strictly energy-based limit condition by Rychlewski as a quadratic function of stress cannot account for the strength differential. In case of Burzyński's hypothesis, some misstatements in the final formulation of the limit condition for anisotropic solids were recently found and discussed in [18].

1.2. General proposition of a yield criterion for anisotropic bodies exhibiting SD effect

In [11] the authors have introduced a new proposition of a limit condition for anisotropic materials with asymmetric elastic range. It was directly motivated by ideas of spectral decomposition of compliance tensor $\mathbf{C}$ and elastic energy decompositions introduced by Rychlewski [15, 16] and the idea of stress state dependent influence functions introduced by Burzyński [2], which enabled him improvement of the classical Huber–Mises [6] condition so that it accounted for the SD effect. It is stated that as a measure of material effort one can consider the following combination:

$$\eta_1 \Phi(\boldsymbol{\sigma}_1) + ... + \eta_\chi \Phi(\boldsymbol{\sigma}_\chi), \qquad \chi \leqslant 6 \tag{1.1}$$

such that:

$$\begin{cases} \mathscr{T}^2_{\mathrm{sym}} = \mathscr{H}_1 \oplus ... \oplus \mathscr{H}_\chi \\ \mathscr{H}_\alpha \dot{\perp} \mathscr{H}_\beta \quad \text{for} \quad \alpha \neq \beta \end{cases} \Rightarrow \begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + ... + \boldsymbol{\sigma}_\chi, \quad \boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha, \\ \boldsymbol{\sigma}_\alpha \bullet \boldsymbol{\sigma}_\beta = \boldsymbol{\sigma}_\alpha \cdot (\mathbf{C} \cdot \boldsymbol{\sigma}_\beta) = 0 \quad \text{for} \quad \alpha \neq \beta, \end{cases} \tag{1.2}$$

where η_α is a certain stress-state dependent function. To keep mutual independence of the terms of the criterion, it is assumed that it depends only on the stress state component corresponding with the proper elastic energy density $\boldsymbol{\sigma}_\alpha$ (the projection of $\boldsymbol{\sigma}$ onto $\mathscr{H}_\alpha$)

$$\eta_\alpha = \eta_\alpha(\boldsymbol{\sigma}_\alpha) \tag{1.3}$$

and that it is isotropic in its domain (subspace $\mathscr{H}_\alpha$), thus it can be expressed only in terms of invariants of $\boldsymbol{\sigma}_\alpha$

$$\eta_\alpha(\boldsymbol{\sigma}_\alpha) = \eta_\alpha\left(I_1(\boldsymbol{\sigma}_\alpha); I_2(\boldsymbol{\sigma}_\alpha); I_3(\boldsymbol{\sigma}_\alpha)\right). \tag{1.4}$$

More details can be found in [11].

Among all possible energetically orthogonal decompositions of the space of symmetric second order tensors $\mathscr{T}^2_{\mathrm{sym}} = \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_\chi$, the choice of the decomposition into eigensubspaces of compliance tensor $\mathbf{C}$ is the best motivated both physically (due to clear physical interpretation of those subspaces) and mathematically (since it is the only decomposition of $\mathscr{T}^2_{\mathrm{sym}}$ which is both orthogonal and energetically orthogonal).

2. General limit condition for pressure-sensitive, Lode angle dependent isotropic bodies exhibiting SD effect

Even in case of the simplest materials, namely those macroscopically homogeneous and isotropic, such as modern alloys, many classical yield criteria (i.e. Huber–Mises [6, 9], Burzyński [2], Drucker–Prager [3] etc.) fail to describe them correctly either due to lack of pressure-sensitivity or the Lode angle dependence. The special isotropic case of the yield criterion introduced above is found suitable for accounting for the influence of both the pressure and Lode's angle.

From the spectral decomposition of isotropic compliance tensor we obtain a one-dimensional subspace of spherical tensors (hydrostatic stresses) and five-dimensional subspace of deviators (shears). Energy density is decomposed into energy density of distortion Φ_f and energy density of volume change Φ_v. Yield condition (1.1) can be rewritten in the following form:

$$\tilde{\eta}_v(I_1(\mathbf{A}_\sigma);\ I_2(\mathbf{A}_\sigma);\ I_3(\mathbf{A}_\sigma))\Phi_v + \tilde{\eta}_f(J_1, J_2, J_3)\Phi_f = 1, \tag{2.1}$$

where $\mathbf{A}_\sigma$ is the isotropic component of the stress tensor and J_1, J_2, J_3 are invariants of the stress tensor deviator. It is known that:

$$\begin{aligned} &I_1(\mathbf{A}_\sigma) = 3p, && I_2(\mathbf{A}_\sigma) = 3p^2, && I_3(\mathbf{A}_\sigma) = p^3,\\ &J_1 = 0, && J_2 = \frac{1}{2}q^2, && J_3 = \frac{1}{3\sqrt{6}}q^3\cos(3\theta),\\ &\Phi_v = \frac{p^2}{2K}, && \Phi_f = \frac{q^2}{4G}, && \end{aligned} \tag{2.2}$$

where K is the Helmholtz bulk modulus, G is the Kirchhoff shear modulus, p is the hydrostatic stress, q deviatoric component of stress and θ is the Lode angle – they can be expressed in terms of stress state components in any coordinate system as well as in principal stresses:

$$
\begin{aligned}
p &= \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3),\\
q &= \sqrt{\frac{1}{3}\left[(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+(\sigma_{11}-\sigma_{22})^2+6(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)\right]}\\
&\qquad = \sqrt{\frac{1}{3}\left[(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+(\sigma_1-\sigma_2)^2\right]},\\
\theta &= \frac{1}{3}\arccos\left[\frac{3\sqrt{3}}{2}\frac{J_3}{(J_2)^{3/2}}\right].
\end{aligned}
\tag{2.3}
$$

Please note that the second invariants of the spherical and deviatoric part of the stress tensor are proportional to the volumetric and distortional part of the elastic energy density respectively. Since all invariants of the hydrostatic component of the stress state depend on p, it is enough to state that $\eta_v = \eta_v(p)$. It is also clear that it is the third invariant of the stress tensor deviator which makes the qualitative, not only quantitative, distinction between various modes of shearing, so it is assumed that the influence function corresponding to the distortional part of energy density depends only on Lode angle θ. Including constant parameters (i.e. elastic moduli) in the influence functions, the limit condition (1.1) can be finally obtained in the following form:

$$\eta_v(p)p^2 + \eta_f(\theta)q^2 = 1. \tag{2.4}$$

2.1. Influence functions

Many authors have been already considering various functions describing the influence of pressure or Lode's angle on the material effort. It seems that in case of pressure influence function, the one proposed by Burzyński is one of the most general – it enables description of various relations between hydrostatic and deviatoric stresses – linear, paraboloidal, hyperboloidal and elliptical. It is a two-parameter rational function of the following form:

$$\eta_v(p) = \left(\omega + \frac{\delta}{p}\right). \tag{2.5}$$

There is a large variety of different functions describing the influence of the Lode angle – valuable summary of propositions of Lode angle dependences was made by BARDET and published in [1]. Some other suggestions of the Lode angle influence function were also presented in [12]:

- Two-parameter power function (RANIECKI, MRÓZ [14])

$$\eta_f(\theta) = [1 + \alpha \cos(3\theta)]^{\beta}.$$

- Two-parameter exponential function (RANIECKI, MRÓZ [14])

$$\eta_f(\theta) = 1 + \alpha \left[1 - e^{-\beta(1+\cos(3\theta))}\right].$$

- One-parameter trigonometric function (LEXCELLENT [8])

$$\eta_f(\theta) = \cos\left[\frac{1}{3}\arccos\left[1 - \alpha(1 - \cos(3\theta))\right]\right].$$

- Two-parameter trigonometric function (PODGÓRSKI [13])

$$\eta_f(\theta) = \frac{1}{\cos(30^\circ - \beta)} \cos\left[\frac{1}{3}\arccos\left(\alpha \cdot \cos(3\theta)\right) - \beta\right].$$

It is often assumed that the Drucker's postulates are true – as a consequence of this assumption, the yield surface should be convex. Convexity condition for the limit surface determined by yield condition (2.4) for arbitrary chosen form of influence function was derived and will be published in the second part of the current paper [17].

3. LIMIT CRITERION SPECIFICATION

An attempt to specify the limit condition referring to experimental data available in the literature was made. A series of experiments performed by IYER, LISSENDEN [7] for Inconel 718 alloy was taken as the reference data. Analysis of the results obtained by Iyer and Lissenden, both in experiments and numerical simulation, lead to the choice of Burzyński's pressure influence function with $\omega = 0$ (paraboloid yield surface) and slightly modified Podgórski's Lode angle influence function with $\alpha = 0.8$, $\beta = 30^\circ$ – for details see [12]. The Levenberg–Marquardt algorithm was used to find the values of the rest of parameters of the criterion by optimal fitting the assumed surface to the twelve points obtained from the experiments. The final form of the yield criterion was obtained as follows:

$$\frac{q^2}{\cos(30^\circ - \beta)} \cos\left[\frac{1}{3}\arccos\left[\alpha \cdot \cos(3(\theta - 90^\circ))\right] - \beta\right] + \left(\omega + \frac{\delta}{p}\right) \cdot p^2 - H = 0, \tag{3.1}$$

where $\alpha = 0.8$, $\beta = 30°$, $\delta = 215.95$ MPa, $\omega = 0$, $H = 201670.46$ MPa^2. The plot of the limit surface and its cross-sections at octahedral plane and BURZYŃSKI'S plane [21] are given in Figs. 1–3. In the latter figure the cross-sections of the yield

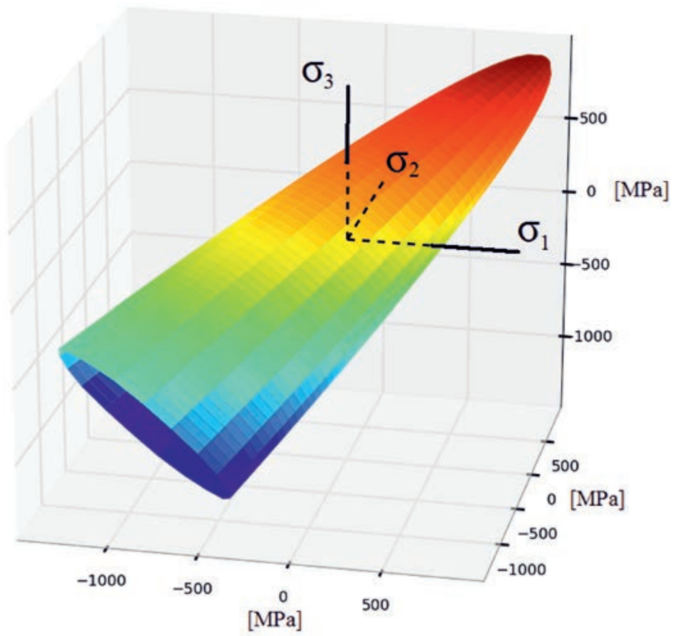

FIG. 1. Yield surface given by Eq. (3.1).

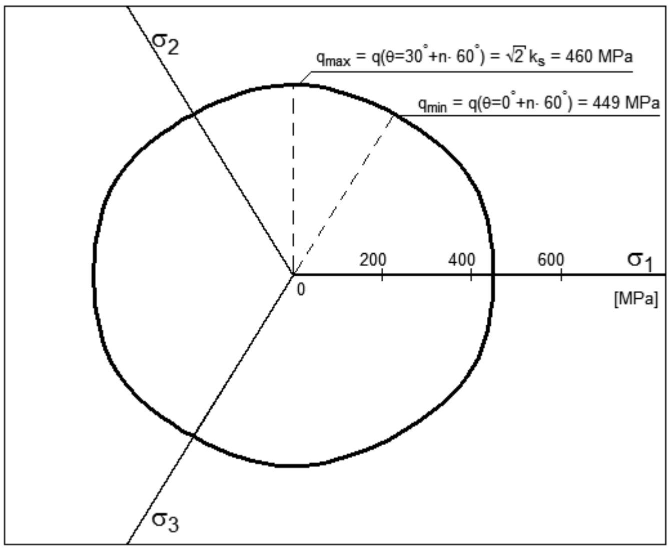

FIG. 2. Lode angle dependence – cross-section of the yield surface given by (3.1) at octahedral plane.

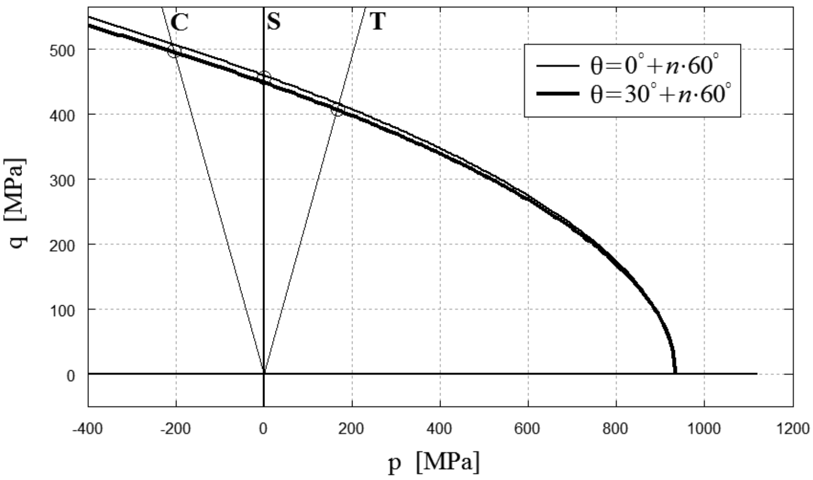

FIG. 3. Pressure-sensitivity – cross-section of the yield surface given by (3.1) at Burzyński's plane. Straight lines denoted C, S and T determine uniaxial compression, shear and uniaxial tensile stress states respectively.

surface (3.1) at Burzyński's plane for two values of the Lode angle is presented – $\theta = 0° + n \cdot 60°$ ($n \in \mathbb{N}$), which corresponds with uniaxial stress states, and $\theta = 30° + n \cdot 60°$ which corresponds with pure shears.

4. Summary

A new proposition of a limit condition for the pressure-sensitive isotropic and homogeneous bodies exhibiting Lode's angle dependency and strength differential effect was presented. The straightforward derivation of the discussed yield criterion from the general idea of an energy-based limit condition for anisotropic bodies with asymmetric elastic range introduced by authors in [11] was shown. Some propositions of the pressure and Lode's angle influence functions were given. Condition for convexity of the yield surface corresponding with the discussed limit condition will be presented in the second part of the paper. Specification of the limit criterion for assumed influence functions referring to the experimental data published in [7] was presented.

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