JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
2, **38**, 2000

ON THE STABILITY OF FRICTION CONTACT

ZENON MRÓZ

Institute of Fundamental Technological Research, Warsaw
e-mail: zmroz@ippt.gov.pl

The stability condition (1.2) for frictional contact expressed in terms of potential energy and the dissipation function is applied to analysis of the stability of rigid-sliding and elastic-sliding contacts obeying the Coulomb friction condition and the non-associated sliding rule. Both static and dynamic modes are considered.

Key words: contact friction, stability, static and dynamic modes

1. Introduction

Stability of frictional contact can be studied for different friction models, including the classical Coulomb-Amontons friction rule for which only rigid-slipping response occurs at the contact. A more general model would be obtained by assuming tangential and normal contact compliance, and also accounting for contact dilatancy or compaction effects.

Certainly, in considering stability problem, we have to analyse the whole system of elastic or elasto-plastic structure interacting with the frictional foundation. The critical state can then be reached either through the loss of stability of the structure or due to destabilizing action of the contact. General stability conditions for a discrete elastic system with frictional interaction were derived by Mróz and Plaut (1992), and Nguyen (1992). A more general class of friction and slip rules was discussed by Jarzębowski and Mróz (1994) and Mróz and Giambanco (1996). The conservative and non-conservative stability problems of truss and frame structures were extensively discussed by Życzkowski (1998) who, however, did not consider the friction effects.

Consider a discrete system with n degrees of freedom. Assuming q_i as the generalized coordinate vector and specifying the potential energy $V = V(q_i)$

and the dissipation function $D = D(q_k, \dot{q}_i)$, the equilibrium equations are

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0 \qquad \text{or} \qquad V_i + D_{(i)} = 0 \quad i = 1, 2, ..., n \tag{1.1}$$

where the subscript i denotes the derivative with respect to the configuration coordinate q_i and the subscript (i) denotes the derivative with respect to the rate $\dot{q}_i$.

The stability conditions were derived in the form

$$\frac{\partial^2 V}{\partial q_i \partial q_j} + \frac{\partial D}{\partial q_k}\dot{q}_k > 0 \qquad \text{or} \qquad V_{ij}\dot{q}_i\dot{q}_j + D_k\dot{q}_k > 0 \tag{1.2}$$

Thus, the sufficient conditions of stability are satisfied when the inequality (1.2) occurs. In particular, when there is no frictional dissipation, the well-known condition is obtained

$$V_{ij}\dot{q}_i\dot{q}_j > 0 \tag{1.3}$$

requiring the potential energy to be a positive-definite function. On the other hand, when the rigid-frictional system is considered, the stability condition is expressed in terms of the dissipation function

$$D_i\dot{q}_i > 0 \tag{1.4}$$

In our further analysis, we shall refer to these conditions.

2. Rigid-slip model of contact

Consider first the rigid-slip model of contact. For the contact stress within the domain bounded by the friction condition there is neither slip nor deformation. The Coulomb friction condition can be expressed in the form

$$f(\boldsymbol{T}, N) = \sqrt{T_iT_i} - \mu N \leqslant 0 \qquad \text{or} \qquad \|\boldsymbol{T}\| - \mu N \leqslant 0 \tag{2.1}$$

and the slip rule in the tangential direction within the contact plane is

$$v_i^s = \dot{\lambda}\frac{T_i}{\|\boldsymbol{T}\|} \qquad v_N^s = 0 \qquad \dot{\lambda} = \sqrt{v_i^s v_i^s} = v^s > 0 \tag{2.2}$$

where v_i^s, $i = 1, 2$ and v_N^s are the slip components in the Cartesian reference frame, where the x_3-axis coincides with the normal direction.

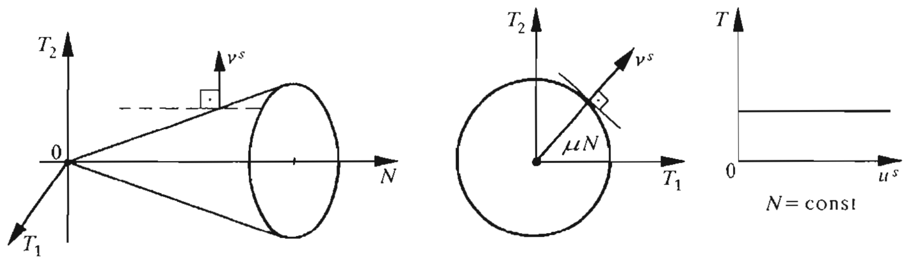

Fig. 1. Friction surface and sliding rule

The dissipation function is expressed as follows

$$D = T_i v_i^s = \mu N \|\boldsymbol{v}^s\| = \mu N \sqrt{(v_1^s)^2 + (v_2^s)^2} \tag{2.3}$$

so the inverse relations to Eq (2.1) are

$$T_i = \frac{\partial D}{\partial v_i^s} = \mu N \frac{v_i^s}{\|\boldsymbol{v}^s\|} \tag{2.4}$$

The rate of variation of T_i equals

$$\begin{aligned} \dot{T}_i &= \mu \dot{N} \frac{v_i^s}{\|\boldsymbol{v}^s\|} + \mu N \frac{\dot{v}_i^s \|\boldsymbol{v}^s\| - v_i^s \frac{v_k^s \dot{v}_k^s}{\|\boldsymbol{v}^s\|}}{v_k^s v_k^s} = \\ &= \mu \dot{N} \frac{v_i^s}{\|\boldsymbol{v}^s\|} + \mu N \Big(\frac{\dot{v}_i^s}{\|\boldsymbol{v}^s\|} - \frac{v_i^s v_k^s \dot{v}_k^s}{\|\boldsymbol{v}^s\|^3} \Big) = \dot{T}_i' + \dot{T}_i'' \end{aligned} \tag{2.5}$$

where the first term $\dot{T}_i'$ is coaxial with the slip velocity and the second term is normal to the slip velocity and represents the *force reorientation* term, Fig.2a.

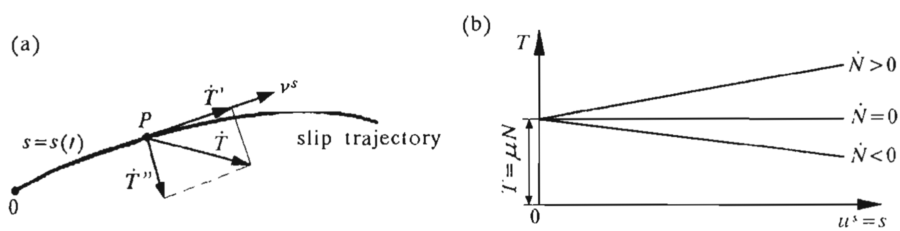

Fig. 2. Friction forces varying along the slip trajectory – (a), and with the normal traction – (b)

In fact, we have

$$\dot{T}_i v_i^s = \dot{T}_i' v_i^s \qquad\qquad \dot{T}_i'' v_i^s = 0 \tag{2.6}$$

Let us note that the term $\dot{T}_i'$ corresponds to frictional hardening or softening due to variation of normal traction N. In fact, we have the incremental work rate

$$\dot{L} = \dot{T}_i v_i^s = \dot{T}_i' v_i^s = \mu \dot{N} \|\boldsymbol{v}^s\| \tag{2.7}$$

Hence the stability condition is

$$\dot{L} = \mu \dot{N} \|\boldsymbol{v}^s\| > 0 \tag{2.8}$$

or

$$\mu \dot{N} > 0 \tag{2.9}$$

since $\|\boldsymbol{v}^s\|$ is always positive. Consider slip along the trajectory $s = s(t)$, where $\dot{s} = \|\boldsymbol{v}^s\|$. As $N = N(s)$, then we may state the stable and unstable slip portions as

$$\frac{dN}{ds} > 0 \ \text{– stable slip} \qquad\qquad \frac{dN}{ds} < 0 \ \text{– unstable slip} \tag{2.10}$$

Let us note that the conditions (2.9) follow directly from the general condition (1.4). In fact, the generalized coordinate rate $\dot{q} = \dot{s}$ and Eq (1.4) provides

$$D_i \dot{q}_i = \frac{\partial D}{\partial N}\frac{dN}{ds} = \mu \dot{N} \|\boldsymbol{v}^s\| > 0 \tag{2.11}$$

where $\dot{N} = dN/ds$. The condition (2.10) is thus equivalent to the requirement that $\dot{L} > 0$.

The stability conditions can easily be generalized for a distributed contact pressure $\sigma_N = -\boldsymbol{n}\sigma\boldsymbol{n}$ over the contact area S_c. In fact, we have

$$\dot{L} = \int \mu \dot{\sigma}_N \|\boldsymbol{v}^s\| \, dS_c > 0 \tag{2.12}$$

Consider the translation and rotation of a rigid body contacting over the area S_c with the other body. Assume the slip distribution in the form, Fig.3

$$\boldsymbol{v}^s = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r} \tag{2.13}$$

or

$$\begin{gathered} v_x^s = v_0 + r\omega \sin\alpha \qquad\qquad v_y^s = -r\omega\cos\alpha \\ \|\boldsymbol{v}^s\| = \sqrt{v_0^2 + r^2\omega^2 + 2v_0\omega r \sin\alpha} \end{gathered} \tag{2.14}$$

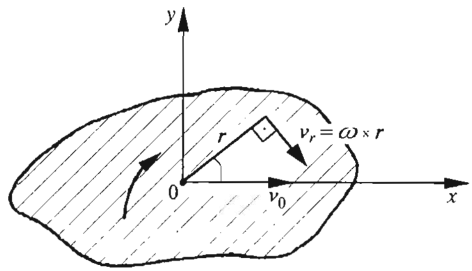

Fig. 3. Slip over the distributed contact area S_c

Then, the condition (2.11) becomes

$$\dot{L} = \int \mu \dot{\sigma}_N r \sqrt{v_0^2 + r^2\omega^2 + 2v_0\omega r \sin\alpha} \; dr d\alpha > 0 \tag{2.15}$$

In the particular case $\omega = 0$, there is a pure translation mode, so Eq (2.15) provides

$$\dot{L} = \int \mu \dot{\sigma}_N \; dS_c = \mu \dot{N} > 0 \tag{2.16}$$

where N is the resultant force transferred through the contact area.

3. Rigid-slip model with elastic interaction

Consider now a more complex case when the tangential and normal tractions are applied through elastic system. Consider first an illustrating example shown in Fig.4a. The block sliding along horizontal foundation is acted on by a normal force N exerted by a spring sliding along the inclined plane. The horizontal force is applied through the horizontal spring.

Denoting by s the frictional sliding distance and by u the total displacement of the point A of the system, we have

$$\begin{aligned} &\dot{u} = \dot{u}^e + \dot{s} \qquad && \dot{u}^e = \frac{\dot{T}}{E} \qquad && \dot{s} = \dot{\lambda}\,\mathrm{sgn}\,T \\ &T - \mu N \leqslant 0 && \dot{T} - \mu\dot{N} \leqslant 0 && \dot{\lambda} > 0 \end{aligned} \tag{3.1}$$

where u^e denotes the spring elongation. Assume that

$$N = N_0 - ks\tan\alpha = N_0 - Ks \qquad \dot{N} = -K\dot{s} \tag{3.2}$$

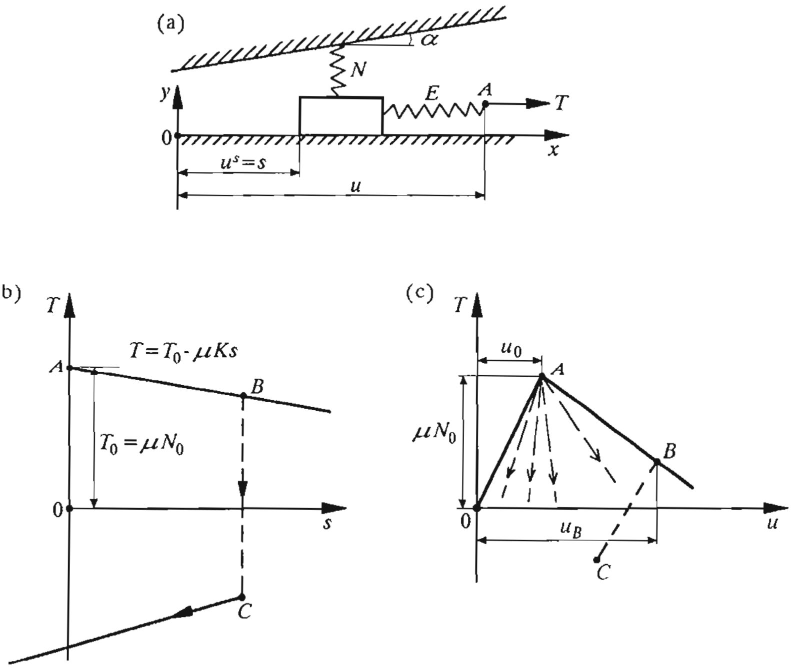

Fig. 4. (a) Block with elastic interaction, (b) force variation with frictional slip s, (c) force variation with total displacement

where k denotes the vertical spring stiffness and $K = k\tan\alpha$. In view of Eqs (3.1) and (3.2), we have for the sliding regime

$$\begin{aligned} &\dot{u} = \frac{\dot{T}}{E} + \dot{s} = -\frac{\mu K\dot{s}}{E} + \dot{s} = \Big(1 - \frac{\mu K}{E}\Big)\dot{s} \qquad \dot{s} > 0 \\ &\dot{T} = \mu\dot{N} = -\mu K\dot{s} = -\frac{\mu K}{1 - \frac{\mu K}{E}}\dot{u} = -E_s\dot{u} \end{aligned} \tag{3.3}$$

and in the elastic regime there is

$$\dot{u} = \dot{u}^e = \frac{\dot{T}}{E} \qquad\qquad T - \mu N < 0 \tag{3.4}$$

Integrating Eq (3.3), we obtain the force-displacement relations

$$
\begin{aligned}
&T = Eu && u < \frac{T_0}{E} = u_0 \\
&T = T_0 - \frac{\mu K}{1 - \frac{\mu K}{E}}(u - u_0) && u > u_0 \qquad \dot{u} > 0
\end{aligned}
\tag{3.5}
$$

It is seen that for $\mu K < E$ the block sliding is associated with friction *softening*. The softening modulus E_s depends on the stiffness ratio K/E. In fact, when $\mu K = E$, the *snap-back* phenomenon occurs and the sliding process cannot be controlled by imposed displacements when $\mu K > E$. Thus

for $\mu K < E$ – process is unstable under force control
for $\mu K > E$ – process is unstable under displacement control

Let us now apply the general stability condition (1.2). Consider the displacement controlled motion. The potential energy of system is

$$
\Pi = \frac{k}{2}(u_{NO} - s\tan\alpha)^2 + \frac{1}{2}E(u - s)^2 \tag{3.6}
$$

where u_{NO} is the initial displacement of vertical spring. Differentiating Eq (3.6) with respect to u and s, we have

$$
T = \frac{\partial \Pi}{\partial u} = E(u - s)
$$

and the dissipative force is

$$
T_d^{\Pi} = -\frac{\partial \Pi}{\partial s} = k(u_{NO} - s\tan\alpha)\tan\alpha + E(u - s) = N\tan\alpha + T \tag{3.7}
$$

Fig.5 presents the force distribution at the constraining surface. The vertical force N in the spring induces the tangential dissipative component $N\sin\alpha$. There are two dissipative forces μN and $N\tan\alpha$ associated with the horizontal slip rate $\dot{s}$, thus

$$
D = D_1 + D_2 = \mu N\dot{s} + N\dot{s}\tan\alpha \tag{3.8}
$$

and the dissipative force is

$$
T_d^{D} = \frac{\partial D}{\partial \dot{s}} = \mu N + N\tan\alpha = (\mu + \tan\alpha)k(u_{NO} - s\tan\alpha) \tag{3.9}
$$

Comparing Eqs (3.7) and (3.9), we obtain $T = \mu N = \mu k(u_{NO} - s\tan\alpha)$.

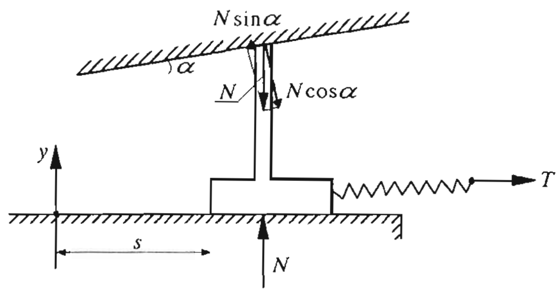

Fig. 5. Force distribution at the contact

The stability condition now is

$$\frac{\partial^2 \Pi}{\partial s^2}\dot{s}^2 + \frac{\partial D}{\partial s}\dot{s} > 0 \tag{3.10}$$

or

$$k\dot{s}\tan^2\alpha + E + (\mu + \tan\alpha)\dot{N} > 0 \tag{3.11}$$

and noting that $\dot{N} = -k\dot{s}\tan\alpha$, we obtain

$$E - \mu k\tan\alpha = E - K\mu > 0 \tag{3.12}$$

Let us note that the stability condition can be written as follows

$$\frac{\partial T_d^D}{\partial s} > \frac{\partial T_d^\Pi}{\partial s} \qquad\qquad T_d^D = T_d^\Pi \tag{3.13}$$

Thus, the dissipative forces calculated from the potential energy and dissipation function are equal, however, their rates with respect to the dissipative displacement s can be different. The stable situation occurs when the dissipative force T_d^D exceeds the potential energy release force T_d^Π near the equilirium state.

Let us consider now the dynamic response of the system. The equation of motion of the block is expressed as follows

$$M\ddot{s} + \mu N = T = -E(u - s) \tag{3.14}$$

Consider the perturbation $z = z(t)$ satisfying the condition $\dot{u}(t) = 0$, so we have

$$M\ddot{z} - \mu K z = -Ez \tag{3.15}$$

or

$$M\ddot{z} + (E - \mu K)z = 0 \tag{3.16}$$

The stable equilibrium occurs when $E - \mu K$. Assuming

$$z = z_0 e^{i\omega t} \qquad \dot{z} = i\omega z_0 e^{i\omega t} \qquad \ddot{z} = -\omega^2 z_0 e^{i\omega t} \tag{3.17}$$

and substituting into Eq (3.15), we obtain

$$M\omega^2 = E - \mu K \tag{3.18}$$

and

$$\omega^2 = \frac{E - \mu K}{M} > 0 \tag{3.19}$$

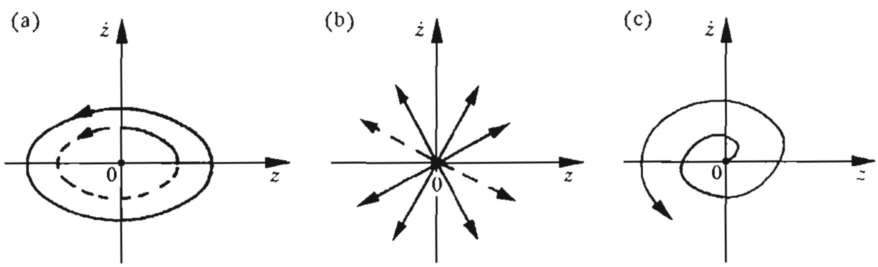

Fig. 6. Phase diagrams of (a) stability, (b) divergence instability, (c) flutter instability

Thus the system will undergo free harmonic vibrations near the equilibrium point with the frequency ω specified by Eq (3.18), cf Fig.6a. On the other hand, when $E - \mu K < 0$, the solution of Eq (3.15) is

$$z = z_0 e^{\lambda t} \qquad \ddot{z} = z_0 \lambda^2 e^{\lambda t}$$

where

$$\lambda^2 = \frac{\mu K - E}{M} > 0 \tag{3.20}$$

so the system exhibits also *divergence instability*, cf Fig.6b. When force control is applied, then $\dot{T} = 0$ and the perturbed equation of motion is

$$M\ddot{z} - \mu K z = 0 \tag{3.21}$$

and the system exhibits divergence instability: $z = z_0 e^{\lambda t}$, where

$$\lambda^2 = \frac{\mu K}{M} > 0 \tag{3.22}$$

4. Elastic slip model with dilatancy effect

Let us now discuss a more complex model when the elastic compliances of the contact are accounted for and the slip dilatancy effect is introduced. The total velocities are now composed of elastic and slip portions, so that

$$v_i = v_i^e + v_i^s \qquad v_N = v_N^e + v_N^s \qquad i = 1,2 \tag{4.1}$$

where

$$v_i^e = \frac{\dot{T}_i}{K_T} \qquad v_N^e = \frac{\dot{N}}{K_N} \tag{4.2}$$

and

$$\begin{bmatrix} v_i^e \\ v_N^e \end{bmatrix} = \begin{bmatrix} \dfrac{1}{K_T} & 0 \\ 0 & \dfrac{1}{K_N} \end{bmatrix} \begin{bmatrix} \dot{T}_i \\ \dot{N} \end{bmatrix} \tag{4.3}$$

The friction condition and the slip rule are

$$\begin{gathered} F(T,N) = \sqrt{T_i T_i} - \mu N = \|\boldsymbol{T}\| - \mu N \leqslant 0 \\ v_i^s = \dot{\lambda}\frac{T_i}{\|\boldsymbol{T}\|} \qquad v_N^s = -\dot{\lambda}\overline{\mu} \qquad \dot{\lambda} = \sqrt{v_i^s v_i^s} > 0 \end{gathered} \tag{4.4}$$

The total velocities now are

$$v_i = \frac{\dot{T}_i}{K_T} + \dot{\lambda}\frac{T_i}{\|\boldsymbol{T}\|} \qquad v_N^e = \frac{\dot{N}}{K_N} - \dot{\lambda}\overline{\mu} \tag{4.5}$$

where $\overline{\mu}$ denotes the dilatancy factor. Using the consistency condition

$$\frac{T_i \dot{T}_i}{\|\boldsymbol{T}\|} - \mu\dot{N} = 0 \tag{4.6}$$

the following rate constitutive equations are derived

$$\begin{bmatrix} \dot{T}_i \\ \dot{N} \end{bmatrix} = \begin{bmatrix} K_T\delta_{ik} - K_T^2\dfrac{T_i T_k}{\|\boldsymbol{T}\|^2 H} & \mu\dfrac{K_T K_N}{H}\dfrac{T_i}{\|\boldsymbol{T}\|} \\ \overline{\mu}\dfrac{K_T K_N}{H}\dfrac{T_k}{\|\boldsymbol{T}\|} & K_N - \dfrac{K_N^2}{H}\mu\overline{\mu} \end{bmatrix} \begin{bmatrix} v_k \\ v_N \end{bmatrix} \tag{4.7}$$

where

$$H = K_T + \mu K_N \overline{\mu} \tag{4.8}$$

When $\bar{\mu} = 0$, that is $v_N^s = 0$, from Eq (4.7) we obtain

$$\begin{bmatrix} \dot{T}_i \\ \dot{N} \end{bmatrix} = \begin{bmatrix} K_T\delta_{ik} - K_T\dfrac{T_iT_k}{\|\boldsymbol{T}\|^2} & \mu K_N\dfrac{T_i}{\|\boldsymbol{T}\|} \\ 0 & K_N \end{bmatrix} \begin{bmatrix} v_k \\ v_N \end{bmatrix} \tag{4.9}$$

Consider the second order work rate

$$\dot{L} = \dot{T}_iv_i + \dot{N}v_N = \frac{\mu\bar{\mu}K_NK_T}{H}v^2 + \frac{K_NK_T}{H}(\mu+\bar{\mu})vv_N + \frac{K_TK_N}{H}v_N^2 > 0 \tag{4.10}$$

where $v = \sqrt{v_iv_i} = \|\boldsymbol{v}\|$. Requiring $\dot{L}$ to be positive definite, we obtain the *static stability condition*

$$S = \mu\bar{\mu}v^2 + (\mu+\bar{\mu})vv_N + v_N^2 > 0 \tag{4.11}$$

When $\bar{\mu} = 0$ we obtain

$$v_N(v_N + \mu v) > 0 \tag{4.12}$$

On the other hand, when $\mu = \bar{\mu}$, Eq (4.10) provides

$$(v_N + \mu v)^2 > 0 \tag{4.13}$$

i.e., the displacement controlled deformation is stable for the associated sliding rule.

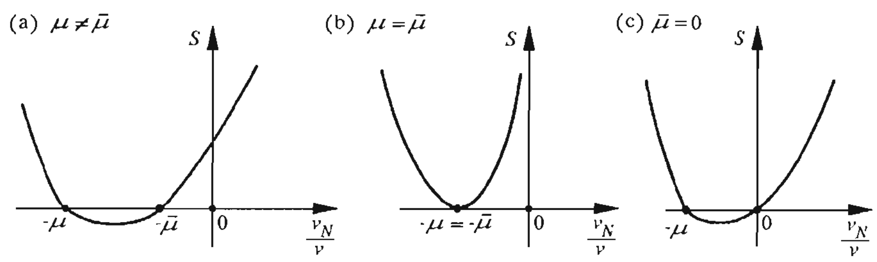

Fig. 7. Stability diagram: (a) non-associated sliding rule, (b) associated sliding rule, (c) vanishing dilatancy effect

Figure 7 presents the diagram of S specified by Eq (4.11) in the plane $S, v_N/v$. It is seen that $S > 0$ when $v_N/v > -\mu$ or $v_N/v < -\bar{\mu}$. For the case of associated slip rule the contact response is stable. When only tangential slip occurs, $\bar{\mu} = 0$, the stable response occurs for $v_N/v > 0$ and $S > 0$ occurs also for $v_N/v < -\mu$.

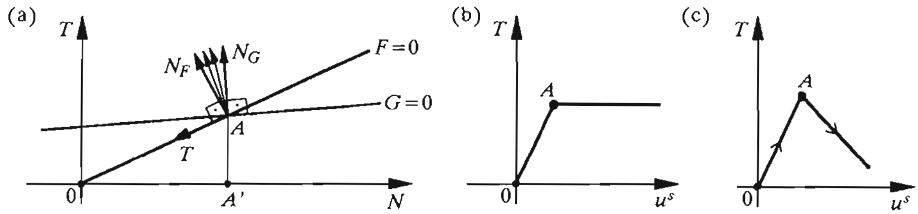

Fig. 8. Illustration of critical states: (a) friction condition and slip potential, (b) sliding at constant T and N, (c) sliding at decreasing T and N

There is a clear physical interpretation of critical points satisfying the condition $S = 0$. The value of S is negative when the velocity vector v_N, v lies within the angular domain bounded by the normals to the limit friction surface $F = 0$ and the sliding potential $G = 0$, Fig.8.

Consider first the situation when $v_N/v = -\overline{\mu}$ and the velocity vector is collinear with N_G. Since the slip vector satisfies the same relation $v_N^s/v^s < -\overline{\mu}$, the elastic strain rate and hence the force rate should vanish. The sliding occurs at constant T and N. On the other hand, when $v_N/v = -\mu$, so the velocity coincides with N_F, the progressive sliding occurs under decreasing T and N. In fact, we have

$$\dot{L} = \dot{T}_i v_i + \dot{T}_N v_N = \frac{\dot{T}_i \dot{T}_i}{K_T} + \frac{\dot{N}^2}{K_N} + \dot{T}_i v_i^s \dot{N} v_N^s = 0 \tag{4.14}$$

Using the consistency condition and the slip rule, Eq (4.14) can be expressed as follows

$$\dot{L} = \frac{\dot{T}^2}{K_T} + \frac{\dot{N}^2}{K_N} + \dot{N} v^s (\mu - \overline{\mu}) = 0 \tag{4.15}$$

Fig.9 presents the domains $\dot{L} < 0$ and $\dot{L} > 0$ in the $\dot{N}\dot{T}$-plane.

Consider now the general stability condition (1.2). The potential energy under displacement control is

$$U = \frac{1}{2}\Big(\frac{u_N^{e^2}}{K_N} + \frac{u_T^{e^2}}{K_T}\Big) = \frac{1}{2}\Big[\frac{(u_N - u_N^s)^2}{K_N} + \frac{\sum_i (u_{T_i} - u_{T_i}^s)^2}{K_T}\Big] \tag{4.16}$$

and the dissipation function is expressed as follows

$$D = T_i v_i^s + N v_N^s = \dot{\lambda}\frac{T_i T_i}{|T|} - N\dot{\lambda}\overline{\mu} = |v^s|(|T| - N\overline{\mu}) = |v^s|(\mu - \overline{\mu})N \tag{4.17}$$

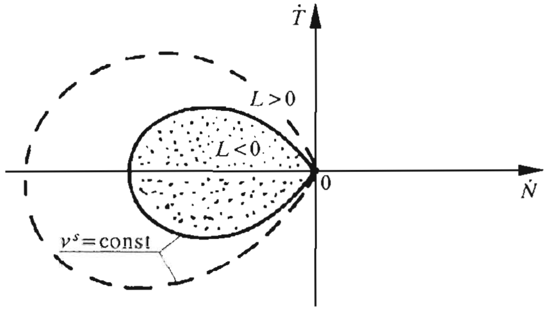

Fig. 9. Instability domain in the $\dot{N}\dot{T}$-plane

The dissipative forces generated by the potential energy are

$$N_d^u = \frac{u_N - u_N^s}{K_N} \qquad\qquad T_{di}^u = \frac{u_{T_i} - u_{T_i}^s}{K_T} \tag{4.18}$$

and the dissipative forces generated by the dissipation function are

$$T_{di}^D = \frac{v_i^s}{|v^s|}(\mu - \overline{\mu})N \tag{4.19}$$

The stability condition can be expressed as follows

$$\dot{N}|v^s|(\mu - \overline{\mu}) \geqslant -\Big[\frac{(v_N^s)^2}{K_N} + \frac{(v^s)^2}{K_T}\Big] = -\Big(\frac{\overline{\mu}^2}{K_N} + \frac{1}{K_T}\Big)(v^s)^2 \tag{4.20}$$

or taking derivatives with respect to tangential translation, there is

$$\dot{N}(\mu - \overline{\mu}) \geqslant -\Big(\frac{\overline{\mu}^2}{K_N} + \frac{1}{K_T}\Big) \tag{4.21}$$

4.1. Dynamic contact response

Consider now the dynamic motion of the block of mass M. The equations of motion are

$$\begin{aligned} \dot{T}_i &= M\ddot{v}_i + [D]_{ij}v_j \\ \dot{N} &= M\ddot{v}_N + [D]_n v_N \end{aligned} \tag{4.22}$$

Considering the perturbation of motion $\boldsymbol{v}+\dot{\boldsymbol{z}}$, the same equations as Eq (4.22) are obtained. Assume the solution of the potential system in the form

$$\dot{z}_i = a_i \mathrm{e}^{\lambda t} \qquad\qquad \dot{z}_n = a_N \mathrm{e}^{\lambda t} \tag{4.23}$$

The following eigenvalue problem is thus obtained

$$\begin{bmatrix} K_T\delta_{ik} - K_T^2 \dfrac{T_iT_k}{\|\boldsymbol{T}\|^2 H} & \mu \dfrac{K_TK_N}{H} \dfrac{T_i}{\|\boldsymbol{T}\|} \\ \overline{\mu}\dfrac{K_TK_N}{H} \dfrac{T_k}{\|\boldsymbol{T}\|} & K_N - \dfrac{K_N^2}{H}\mu\overline{\mu} + M\lambda^2 \end{bmatrix} \begin{bmatrix} a_k \\ a_N \end{bmatrix} = 0 \tag{4.24}$$

requiring the determinant of the non-symmetric matrix to vanish.

Consider the motion within the plane x_1n, so that $v_2 = z_2 = 0$. Eq (4.24) now becomes

$$\begin{bmatrix} K_T - \dfrac{K_T^2}{H} + M\lambda^2 & \mu\dfrac{K_TK_N}{H} \\ \overline{\mu}\dfrac{K_TK_N}{H} & K_N - \dfrac{K_N^2}{H}\mu\overline{\mu} + M\lambda^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_N \end{bmatrix} = 0 \tag{4.25}$$

and we have

$$\Big(K_T - \frac{K_T^2}{H} + M\lambda^2\Big)\Big(K_N - \frac{K_N^2}{H}\mu\overline{\mu} + M\lambda^2\Big) - \mu\overline{\mu}\frac{K_T^2K_N^2}{H^2} = 0 \tag{4.26}$$

which provides the quadratic equation

$$M^2x^2 + Mx\Big[K_T^2\Big(1 - \frac{K_T}{H}\Big) + K_N\Big(1 - \frac{K_N}{H}\mu\overline{\mu}\Big) + K_TK_N\Big(1 - \frac{K_T}{H} - \frac{K_N}{H}\mu\overline{\mu}\Big) = 0 \tag{4.27}$$

where $x = \lambda^2$. The discrimant of this equation equals

$$\begin{aligned} \varDelta = M^2\Big[K_T^2\Big(1 - \frac{K_T}{H}\Big)^2 + K_N\Big(1 - \frac{K_N}{H}\mu\overline{\mu}\Big)^2 + \\ +2K_TK_N\Big(1 - \frac{K_T}{H}\Big)\Big(1 - \frac{K_N}{H}\mu\overline{\mu}\Big)\Big] - 4M^2K_TK_N\Big(1 - \frac{K_T}{H} - \frac{K_N}{H}\mu\overline{\mu}\Big) \end{aligned} \tag{4.28}$$

and

$$\lambda_{1,2}^2 = x = \frac{-M\Big[K_T\Big(1 - \frac{K_T}{H}\Big) + K_N\Big(1 - \frac{K_N}{H}\mu\overline{\mu}\Big)\Big] \pm \sqrt{\varDelta}}{2M^2} \tag{4.29}$$

Thus, the eigenvalues can be either real when $\varDelta > 0$ or complex conjugate, when $\varDelta < 0$. In the latter case the flutter vibrations may develop. In the particular case, when $\overline{\mu} = 0$, we have

$$\lambda_1^2 = -\frac{K_T\Big(1 - \frac{K_T}{H}\Big)}{M} \qquad \lambda_2^2 = -\frac{K_N}{M} > 0 \tag{4.30}$$

When $K_T < H$, then λ_1 is imaginary and the system undergoes harmonic vibrations, when $K_T > H$ there is *divergence* in the system in the tangent direction.

5. Concluding remarks

The present paper provides the analysis of contact stability using the general conditions (1.2). The effect of dilatancy and elastic compliance is clarified and dynamic response is briefly outlined, indicating the possibility of flutter type instability. The analysis pertains to simple cases and is aimed at clarification of instability modes. More complex cases of stability of elastic systems with frictional interaction will be discussed separately.

Acknowledgments
The research has been supported by the State Committee for Scientific Research (KBN), grant No. 7 T07 A01514.

References

1. MRÓZ Z., PLAUT R.H., 1992, On the Stability and Post-Critical Behaviour of Elastic Structures with Dry Friction, *Int. J. Solids Strut.*, **29**, 1241-1253
2. JARZĘBOWSKI A., MRÓZ Z., 1994, On Slip and Memory Rules in Elastic, Friction Contact Problems, *Acta Mech.*, **102**, 199-216
3. MRÓZ Z., GIAMBANCO G., 1996, An Interface Model for Analysis of Deformation Behaviour of Discontinuities, *Int. J. Num. Anal. Meth. Geom.*, **20**, 1-33
4. NGUYEN Q.S., 1992, Bifurcation and Stability in Dissipative Media (Plasticity, Friction, Fracture), *ASME Appl. Mech. Revs.*, **47**, 1, 1241-1253
5. ŻYCZKOWSKI M., 1988, Stability of Truss Systems, in *Strength of Structural Elements*, Edit. M. Życzkowski, PWN

O stateczności kontaktu z tarciem

Streszczenie

Warunek (1.2) stateczności układu sprężystego z ciernym kontaktem wyrażony jest przez funkcję energii potencjalnej $V(q_i)$ i funkcję dysypacji $D(\dot{q}_i, q_k)$. Rozpatrzono prawo tarcia Coulomba i niestowarzyszone prawo poślizgu i zastosowano warunek stateczności do kontaktu sztywnego i z podatnością sprężystą. Określono statyczne i dynamiczne formy utraty stateczności.

Manuscript received October 27, 1999; accepted for print November 29, 1999