Research Paper

Struct Multidisc Optim 29, 1–18 (2005)
DOI 10.1007/s00158-004-0474-0

Optimal loading conditions in the design and identification of structures. Part 1: discrete formulation

Z. Mróz and A. Garstecki

Abstract The paper is concerned with a class of structural optimization problems for which loading distribution and orientation are unspecified. The optimal loading conditions correspond to the extremal structural response, which can be used in assessment of structural safety or in generating the maximum structure stiffness or compliance. In identification problems the optimal load distribution is selected in order to minimize the distance norm between model prediction and experimental data. The sensitivity derivatives and optimality conditions are derived in the paper using discretized formulations. The generalized coaxiality conditions of loading and displacement or adjoint displacement vectors generate eigenvalue problems specifying stationary solutions. The paper is illustrated by examples of optimal loading distribution in structure design and identification.

Key words optimization, sensitivity analysis, structural identification, variable loading conditions

1 Introduction

Problems of optimal loading distribution are faced frequently in engineering practice. For instance, when multiple loads occur, the worst loading case is selected in assessing lower bound on the structure failure factor. On the other hand, when the load can be controlled, it is selected in order to maximize the structure safety. For an elastic structure, the extremal loading distribution may correspond either to the minimum structure compliance (maximum stiffness) or to the maximum compliance. For a rigid, perfectly plastic structure, the critical load factor corresponding to onset of failure may be maximized with respect to loading distribution. In structural design problems this factor is usually maximized. However, in technological problems such as metal forming, the optimal load distribution should correspond to the lowest load factor associated with the induced deformation mode. In solving identification problems, the distance norm between experimental data and model prediction is minimized with respect to material parameters but it can be maximized with respect to load distribution in order to increase the discrepancy between the model and the actual structure responses.

The problems of optimal loading control discussed in the present paper are closely related to the optimal design of structure supports. Problems of this type were first formulated in terms of structural optimization by Mróz and Rozvany (1975), who studied the support location providing maximum stiffness of elastic beams or maximum limit loads of rigid plastic beams. A little later, papers by Rozvany and Mróz (1975, 1977) presented generalized formulation, where the objective function was assumed to be the cost of supports in beams and columns, respectively. Optimal supporting conditions in frame structures were discussed by Szeląg and Mróz (1978) and recently were generalized by Bojczuk and Mróz (1998) by consideration of support and joint conditions as topological variables. Optimality conditions for elastic supports were first derived by Szelàg and Mróz (1979). Åkesson and Olhoff (1988) and Olhoff and Åkesson (1991) studied the problems of minimal stiffness of optimally located supports providing maximal eigenfrequencies in beam structures and maximum values of buckling loads in columns, respectively.

The problem of optimal load distribution was first formulated by Mróz and Garstecki (1976) and general theorems were provided by Mróz (1980). The optimality conditions derived in these papers ensured the best structural response. Lombardi and Haftka (1998) presented the opposite formulation, called antioptimization, where the aim was to find the worst loading conditions with the

Received: 2 February 2004
Revised manuscript received: 19 May 2004
Published online: 30 September 2004

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application to design of composite structures. This idea was further developed by Cherkaev and Cherkaev (1999a, 1999b) and Cherkaev and Kucuk (1999). Optimal load distribution in identification of structural models was discussed by Gangadharan *et al.* (1999), where elastic energy was assumed as the distance measure. Lee *et al.* (1994) took up the problem of detecting delamination using antioptimization of loading. A general review of optimization in identification problems and design of experiments was provided by Haftka *et al.* (1998). The application of genetic algorithms to design for the worst loading condition was discussed by Venter and Haftka (2000). Maximization of different distance norms with respect to loading parameters was discussed in this study. Mróz and Garstecki (1997) proposed simultaneous minimization of the distance norm with respect to model parameters and maximization with respect to load parameters. Dems and Mróz (2001) proposed an enhanced structural model for damage identification by introducing variable support or concentrated mass positions in dynamic tests.

In this paper we shall present the uniform formulation of the optimal loading problem with the purpose to provide application to various design and identification problems. The previous results will be generalized and the examples will illustrate the applicability of the derived optimality conditions.

2 Extremal force action on a structure

2.1 Extrema of potential or complementary energy

Consider a discrete model of a structure rigidly supported and loaded on its boundary

$$\mathbf{Ku} - \mathbf{f} = \mathbf{0}\,, \tag{1}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the vector of nodal displacements and $\mathbf{f}$ is the vector of nodal forces. The global potential energy is

$$\Pi(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{Ku} - \mathbf{f}^T\mathbf{u} = -\frac{1}{2}\mathbf{u}^T\mathbf{Ku}\,, \tag{2}$$

and the complementary energy is expressed in terms of the force vector $\mathbf{f}$, namely

$$\tilde{\Pi}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{Df}\,, \tag{3}$$

where $\mathbf{D} = \mathbf{K}^{-1}$ is the global compliance matrix and

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{f} = \mathbf{Df}\,. \tag{4}$$

The global structure compliance C can be assumed to be equivalent to the complementary energy, thus

$$\mathrm{C} = \tilde{\Pi}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{u} = \frac{1}{2}\mathbf{f}^T\mathbf{Df}\,, \tag{5}$$

and the global stiffness S is assumed as equivalent to the potential energy, so that

$$\mathrm{S} = -\mathrm{C} = \Pi(\mathbf{u},\mathbf{f}) = -\frac{1}{2}\mathbf{u}^T\mathbf{Ku}\,. \tag{6}$$

The optimal loading problem can be stated as follows:

$$\begin{aligned}&\text{maximize global stiffness} && \max \Pi(\mathbf{u},\mathbf{f}) = \frac{1}{2}\mathbf{u}^T\mathbf{Ku} - \mathbf{f}^T\mathbf{u}\\ &\text{subject to} && \mathbf{f}^T\mathbf{f} - \rho_0^2 \geq 0\,,\end{aligned} \tag{7}$$

or

$$\begin{aligned}&\text{minimize global compliance} && \min \mathrm{C}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{Df}\\ &\text{subject to} && \mathbf{f}^T\mathbf{f} - \rho_0^2 \geq 0\,.\end{aligned} \tag{8}$$

The admissible load vector $\mathbf{f}$ is shown in Fig. 1.

The Lagrange function associated with (7) is

$$\Pi^{\mathrm{L}}(\mathbf{u},\mathbf{f},\eta^2) = \frac{1}{2}\mathbf{u}^T\mathbf{Ku} - \mathbf{f}^T\mathbf{u} + \frac{1}{2}\eta^2\left(\mathbf{f}^T\mathbf{f} - \rho_0^2\right), \tag{9}$$

and its variation

$$\delta\Pi^{\mathrm{L}} = \delta\mathbf{u}^T(\mathbf{Ku} - \mathbf{f}) - \delta\mathbf{f}^T(\mathbf{u} - \eta^2\mathbf{f}) = 0\,. \tag{10}$$

As the first term of (10) vanishes, the stationarity condition is

$$\mathbf{u} = \eta^2\mathbf{f}\,, \tag{11}$$

where η^2 is the positive Lagrange multiplier. The condition (11) provides the *coaxiality rule* between optimal load and displacement vectors (Fig. 2(i)). This rule can also be expressed as the *eigenvalue problem*

$$\mathbf{Ku} = \frac{1}{\eta^2}\mathbf{u}\,, \tag{12}$$

or

$$\mathbf{Df} = \eta^2\mathbf{f} \tag{13}$$

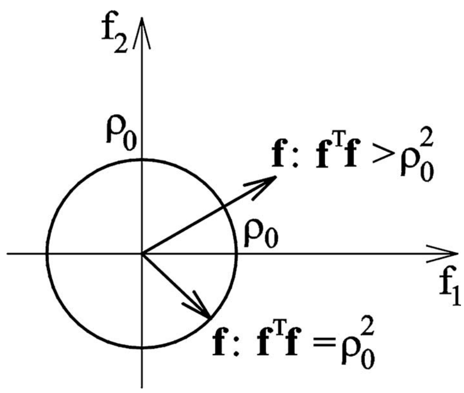

Fig. 1 Graphical illustration of the constraint $\mathbf{f}^T\mathbf{f} \geq \rho_0^2$

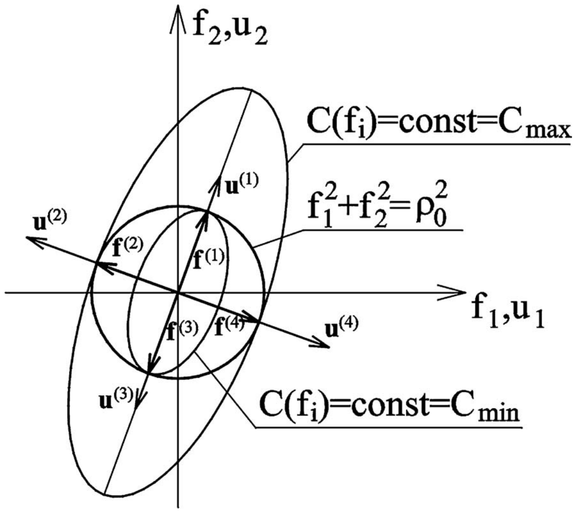

Fig. 2 Four load vectors $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \mathbf{f}^{(4)}$ satisfying the stationarity condition (11). (i) Vectors $\mathbf{f}^{(1)}, \mathbf{f}^{(3)}$ provide $\min \mathrm{C} = \max \Pi$, vectors $\mathbf{f}^{(2)}, \mathbf{f}^{(4)}$ provide $\max \mathrm{C} = \min \Pi$, for prescribed ρ_0. (ii) Vectors $\mathbf{f}^{(1)}, \mathbf{f}^{(3)}$ provide $\min \rho_0$, vectors $\mathbf{f}^{(2)}, \mathbf{f}^{(4)}$ provide $\max \rho_0$, for prescribed compliance C_0

expressed in terms of nodal displacement or force vectors. The same optimality condition is obtained when the variation of (8) is considered, thus

$$\mathrm{C}^{\mathrm{L}} = \tilde{\Pi}^{\mathrm{L}}\left(\mathbf{f}, \eta^2\right) = \frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f} - \frac{1}{2}\eta^2\left(\mathbf{f}^T\mathbf{f} - \rho_0^2\right), \tag{14}$$

and

$$\delta\tilde{\Pi}^{\mathrm{L}} = \delta\mathbf{f}^T\left(\mathbf{D}\mathbf{f} - \eta^2\mathbf{f}\right) = 0, \tag{15}$$

provides the optimality condition (13). Let us note that the force constraint $(8)^2$ provides the Lagrangian multiplier, thus

$$\left(\mathbf{u}^T\mathbf{u}\right)^{\frac{1}{2}} = \|\mathbf{u}\| = \eta^2\rho_0 \tag{16}$$

and

$$\eta^2 = \frac{\|\mathbf{u}\|}{\rho_0}. \tag{17}$$

Alternatively to (7) or (8) the problem can be formulated as:

minimize the load $\min \rho_0^2 = \mathbf{f}^T\mathbf{f}$

subject to $\quad \mathrm{C}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f} \geq \mathrm{C}_0, \quad (18)$

or

maximize the load $\min \rho_0^2 = \mathbf{f}^T\mathbf{f}$

subject to $\quad \mathrm{C}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f} \leq \mathrm{C}_0. \quad (19)$

Using (18) or (19) we again arrive at optimality conditions (11) and (13). The solution is illustrated in Fig. 2(ii).

Let us note that (12) or (13) are identical to free vibration equations with the mass matrix $\mathbf{M} = \mathbf{1}$. Thus, there is a set of conjugate eigenvalues $\frac{1}{\eta_1^2}, \frac{1}{\eta_2^2}, \ldots, \frac{1}{\eta_n^2}$. The lowest value of η_1 now corresponds to the lowest structure compliance.

The problem can easily be generalized by assuming the constraint $(7)^2$ in the form

$$\mathbf{f}^T\mathbf{M}\mathbf{f} - \rho_0^2 \geq 0 \tag{20}$$

and instead of (12) and (13) the resulting eigenvalue problems are

$$\mathbf{K}\mathbf{u} - \frac{1}{\eta^2}\mathbf{M}^{-1}\mathbf{u} = \mathbf{0}, \quad \mathbf{D}\mathbf{f} - \eta^2\mathbf{M}\mathbf{f} = \mathbf{0}, \tag{21}$$

where $\mathbf{M} = \mathbf{M}^T$ is the weighting matrix, specifying the relative significance of particular load components. This generalized formulation was recently discussed by Cherkaev and Cherkaev (1999a,b).

Let us note that when the constraint $(8)^2$ is expressed in terms of the norm of $\mathbf{f}$, thus

$$\|\mathbf{f}\| = -\rho_0 \geq 0 \tag{22}$$

and the optimality conditions provide the coaxiality relation

$$\mathbf{u} = \mathbf{D}\mathbf{f} = \eta^2\frac{\mathbf{f}}{\|\mathbf{f}\|}, \tag{23}$$

or

$$\frac{\mathbf{K}\mathbf{u}}{\|\mathbf{K}\mathbf{u}\|} = \frac{1}{\eta^2}\mathbf{u}. \tag{24}$$

This provides the formulation of the nonlinear eigenvalue problem for the vector $\mathbf{f}$.

2.2
Load superposed on constant loading

In many practical design problems, or in planning experiments for structural identification, a structure is subjected to a constant (dead) loading and the next load is superposed on it. The designer's concern is to find the best or the worst superposed load configuration, which is associated with an extremal structural response.

Consider a discrete model of a structure (1). The problem of optimal load can be formulated similarly to (7) or (8), but now the constraint takes the form

$$\mathrm{G}(\mathbf{f}) = (\mathbf{f} - \mathbf{f}_0)^T(\mathbf{f} - \mathbf{f}_0) - \rho_0^2 = 0, \tag{25}$$

where $\mathbf{f}_0$ is a vector representing the dead load (Fig. 3). Then, we have the following Lagrange function and its variation

$$\Pi^{\mathrm{L}} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{u} + \frac{1}{2}\eta\left((\mathbf{f} - \mathbf{f}_0)^T(\mathbf{f} - \mathbf{f}_0) - \rho_0^2\right), \tag{26}$$

$$\delta\Pi^{\mathrm{L}} = \left(\mathbf{u}^T\mathbf{K} - \mathbf{f}^T\right)\delta\mathbf{u} - \delta\mathbf{f}^T\mathbf{u} + \eta\delta\mathbf{f}^T(\mathbf{f} - \mathbf{f}_0) =$$
$$\delta\mathbf{f}^T(-\mathbf{u} + \eta(\mathbf{f} - \mathbf{f}_0)) = 0. \tag{27}$$

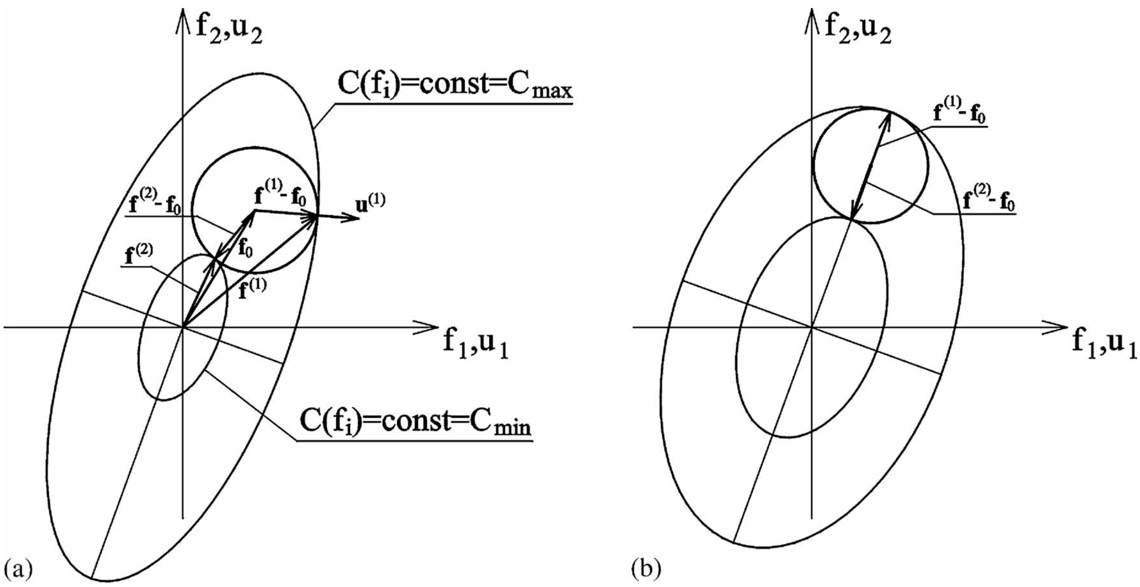

Fig. 3 Load superposed on constant loading $\mathbf{f}_0$. (a) Illustration of two optimal solutions. (b) Two coaxial optimal solutions

Hence we arrive again at the stationarity condition in the form of coaxiality rule

$$\mathbf{u} = \eta(\mathbf{f} - \mathbf{f}_0)\,. \tag{28}$$

Substituting (4) for $\mathbf{u}$ in (28) we have

$$\mathbf{D}\mathbf{f} = \eta\mathbf{f} - \eta\mathbf{f}_0 \tag{29}$$

$$(\mathbf{D} - \eta\mathbf{1})\mathbf{f} = -\eta\mathbf{f}_0 \tag{30}$$

and an alternative solution to (28) for the optimal load vector $\mathbf{f}$ is

$$\mathbf{f} = -\eta[\mathbf{D} - \eta\mathbf{1}]^{-1}\mathbf{f}_0\,. \tag{31}$$

Let us note that there are two solutions for η, one corresponding to minimal, the other to maximal compliance (Fig. 3). Substituting (31) into the constraint (25), we obtain the equation specifying η

$$\eta^2[(\mathbf{D} - \eta\mathbf{1})^{-1} - 1]^T[(\mathbf{D} - \eta\mathbf{1})^{-1} - 1]\mathbf{f}_0^T\mathbf{f}_0 = \rho_0^2\,. \tag{32}$$

The constraint (25) can be expressed in the equivalent form

$$\mathrm{G}(\mathbf{f}) = \left[(\mathbf{f} - \mathbf{f}_0)^T(\mathbf{f} - \mathbf{f}_0)\right]^{\frac{1}{2}} - \rho_0 = 0 \quad \text{or}$$

$$\mathrm{G}(\mathbf{f}) = \|\mathbf{f} - \mathbf{f}_0\| - \rho_0 = 0\,. \tag{33}$$

Then, we arrive at the following Lagrange function and stationarity condition

$$\Pi^{\mathrm{L}} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{u} + \eta(\|\mathbf{f} - \mathbf{f}_0\| - \rho_0)\,, \quad \rho_0 > 0 \tag{34}$$

$$\delta\Pi^{\mathrm{L}} = -\delta\mathbf{f}^T\mathbf{u} + \eta\delta\mathbf{f}^T\frac{\mathbf{f} - \mathbf{f}_0}{\|\mathbf{f} - \mathbf{f}_0\|} = 0\,. \tag{35}$$

The stationarity condition (35) again indicates that extremal live load $\mathbf{f} - \mathbf{f}_0$ is coaxial with the total displacement $\mathbf{u}$, namely

$$\mathbf{u} = \eta\frac{\mathbf{f} - \mathbf{f}_0}{\|\mathbf{f} - \mathbf{f}_0\|}\,. \tag{36}$$

Here we do not provide a general solution to the problem of optimal load. However, when we modify the constraint (25) by introduction the compliance matrix $\mathbf{D}$ as the weighting matrix

$$\mathrm{G}(\mathbf{f}) = (\mathbf{f} - \mathbf{f}_0)^T\mathbf{D}(\mathbf{f} - \mathbf{f}_0) - \rho_0^2 = 0\,, \tag{37}$$

then

$$\Pi^{\mathrm{L}} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{f}^T\mathbf{u} + \frac{1}{2}\eta\left((\mathbf{f} - \mathbf{f}_0)^T\mathbf{D}(\mathbf{f} - \mathbf{f}_0) - \rho_0^2\right) \tag{38}$$

$$\delta\Pi^{\mathrm{L}} = \left(\mathbf{u}^T\mathbf{K} - \mathbf{f}^T\right)\delta\mathbf{u} - \delta\mathbf{f}^T\mathbf{u} + \eta\delta\mathbf{f}^T\mathbf{D}(\mathbf{f} - \mathbf{f}_0) =$$

$$\delta\mathbf{f}^T(-\mathbf{u} + \eta\mathbf{D}(\mathbf{f} - \mathbf{f}_0)) = 0 \tag{39}$$

and the optimality condition has the form

$$\mathbf{u} = \eta\mathbf{D}(\mathbf{f} - \mathbf{f}_0)\,, \text{ or } \mathbf{D}\mathbf{f} = \eta\mathbf{D}(\mathbf{f} - \mathbf{f}_0)\,, \text{ or } \mathbf{u} = \eta(\mathbf{u} - \mathbf{u}_0)\,. \tag{40}$$

$$\mathbf{D}\mathbf{f} = \eta\mathbf{D}\mathbf{f} - \eta\mathbf{D}\mathbf{f}_0 \quad \text{ or } \mathbf{D}\mathbf{f}(1 - \eta) = -\eta\mathbf{D}\mathbf{f}_0 \tag{41}$$

and

$$\mathbf{f} = -\frac{\eta}{1 - \eta}\mathbf{f}_0\,, \quad \mathbf{f} - \mathbf{f}_0 = \frac{1}{1 - \eta}\mathbf{f}_0\,, \tag{42}$$

$$\frac{1}{(1 - \eta)^2}\mathbf{f}_0^T\mathbf{D}\mathbf{f}_0 = \rho_0^2\,. \tag{43}$$

Optimality condition (43) can be expressed by energy norm $\left\|\tilde{\mathbf{f}}_0\right\|$

$$\frac{1}{1-\eta} = \pm\frac{\rho_0}{\left\|\tilde{\mathbf{f}}_0\right\|}, \quad 1-\eta = \pm\frac{\left\|\tilde{\mathbf{f}}_0\right\|}{\rho_0}, \quad \left\|\tilde{\mathbf{f}}_0\right\| = \mathbf{f}_0^T\mathbf{D}\mathbf{f}_0 \quad (44)$$

and

$$\eta = 1 \mp \frac{\left\|\tilde{\mathbf{f}}_0\right\|}{\rho_0}. \quad (45)$$

2.3 Linear combination of load sets

The dimension of the considered load vector $\mathbf{f}$ was equivalent to the dimensions of the discrete model (1) of a structure. Usually these dimensions are large, whereas the number of independent parameters by which the load can be controlled is much smaller.

Consider a linear combination of specified load sets $\mathbf{f}_\mathrm{i}^0, \mathrm{i} = 1, 2, \ldots, \mathrm{m}$

$$\mathbf{f} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mathbf{f}_\mathrm{i}^0 \quad (46)$$

and a quadratic constraint

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - (\mu^0)^2 \geq 0\,. \quad (47)$$

The problem of extremal load combination can be formulated as follows: find the optimal load multipliers μ_i that satisfy (47) and provide minimum or maximum compliance C of the structure modeled by (5). The respective Lagrange function and its variation are

$$\mathrm{C^L}\left(\mu_\mathrm{i}, \eta^2\right) = \frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f} - \frac{1}{2}\eta^2\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - \left(\mu^0\right)^2\right) \quad (48)$$

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \left(\mathbf{f}_\mathrm{i}^{0\,T}\mathbf{D}\mathbf{f} - \eta^2\mu_\mathrm{i}\right)\delta\mu_\mathrm{i} - \eta\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - \left(\mu^0\right)^2\right)\delta\eta = 0\,. \quad (49)$$

The Kuhn–Tucker stationarity conditions are

$$\mathbf{f}_\mathrm{i}^{0\,T}\mathbf{D}\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathbf{f}_\mathrm{j}^0\mu_\mathrm{j} - \eta^2\mu_\mathrm{i} = 0 \quad \forall\,\mathrm{i} \quad (50a)$$

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - \left(\mu^0\right)^2 \geq 0 \quad (50b)$$

$$\eta\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - \left(\mu^0\right)^2\right) = 0\,. \quad (50c)$$

The switching condition (50c) provides two solutions. The first one, $\eta = 0$, leads to a trivial solution $\mu_\mathrm{i} = 0$, therefore the equality sign must appear in (50b). The stationarity conditions take the form

$$\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathbf{D}_\mathrm{ij}^0\mu_\mathrm{j} - \eta^2\mu_\mathrm{i} = 0 \quad \forall\,\mathrm{i} \quad (51a)$$

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mu_\mathrm{i}\mu_\mathrm{i} - \left(\mu^0\right)^2 = 0 \quad (51b)$$

where

$$\mathbf{D}_\mathrm{ij}^0 = \mathbf{f}_\mathrm{i}^{0\,T}\mathbf{D}\mathbf{f}_\mathrm{j}^0\,. \quad (52)$$

Note that (51a) is the equation for a linear eigenvalue problem

$$(\mathbf{D}^0 - \eta^2\mathbf{1})\boldsymbol{\mu} = \mathbf{0}\,. \quad (53)$$

Here $\mathbf{D}^0$ is a linear, positive definite operator represented by a quadratic matrix with dimensions $\mathrm{m} \times \mathrm{m}$. Note that the matrices $\mathbf{K}$ and $\mathbf{D}$ had larger dimensions namely $\mathrm{n} \times \mathrm{n}$, where n represents the number of degrees of freedom of the discretized structure. Thus, by definition (52) we have contracted our problem from dimensions $\mathrm{n} \times \mathrm{n}$ to $\mathrm{m} \times \mathrm{m}$. The positive definiteness of $\mathbf{D}^0$ results directly from (52) since $\boldsymbol{\mu}^T\mathbf{D}^0\boldsymbol{\mu} = \mathbf{f}^T\mathbf{D}\mathbf{f} = \mathbf{f}^T\mathbf{u} = 2\mathrm{C} > 0$ for all $\boldsymbol{\mu} \neq \mathbf{0}$. Hence the eigenvalue problem (53) has m real solutions and the eigenvectors $\boldsymbol{\mu}$ are mutually orthogonal. Figure 4 shows the extremal load vector $\boldsymbol{\mu}^{(1)}$ as the solution of the formulated above problem of minimum compliance C.

In structural identification or in structural design the problem can be formulated alternatively: find the worst load combination that provides maximum compliance C from the set of admissible load combinations that satisfy the constraint

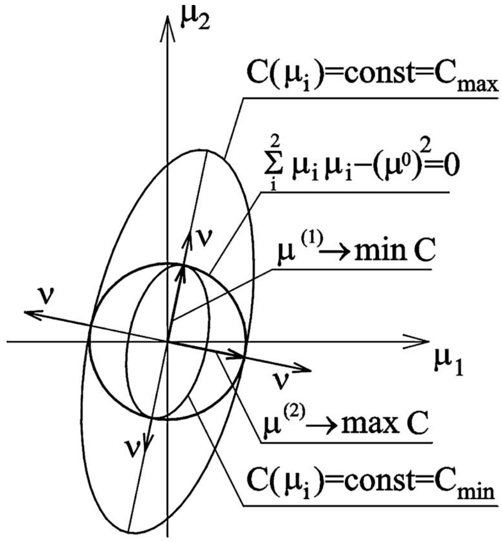

Fig. 4 Optimal generalized loads $\boldsymbol{\mu}$ providing extrema of C (the dimension of vectors $\boldsymbol{\mu}$ is decisively smaller than the dimension of vectors $\mathbf{f}$ illustrated in Fig. 2)

$$\sum_{i=1}^{m} \mu_i\mu_i - \left(\mu^0\right)^2 \le 0\,. \tag{54}$$

The constraint (54) can be illustrated by a load vector remaining within the circle $\mu_1^2+\mu_2^2=\left(\mu^0\right)^2$ shown in Fig. 4 for the case of $m=2$. The constant structural compliance measured by the complementary energy is represented in the force plane by an ellipse $\tilde{\Pi}=C(\mathbf{f})=\text{const}$. Figure 4 presents the optimal load vectors $\boldsymbol{\mu}$. The coaxiality occurs at points of tangency of the constant compliance and constant norm curves. It is seen that there are two extrema, one for minimum compliance with μ_1 and μ_2 of the same sign and the other for maximal compliance with μ_1 and μ_2 of the opposite signs. For m independent loads (or load systems) there are m solutions satisfying the optimality conditions (51) and the respective load vectors $\boldsymbol{\mu}$ are mutually orthogonal.

Since $\boldsymbol{\mu}$ is interpreted as a load vector in the space R^m, we can introduce a conjugate displacement vector $\boldsymbol{\nu}$ in the same space R^m, so that there is the equivalence of the work in the space R^n and in the subspace R^m, namely $\mathbf{f}^T\mathbf{u}=\boldsymbol{\mu}^T\boldsymbol{\nu}$. Hence, we arrive at the definition of the displacement $\boldsymbol{\nu}$

$$\boldsymbol{\nu}_i = \sum_{j=1}^{m} D^0_{ij}\mu_j\,. \tag{55}$$

Introducing (55) into (51a) we obtain the stationarity condition in the form of coaxiality rule

$$\boldsymbol{\nu}_i = \eta^2\boldsymbol{\mu}_i \quad i=1,2,\ldots,m \quad \text{or} \quad \boldsymbol{\nu}=\eta^2\boldsymbol{\mu}\,. \tag{56}$$

It should be noted, however, that when a *linear constraint on* $\boldsymbol{f}$ is imposed, so that

$$\mathbf{f}=\sum_{i=1}^{m}\mu_i\mathbf{f}_i^0 \quad \text{and} \quad \sum_{i=1}^{m}\mu_i \ge \mu^0 \tag{57}$$

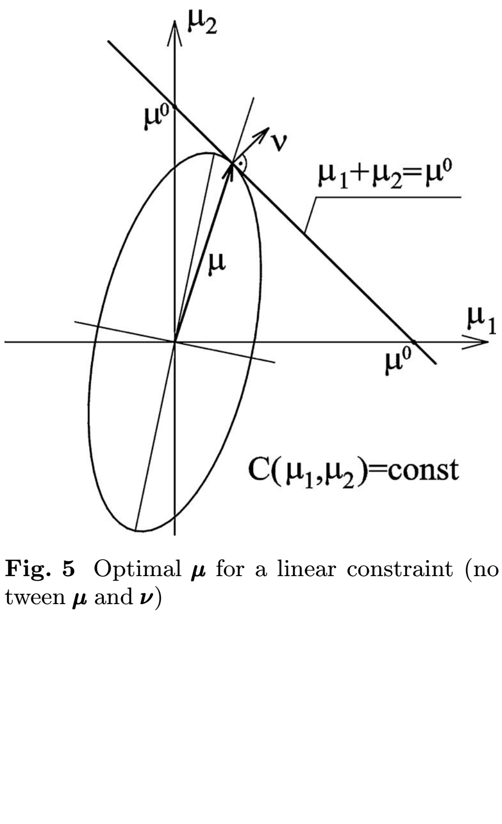

Fig. 5 Optimal $\boldsymbol{\mu}$ for a linear constraint (no coaxiality between $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$)

(Fig. 5), then minimization of C with respect to μ_i provides the Lagrange function C^L and its variation

$$C^L\left(\mu_i,\eta^2\right)=\frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f}-\eta^2\left(\sum_{i=1}^{m}\mu_i-\mu^0\right)\,. \tag{58}$$

Here $\mathbf{f}^0=[f_1^0,f_2^0,\ldots,f_m^0]^T$ is a specified load system and $\boldsymbol{\mu}=[\mu_1,\mu_2,\ldots,\mu_m]^T$ is a vector of load multipliers that can vary. The variation of (58) with respect to μ_i now equals

$$\delta C^L=\sum_{i=1}^{m}\delta\mu_i\left(\mathbf{f}_i^{0\,T}\mathbf{D}\mathbf{f}-\eta^2\right)=0\,. \tag{59}$$

Introducing (52) into (59) we arrive at the optimality condition

$$\sum_{j=1}^{m}D^0_{ij}\mu_j-\eta^2=0 \quad \forall\, i \text{ and } \sum_{i=1}^{m}\mu_i=\mu^0 \tag{60}$$

or

$$\mathbf{D}^0\boldsymbol{\mu}-\eta^2\mathbf{1}=\mathbf{0} \quad \text{and} \quad \sum_{i=1}^{m}\mu_i=\mu^0\,. \tag{61}$$

The optimal vector $\boldsymbol{\mu}$ is presented in Fig. 5. In the present case the coaxiality of the optimal load vector $\boldsymbol{\mu}$ and the respective displacement $\boldsymbol{\nu}$ does not occur.

2.4 Optimal displacement control

Consider now the case when the displacement vector $\mathbf{u}$ is specified on a structure boundary, so that the complementary energy is expressed as follows

$$C=\tilde{\Pi}(\mathbf{f},\mathbf{u})=\frac{1}{2}\mathbf{f}^T\mathbf{D}\mathbf{f}-\mathbf{u}^T\mathbf{f}\,, \tag{62}$$

where $\mathbf{f}$ is the induced load vector. Assume that the displacement control can be executed in order to deform the structure at minimal work or stored elastic energy. The displacement constraint is specified in the form

$$\mathbf{u}^T\mathbf{u}-\hat{\rho}_0^2\ge 0\,, \tag{63}$$

or

$$\|\mathbf{u}\|-\hat{\rho}_0\ge 0\,. \tag{64}$$

Maximizing the complementary energy with the constraint (63), we obtain the optimality conditions

$$\mathbf{f}=\lambda\mathbf{u} \quad \text{and } \mathbf{K}\mathbf{u}=\lambda\mathbf{u}\,, \tag{65}$$

or

$$\mathbf{f}=\lambda\frac{\mathbf{u}}{\|\mathbf{u}\|} \quad \text{and } \mathbf{K}\mathbf{u}=\lambda\frac{\mathbf{u}}{\|\mathbf{u}\|}\,, \tag{66}$$

where λ is the Lagrange multiplier specified from the constraint condition $\mathbf{f}^T\mathbf{f}=\lambda^2\hat{\rho}_0^2$, or $\|\mathbf{f}\|=\lambda$. It is seen that the same coaxiality conditions are obtained and the optimal load vector is characterized by a constraint norm. When the constraint (60) is applied, a nonlinear eigenvalue problem is generated, known in the mathematical literature as the Steklov problem, cf. Cherkaev, E. and Cherkaev, A. (1999a,b).

2.5
Extremum conditions of an arbitrary functional

Consider now an arbitrary response functional $\mathrm{G} = \mathrm{G}(\mathbf{u})$ whose extremum with respect to $\mathbf{f}$ is to be determined. We have the following formulation

$$\mathrm{G} = \mathrm{g}(\mathbf{u}) \to \min_{\mathbf{f}}, \quad \text{or } \mathrm{G} = \mathrm{g}(\mathbf{u}) \to \max_{\mathbf{f}} \tag{67}$$

subject to: $\mathbf{Ku} - \mathbf{f} = \mathbf{0}$

$$\mathbf{f}^T\mathbf{f} - \rho_0^2 \geq 0\,. \tag{68}$$

Introducing the Lagrange multipliers $\mathbf{u}^{\mathrm{a}}$ and η^2, the augmented functional has the form

$$\mathrm{G}^{\mathrm{L}}\left(\mathbf{u}, \mathbf{f}, \mathbf{u}^{\mathrm{a}}, \eta^2\right) = \mathrm{g}(\mathbf{u}) - (\mathbf{u}^{\mathrm{a}})^T(\mathbf{Ku} - \mathbf{f}) - \frac{1}{2}\eta^2\left(\mathbf{f}^T\mathbf{f} - \rho_0^2\right) \tag{69}$$

and the stationarity condition can be expressed as follows

$$\delta\mathrm{G}^{\mathrm{L}} = \delta\mathbf{u}^T\left(\frac{\partial \mathrm{g}}{\partial \mathbf{u}} - \mathbf{Ku}^{\mathrm{a}}\right) + \delta\mathbf{f}^T\left(\mathbf{u}^{\mathrm{a}} - \eta^2\mathbf{f}\right) = 0\,. \tag{70}$$

Assuming $\delta\mathbf{u}$ and $\delta\mathbf{f}$ as independent vectors, the necessary stationarity conditions are

$$\mathbf{Ku}^{\mathrm{a}} = \frac{\partial \mathrm{g}}{\partial \mathbf{u}} = \mathbf{f}^{\mathrm{a}}(\mathbf{f})$$

$$\mathbf{u}^{\mathrm{a}} = \eta^2\mathbf{f}\,. \tag{71}$$

The first condition (71) specifies the adjoint problem and the second provides the *coaxiality rule* between the adjoint displacement and the primary load vectors

$$\mathbf{u}^{\mathrm{a}} = \mathbf{Df}^{\mathrm{a}} = \mathbf{K}^{-1}\frac{\partial \mathrm{g}}{\partial \mathbf{u}} \tag{72}$$

and the coaxiality condition can be expressed as a *generalized eigenvalue problem*

$$\mathbf{Df}^{\mathrm{a}}(\mathbf{f}) = \eta^2\mathbf{f}\,, \quad \text{or } \mathbf{Ku} = \frac{1}{\eta^2}\mathbf{u}^{\mathrm{a}}(\mathbf{u})\,. \tag{73}$$

As $\mathbf{f}^{\mathrm{a}}$ depends on $\mathbf{u}(\mathbf{f})$ according to (71), the eigenvalue problem may in general be linear or nonlinear. It is also an implicit problem as $\mathbf{u}^{\mathrm{a}}$ is an implicit function of $\mathbf{u}$ generated by the solution of the boundary problem (71)[1].

Consider a special case when $\mathrm{g}(\mathbf{u})$ is a quadratic function, thus

$$\mathrm{g}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{Au}\,, \tag{74}$$

where $\mathbf{A}$ is a symmetric, positive definite matrix. Then

$$\mathbf{f}^{\mathrm{a}} = \frac{\partial \mathrm{g}}{\partial \mathbf{u}} = \mathbf{Au} = \mathbf{ADf} \tag{75}$$

and the corresponding eigenvalue problem (73) becomes linear and is expressed as follows

$$\mathbf{DADf} = \eta^2\mathbf{f} \quad \text{or } \mathbf{Bf} = \eta^2\mathbf{f}\,, \quad \mathbf{B} = \mathbf{DAD}\,, \tag{76}$$

$$\mathbf{DAu} = \eta^2\mathbf{Ku} \tag{77}$$

and

$$\mathrm{g}(\mathbf{u}(\mathbf{f})) = \bar{\mathrm{g}}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{Bf}\,. \tag{78}$$

Alternatively, the generalized eigenvalue problem can be expressed in terms of displacement vector, namely

$$\mathbf{Ku} = \frac{1}{\eta^2}(\mathbf{DA})\mathbf{u} = \frac{1}{\eta^2}\mathbf{Nu}\,, \quad \mathbf{N} = \mathbf{DA}\,, \tag{79}$$

where in general the matrix $\mathbf{N}$ is not symmetric.

Specifically, when $\mathbf{A} = \mathbf{I}$, $\mathrm{g}(\mathbf{u}) = (1/2)\mathbf{u}^T\mathbf{u}$, the eigenvalue problem (76) takes the form

$$\mathbf{D}^2\mathbf{f} = \eta^2\mathbf{f}\,, \quad \bar{\mathrm{g}}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{D}^2\mathbf{f} \tag{80}$$

and when $\mathbf{A} = \mathbf{K} = \mathbf{D}^{-1}$, that is, the elastic energy control is considered, we obtain

$$\mathbf{Df} = \eta^2\mathbf{f}\,, \quad \bar{\mathrm{g}}(\mathbf{f}) = \frac{1}{2}\mathbf{f}^T\mathbf{Df}\,, \tag{81}$$

which is equivalent to (13).

Figure 6 illustrates the eigenvalue problem (76). In the plane $\mathrm{f}_1, \mathrm{f}_2$, the objective function $\bar{\mathrm{g}} = \frac{1}{2}\mathbf{f}^T\mathbf{Bf} = \text{const.}$ is presented as an ellipse. The coaxiality rule requires the gradient vector of the ellipse to be coaxial with the load vector $\mathbf{f}$. This rule specifies two solutions corresponding to tangency points of the constraint line $\mathbf{f}^T\mathbf{f} - \rho_0^2 = 0$ and the ellipses $\bar{\mathrm{g}} = \mathrm{g}_1, \bar{\mathrm{g}} = \mathrm{g}_2$, where $\mathrm{g}_1, \mathrm{g}_2$ are the minimal and maximal values of $\bar{\mathrm{g}}$.

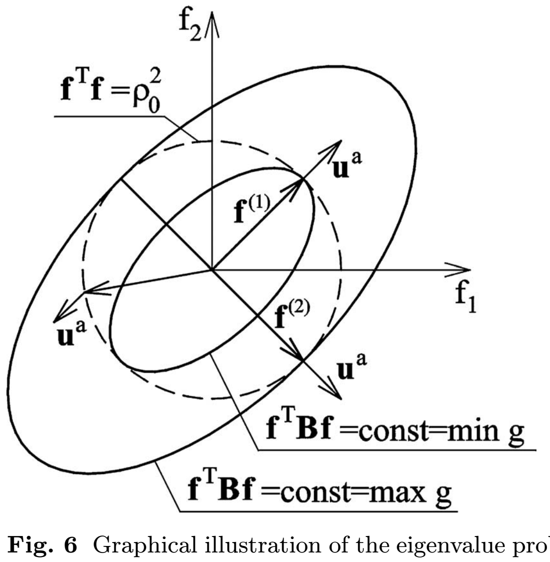

Fig. 6 Graphical illustration of the eigenvalue problem (76)

2.6 Extension to non-linear elastic structures

Now consider a non-linear elastic structure for which the potential energy is of the form

$$\Pi(\mathbf{u},\mathbf{f}) = \mathrm{U}(\mathbf{u}) - \mathbf{f}^T\mathbf{u}\,, \tag{82}$$

where $\mathrm{U}(\mathbf{u})$ is a non-linear energy function of nodal displacements generating the internal nodal forces

$$\mathbf{F} = \frac{\partial \mathrm{U}}{\partial \mathbf{u}} \tag{83}$$

satisfying the equilibrium equations

$$\frac{\partial \Pi}{\partial \mathbf{u}} = \mathbf{F} - \mathbf{f} = \mathbf{0}\,. \tag{84}$$

Note that when $\mathrm{U}(\mathbf{u})$ is a homogeneous function of degree $\mathrm{n}+1$, then we have

$$\mathbf{f}^T\mathbf{u} = \mathbf{F}^T\mathbf{u} = \left(\frac{\partial \mathrm{U}}{\partial \mathbf{u}}\right)^T \mathbf{u} = (\mathrm{n}+1)\mathrm{U}(\mathbf{u})$$

$$\Pi = \mathrm{U}(\mathbf{u}) - \mathbf{f}^T\mathbf{u} = -\mathrm{n}\mathrm{U}(\mathbf{u}) = -\frac{\mathrm{n}}{\mathrm{n}+1}\mathbf{f}^T\mathbf{u} \tag{85}$$

and the potential energy can be expressed in terms of the work of external forces on conjugate displacements. In this case the global stiffness can be represented by the potential energy Π.

The previous analysis of optimality conditions can now be easily extended to the non-linear case. Considering the minimization of the potential energy with a constraint set on the force norm, instead of (10), we now have

$$\delta\Pi^{\mathrm{L}} = \delta\mathbf{u}^T\left(\frac{\partial \Pi}{\partial \mathbf{u}}\right) - \mathrm{d}\mathbf{f}^T\left(\mathbf{u} - \eta^2\mathbf{f}\right) = 0 \tag{86}$$

and in view of (84) the first term of (86) vanishes and the coaxiality condition (11) is obtained. Similarly, considering an arbitrary displacement functional (67), instead of (71), we obtain the optimality condition in the form

$$\mathbf{K}^{\mathrm{t}}\mathbf{u}^{\mathrm{a}} = \frac{\partial \mathrm{g}}{\partial \mathbf{u}} = \mathbf{f}^{\mathrm{a}}\,, \quad \mathbf{K}^{\mathrm{t}} = \frac{\partial^2 \mathrm{U}}{\partial \mathbf{u}\partial \mathbf{u}}$$

$$\mathbf{u}^{\mathrm{a}} = \eta^2\mathbf{f} \tag{87}$$

where $\mathbf{K}^{\mathrm{t}}$ is the tangent stiffness matrix.

3 Optimal load vector for identification problem

Assume the same structure to be described by two stiffness matrices $\mathbf{K}_1$ and $\mathbf{K}_2$, so that the equilibrium equations are

$$\mathbf{K}_1\mathbf{u}_1 - \mathbf{f} = 0\,, \quad \mathbf{K}_2\mathbf{u}_2 - \mathbf{f} = 0\,. \tag{88}$$

Let $\mathbf{K}_1$ be the stiffness matrix of the actual structure specified from experimental data. The stiffness matrix $\mathbf{K}_2$ results from the assumed structure model. Specify the distance of solutions by a positive – definite distance norm

$$\mathrm{I} = \Psi(\mathbf{u}_1,\mathbf{u}_2) = \Psi(\mathbf{u}_2 - \mathbf{u}_1)\,, \quad \Psi(0) = 0\,. \tag{89}$$

Assume the load $\mathbf{f}$ to be controllable and to satisfy the constraint

$$\mathbf{f}^T\mathbf{f} - \rho_0^2 \le 0\,. \tag{90}$$

Now, the distance norm is maximized with respect to load distribution and minimized with respect to structure model parameters.[1] In other words, the structure loading should be selected in order to maximize the response difference between the structure and its model. The associated Lagrangian and its variation take the form

$$\mathrm{I}^{\mathrm{L}} = \mathrm{I} - (\mathbf{u}_1^{\mathrm{a}})^T[\mathbf{K}_1\mathbf{u}_1 - \mathbf{f}] + (\mathbf{u}_2^{\mathrm{a}})^T[\mathbf{K}_2\mathbf{u}_2 - \mathbf{f}] - \frac{1}{2}\mu\left(\mathbf{f}^T\mathbf{f} - \rho_0^2\right) \tag{91}$$

and

$$\delta\mathrm{I}^{\mathrm{L}} = \left(\frac{\partial\Psi}{\partial\mathbf{u}_2}\right)^T\delta\mathbf{u}_2 + \left(\frac{\partial\Psi}{\partial\mathbf{u}_1}\right)^T\delta\mathbf{u}_1 - (\mathbf{u}_1^{\mathrm{a}})^T\mathbf{K}_1\delta\mathbf{u}_1 + (\mathbf{u}_2^{\mathrm{a}})^T\mathbf{K}_2\delta\mathbf{u}_2 + \delta\mathbf{f}^T\left(\mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} - \mu\mathbf{f}\right)\,. \tag{92}$$

Here $\mathbf{u}_1^{\mathrm{a}}$, $\mathbf{u}_2^{\mathrm{a}}$ and μ are the Lagrange multipliers. Introduce the adjoint structures for which the displacement fields $\mathbf{u}_1^{\mathrm{a}}$ and $\mathbf{u}_2^{\mathrm{a}}$ are the Lagrange multipliers in (91) and are specified by the equations

$$\mathbf{K}_1\mathbf{u}_1^{\mathrm{a}} = \frac{\partial\Psi}{\partial\mathbf{u}_1} = \mathbf{f}^{\mathrm{a}}\,, \quad \mathbf{K}_2\mathbf{u}_2^{\mathrm{a}} = -\frac{\partial\Psi}{\partial\mathbf{u}_2} = +\frac{\partial\Psi}{\partial\mathbf{u}_1} = \mathbf{f}^{\mathrm{a}}\,. \tag{93}$$

The optimality condition for load distribution now takes the form

$$\delta\mathbf{f}^T\left(\mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} - \mu\mathbf{f}\right) = 0 \tag{94}$$

and for an arbitrary $\delta\mathbf{f}$ is expressed as the coaxiality rule

$$\mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} = \mu\mathbf{f}\,, \tag{95}$$

or

$$(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f}^{\mathrm{a}} = \mu\mathbf{f}\,, \tag{96}$$

where $\mathbf{D}_2 = \mathbf{K}_2^{-1}$, $\mathbf{D}_1 = \mathbf{K}_1^{-1}$ are the structure compliance matrices.

Thus, the load vector should be coaxial with the displacement difference vector of the adjoint structures.

[1] In a general theory of optimal experiment design for parameter identification, usually the Fisher information matrix is used, cf. Haftka *et al.* (1998). It is composed of sensitivity gradients of measured responses with respect to material parameters. The present analysis provides the sensitivity of the distance functional with respect to load parameters.

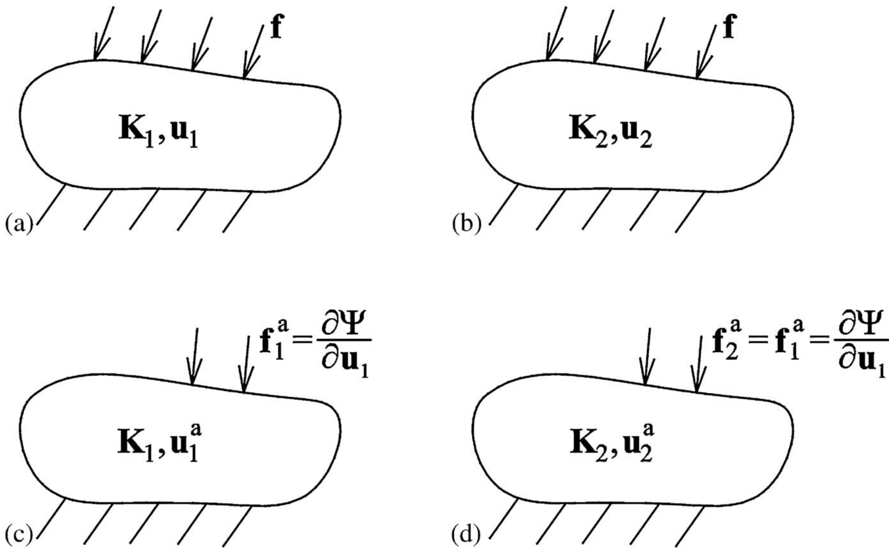

Fig. 7 Structural identification problem. Stiffness matrix $\mathbf{K}_1$ and the measured displacements $\mathbf{u}_1$ refer to the real structure. The matrix $\mathbf{K}_2$ and displacements $\mathbf{u}_2$ follow from model prediction. Adjoint load is $\mathbf{f}_1^{\mathrm{a}} = \mathbf{f}_2^{\mathrm{a}} = \mathbf{f}^{\mathrm{a}}$ and respective adjoint displacements are $\mathbf{u}_1^{\mathrm{a}}$, $\mathbf{u}_2^{\mathrm{a}}$

Equation (96) specifies the generalized eigenvalue problem.

Assume the functional I in the form

$$\mathrm{I}_1 = \frac{1}{2}(\mathbf{u}_2 - \mathbf{u}_1)^T \mathbf{A}(\mathbf{u}_2 - \mathbf{u}_1)\,, \tag{97}$$

where $\mathbf{A} = \mathbf{A}^T$, $\det \mathbf{A} > 0$ and $\mathbf{A}$ is a symmetric weighting matrix.

From (93) we then have

$$\mathbf{f}^{\mathrm{a}} = -\mathbf{A}(\mathbf{u}_2 - \mathbf{u}_1) = -\mathbf{A}(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f} \tag{98}$$

and the eigenvalue problem (96) now takes the form

$$(\mathbf{D}_2 - \mathbf{D}_1)^T \mathbf{A}(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f} = \mu\mathbf{f}\,, \tag{99}$$

or

$$\mathbf{L}\mathbf{f} = \mu\mathbf{f}\,, \tag{100}$$

where $\mathbf{L} = (\mathbf{D}_2 - \mathbf{D}_1)^T \mathbf{A}\,(\mathbf{D}_2 - \mathbf{D}_1)$ is the positive definite and symmetric matrix.

An alternative distance norm can be assumed in the form of the elastic energy difference, so that

$$\mathrm{I}_2 = \frac{1}{2}\mathbf{f}^T(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f}\,. \tag{101}$$

The optimality condition now provides the coaxiality rule

$$(\mathbf{u}_2 - \mathbf{u}_1) = \mu\mathbf{f} \tag{102}$$

expressed by the eigenvalue problem

$$(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f} = \mu\mathbf{f}\,. \tag{103}$$

Following the formulation of Lee *et al.* (1994), the ratio of elastic energies can be assumed as an indicator of difference between states of two structures, thus

$$\mathrm{I}_3 = \lambda = \frac{\frac{1}{2}\mathbf{f}^T\mathbf{D}_2\mathbf{f}}{\frac{1}{2}\mathbf{f}^T\mathbf{D}_1\mathbf{f}} \tag{104}$$

and the stationary condition of I_3 with the constraint (90) now provides

$$(\mathbf{D}_2 - \lambda\mathbf{D}_1)\mathbf{f} = \mu\mathbf{f}\,. \tag{105}$$

Let us note that the matrices $\mathbf{D}_2 - \mathbf{D}_1$ and $\mathbf{D}_2 - \lambda\mathbf{D}_1$ occurring in (103) and (105) may not be positive definite in general so the problem should be properly formulated to obtain positive eigenvalues. On the other hand, the eigenvalue problem (99) is associated with the positive – definite – symmetric matrix $\mathbf{L}$. The examples presented by Lee *et al.* (1994) refer to the antioptimization method aimed at maximization of I_3 or other measures associated with harmonic vibrations. An example of identification presented in the present paper illustrates the application of optimal load distribution specified by (99).

4
Examples

In this section we shall present several illustrative examples of applications of general theorems. The first four examples refer to statically determinate beam-column structures, where closed form solutions will be derived to illustrate in comprehensive way the derived formulae for optimal load in structural problems. Example 5 refers to a propped cantilever beam, where structural identification using a min-max approach will be discussed by applying FEM solutions.

4.1
Example 1

Optimal loading for maximal and minimal structure stiffness

Consider a beam-column structure shown in Fig. 8(a), loaded by two independent concentrated forces f_1, f_2. Find the best and worst loading conditions, using the global stiffness or compliance functions according to (7), (18), or (19). Assume the stiffness ratio $k = 2$.

Introducing unit forces f_1, f_2 and using a virtual work theorem we derive the closed form expression of the compliance matrix

$$\mathbf{D} = \mathbf{K}^{-1} = \frac{a^3}{12EI}\begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} \tag{106}$$

and $\mathbf{u} = \mathbf{Df}$, where $\mathbf{u} = [u_1, u_2]^T$, $\mathbf{f} = [f_1, f_2]^T$. The solutions of the eigenvalue problem (13) are:

$$\text{eigenvalues } \eta^2 = \frac{a^3}{12EI}\begin{bmatrix} 1 \\ 11 \end{bmatrix},$$
$$\text{eigenvectors } \mathbf{f} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \tag{107}$$

The angles of principal directions are

$$\tan 2\alpha_{pr} = \frac{3}{4}, \quad \alpha_{pr} = 18.4° \pm n\, 90° . \tag{108}$$

The complementary energy expressed in terms of force components is

$$C = \tilde{\Pi} = \frac{a^3}{12EI}\left(f_1^2 + 3f_1f_2 + 5f_2^2\right) . \tag{109}$$

The results are illustrated in Fig. 9. Note that the extremal load vectors $\mathbf{f}$ are collinear with the induced displacement vectors $\mathbf{u}$ and are represented by semiaxes of the ellipse (109). The values of the energy C for the extremal loads are

$$\min C = \frac{1}{24}\frac{\rho_0^2 a^3}{EI}, \quad \max C = \frac{11}{24}\frac{\rho_0^2 a^3}{EI} . \tag{110}$$

4.2
Example 2

Optimal loading for minimizing or maximizing local displacement norm

Consider a similar structure as in Example 1, shown in Fig. 8(a), but apply the response function $g(\mathbf{u})$ in the form of (74) as the quadratic norm of $\mathbf{u}$, with $\mathbf{A} = \mathbf{I}$, namely

$$g(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{u}, \tag{111}$$

where $\mathbf{u} = [u_1, u_2]^T$ is the displacement vector at the tip of the beam. Now, we use the adjoint variable method. The adjoint structure is similar to the primal one (Fig. 8(a)), but according to (75) the adjoint load is equal to the primal displacement $\mathbf{u}$ shown in Fig. 8(b), namely $f_1^a = u_1$, $f_2^a = u_2$. These adjoint loads induce the displacements of the tip of the adjoint structure $\mathbf{u}^a = [u_1^a, u_2^a]$.

Introducing $\mathbf{A} = \mathbf{I}$ in the optimality conditions (76)–(78) we compute the optimal load vectors as eigenvectors of the square of the compliance matrix $\mathbf{D}$ from Example 1, namely

$$\mathbf{D}^T\mathbf{D} = \left(\frac{a^3}{12EI}\right)^2\begin{bmatrix} 13 & 36 \\ 36 & 109 \end{bmatrix} \tag{112}$$

and

$$\text{eigenvalues } \eta^2 = \left(\frac{a^3}{12EI}\right)^2\begin{bmatrix} 1 \\ 121 \end{bmatrix},$$
$$\text{eigenvectors } \mathbf{f} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} . \tag{113}$$

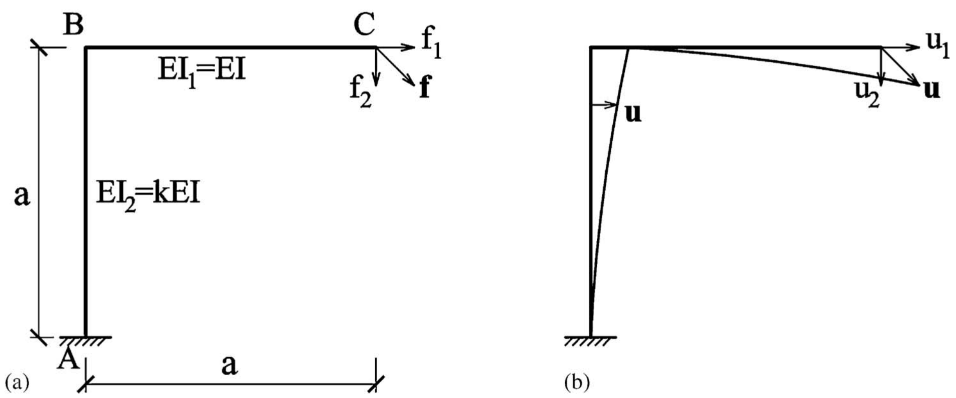

Fig. 8 A beam-column structure: (a) loads (b) displacements

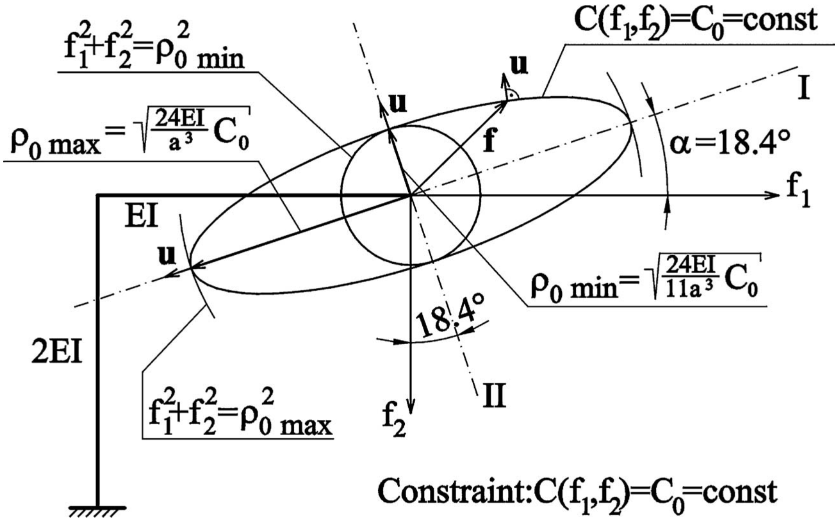

Fig. 9 Extremal loads and the ellipses illustrating the structural response measures

Note that the eigenvectors (113) are identical to (107). The coaxiality of extremal load and the conjugate displacement in Example 1 is also in agreement with the optimality condition in Example 2. In the latter case the extremal load of primal structure, primal displacement, adjoint load and adjoint displacement are all coaxial by definition.

Interesting are the values of the response measure (111) for extremal loads as the principal axes of the ellipses. Introducing eigenvectors into (74) and setting $\mathbf{A}=\mathbf{I}$ we arrive at

$$\min \mathrm{g} = \frac{1}{2}\left(\frac{\rho_0 \mathrm{a}^3}{12\mathrm{EI}}\right)^2 , \quad \max \mathrm{g} = \frac{1}{2}121\left(\frac{\rho_0 \mathrm{a}^3}{12\mathrm{EI}}\right)^2 . \quad (114)$$

The ratio of max g/min g is 121, whereas in Example 1 the ratio was 11. These ratios illustrate the importance of load optimization.

4.3 Example 3

Optimal loading for minimizing or maximizing global displacement norm

Consider again the structure discussed in Examples 1 and 2 with stiffness ratio $k=2$, shown in Fig. 10(a). Our aim is to find the extremal load $\mathbf{f}=[f_1, f_2]$, following the formulation (67), (68). We introduce the response function $g(\mathbf{u})$ as the quadratic norm of $\mathbf{u}$ following from (74), but this time we define the scalar product (111) in the form

$$g(\mathbf{u}) = \frac{1}{2}\sum \int_0^a \mathbf{u}^T \mathbf{u}\,\mathrm{d}x , \quad (115)$$

where the displacement vector $\mathbf{u}$ is now considered to be a continuous function of local coordinates x and $\sum$ denotes the summation over column and beam. Let the horizontal and vertical displacement components be u(x) and v(x) (cf. Fig. 10(b)). We use the adjoint variable method. According to (71) the displacements of the actual (primal) structure play the role of loading of the adjoint structure. To find the closed form solution we consider unit loads of primal structure.

Step 1. Loading $f_1 = 1$ Displacements (adjoint loading) and bending moments:
Column: $u(x) = (1/12EI)(3ax^2 - x^3)$, $v(x) = 0$
Beam: $u(x) = (1/6EI)a^3, v(x) = (1/4EI)a^2x$
Bending moments in the adjoint structure:
Beam: $M(x) = (1/24EI)(2a^5 - 3a^4x + a^2x^3)$
Column: $M(x) = (1/240EI)(71a^5 - 55a^4x + 5ax^4 - x^5)$
Displacements of the tip of the adjoint structure:

$$\text{Horizontal: } u_{11}^a = \frac{1}{(EI)^2}\frac{139}{2520}$$
$$\text{Vertical: } u_{21}^a = \frac{1}{(EI)^2}\frac{83}{720} \quad (116)$$

Step 2. Loading $f_2 = 1$ Displacements (adjoint loading) and bending moments:
Column: $u(x) = (1/4EI)ax^2$, $v(x) = 0$
Beam: $u(x) = (1/4EI)a^3$, $v(x) = (1/6EI)(3a^2x + 3ax^2 - x^3)$
Bending moments in the adjoint structure:
Beam: $M(x) = (1/120EI)(31a^5 - 45a^4x + 10a^2x^3 + 5ax^4 - x^5)$
Column: $M(x) = (1/240EI)(137a^5 - 80a^4x + 5ax^4)$
Displacements of the tip of the adjoint structure:

$$\text{Horizontal: } u_{12}^a = \frac{1}{(EI)^2}\frac{83}{720}$$
$$\text{Vertical: } u_{22}^a = \frac{1}{(EI)^2}\frac{29}{105} \quad (117)$$

The relation between the adjoint displacements of the tip of beam and the primal loads $\mathbf{f}$ is

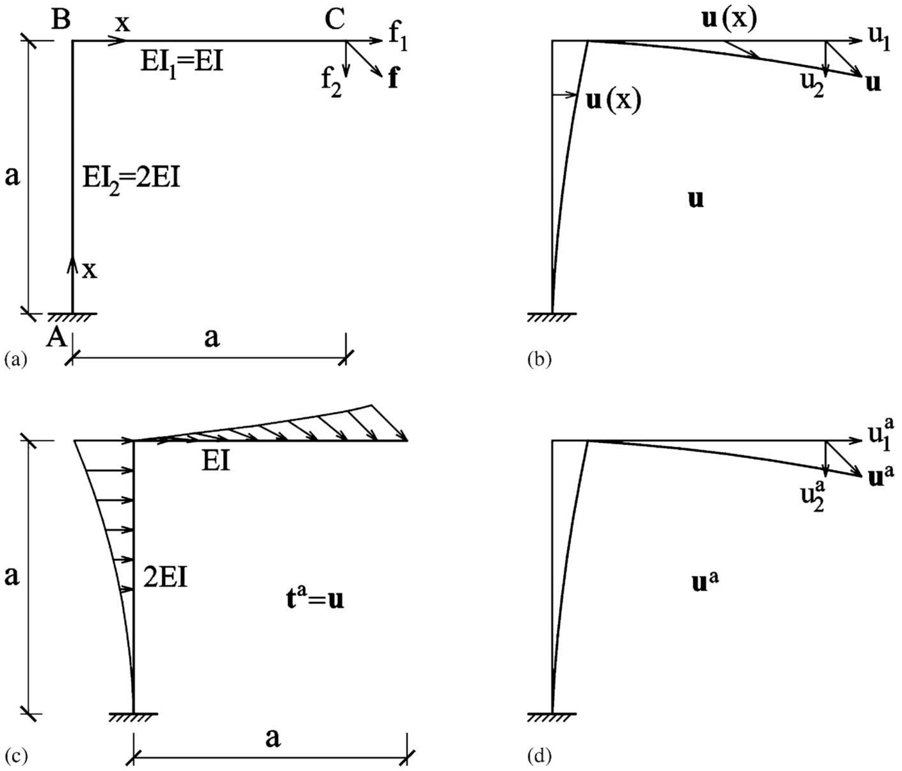

Fig. 10 Optimal loading for the quadratic norm (115). (a) Loading subject to optimization, b) Displacement field $\mathbf{u}(x)$, c) Adjoint structure loaded by $\mathbf{t}^a(x) = \mathbf{u}(x)$, and d) Displacements of the adjoint structure

$$u^a = \begin{bmatrix} u_1^a \\ u_2^a \end{bmatrix} = \frac{a^7}{(EI)^2} \begin{bmatrix} \frac{139}{2520} & \frac{83}{720} \\ \frac{83}{720} & \frac{29}{105} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \text{or } u^a = Bf. \tag{118}$$

The optimality condition (73)[1] now has the form of a linear eigenvalue problem

$$Bf = \eta^2 f, \tag{119}$$

which provides the solution for the extremal loading

$$\eta^2 = \left(\frac{a^7}{(EI)^2}\right) \begin{bmatrix} 5.979 \times 10^{-3} \\ 0.3254 \end{bmatrix},$$

$$\text{eigenvectors } \mathbf{f} = \begin{bmatrix} 0.9198 & 0.3924 \\ -0.3924 & 0.9198 \end{bmatrix}. \tag{120}$$

The angles of principal directions are $\alpha_{pr} = \arctan(0.3924/0.9198) = 23.10° \pm n\,90°$. The values of the response norm (115) for the extremal loading follow from

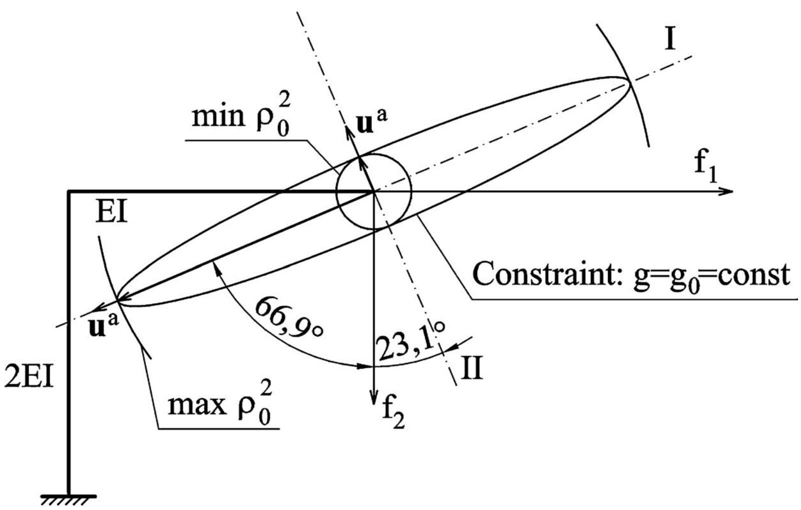

Fig. 11 The "best" loading direction **I**, the "worst" **II** and the associated load measures ρ_0^2

$$g = \frac{1}{2} f^T B f \,. \tag{121}$$

Hence

$$\max g = \frac{1}{2} [0.3924 \;\; 0.9198] [B] \begin{bmatrix} 0.3924 \\ 0.9198 \end{bmatrix} \rho_0^2 = 0.3254 \frac{a^7 \rho_0^2}{2(EI)^2} \tag{122}$$

and similarly for the second eigenvector

$$\min g = 5.979 \times 10^{-3} \frac{a^7 \rho_0^2}{2(EI)^2} \,. \tag{123}$$

Note that the response function (122) is associated with the "worst" and (123) with the "best" loading. Alternatively, when the value of the response function is prescribed $g = g_0$, the solutions (122) and (123) provide the minimum and maximum load, respectively:

$$\min \rho_0^2 = 3.073 \frac{2(EI)^2}{a^7} g_0 \,, \quad \max \rho_0^2 = 167.255 \frac{2(EI)^2}{a^7} g_0 \,. \tag{124}$$

The ellipse (121) and the solutions (124) are shown in Fig. 11. Note the coaxiality of primal load $\mathbf{f}$ and adjoint displacement $\mathbf{u}^a$ for the optimal solutions (124).

4.4
Example 4

Optimal loading maximizing or minimizing limit load for a perfectly plastic structure

For the sake of better illustrating the class of problems of optimal load action, we will briefly demonstrate optimal load of a structure made of a rigid, perfectly plastic material. This example extends the analysis to nonlinear structures discussed in Sect. 2.5. In fact, setting $n = 0$ in (85), we specify the perfectly soft material response equivalent to the rigid, perfectly plastic response. Our aim is to find optimal load $\mathbf{f} = [f_1, f_2]$ corresponding to a maximum of limit load of the structure shown in Fig. 8(a). We shall solve the problem for the ratios of the yield moments $k = (M^Y_{column})/(M^Y_{beam}) = 1$ and $k = 2$.

The problem can be formulated as follows: specify the optimal load vector $\mathbf{f}$, which provides

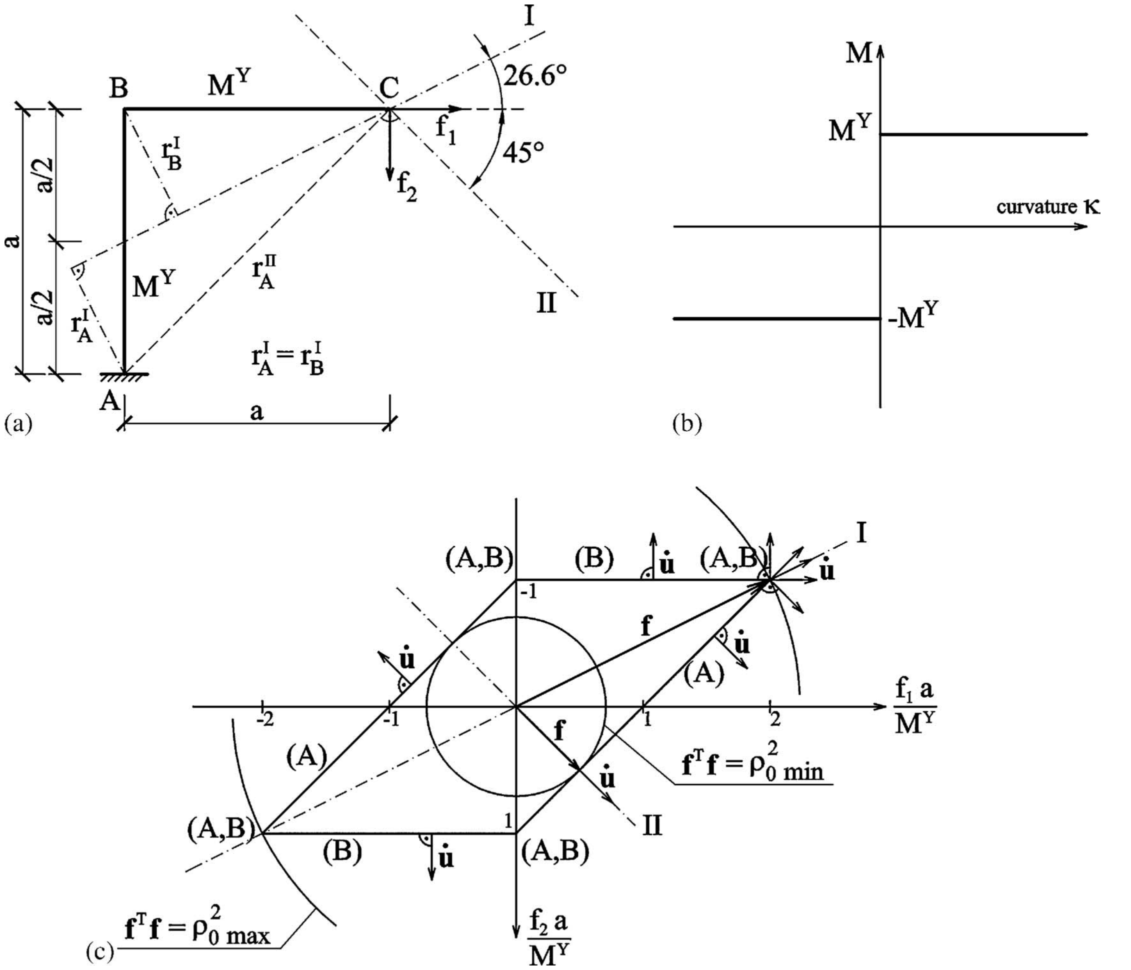

Fig. 12 Rigid plastic structure with constant yield moments ($k = 1$): (a) Principal directions, (b) Moment-curvature relation, and (c) Limit load polygon in non-dimensional coordinates

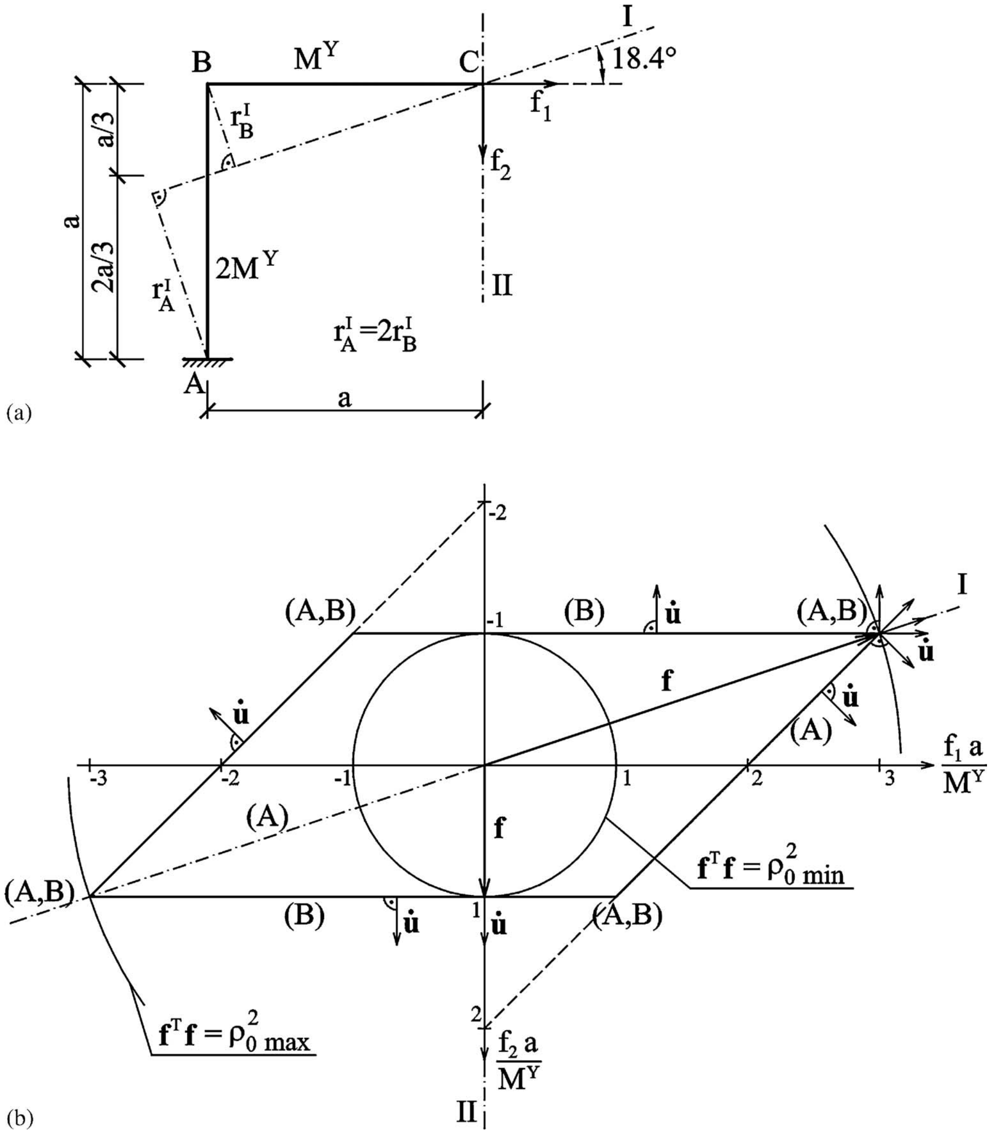

Fig. 13 Rigid plastic structure with the ratio of yield moments k = 2: (a) Principal directions specified by the coaxiality rule, (b) Limit load polygon in non-dimensional coordinates

maximum ρ_0,
subject to constraints

$$\mathrm{f}^T\mathrm{f} \le \rho_0^2\,, \quad |\mathrm{f}_1 + \mathrm{f}_2| \le \frac{\mathrm{k}M^Y}{\mathrm{a}}\,, \quad |\mathrm{f}_2| \le \frac{M^Y}{\mathrm{a}}\,. \tag{125}$$

The constraints $(125)^2$ and $(125)^3$ represent the yield conditions for plastic hinges at the cross-sections A and B, respectively. The solution for k = 1 is illustrated in Fig. 12, whereas Fig. 13 refers to the case k = 2. Note that the principal axes I and II are not orthogonal, as in previous examples. It is interesting that for optimal solutions (the best and the worst loading **f**) the associated plastic displacements **u** are collinear with **f**. It is also interesting that in the space of loads f_1, f_2 the plastic displacement **u** is orthogonal to the limit load curve, similarly as in the classical theory of plasticity in the space of stress.

4.5 Example 5

Optimal loading in structural identification

Consider a propped cantilever elastic beam. Assume that the beam consists of six elements of constant cross-sections with moments of inertia $I_1, I_2, \ldots, I_6$, shown in Fig. 14, which provide bending stiffness coefficients $s_i = EI_i$. Let the vector of the stiffness coefficients of the actual beam be

$$\mathbf{s}^{(1)} = [3;\ 1;\ 1.5;\ 2.5;\ 2;\ 1] \times 10^3\,\mathrm{kN\,m}^2\,. \tag{125}$$

Our aim is to find the unknown vector $\mathbf{s}^{(1)}$ using experimental data and the theory of load optimization discussed in Sect. 3.

Let us assume that in the experiment the structure has been subjected to static load f_1, next to the load f_2

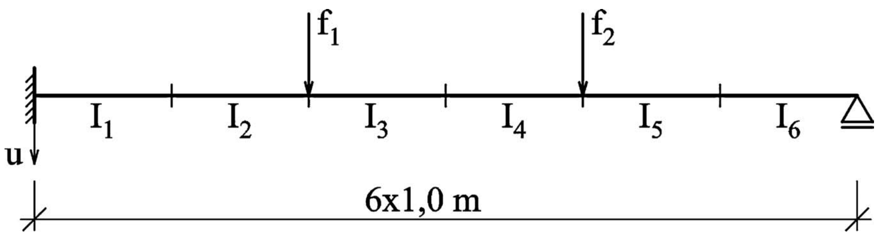

Fig. 14 Propped cantilever beam, applied loading f_1, f_2 and measured displacements u_1, u_2

and assume that displacements u_1 and u_2 (Fig. 14) have been measured for each load case. At this stage of the study we neglect measurement errors. Dividing the measured displacements by the values of forces we arrive at the compliance matrix $\mathbf{D}^{(1)}$ of the actual structure. In this example we have just computed displacements using FEM with high precision for two load cases of the actual structure (125): displacements u_{11} and u_{21} induced by the force $f_1 = 1$ and displacements u_{12} and u_{22} induced by $f_2 = 1$. Thus, the starting point for structural identification is the experimentally measured compliance matrix

$$\mathbf{D}^{(1)} = \begin{bmatrix} 4.993 & 4.937 \\ 4.937 & 9.354 \end{bmatrix} \times 10^{-4} \, \frac{\mathrm{m}}{\mathrm{kN}} \,. \tag{126}$$

Note that we aim at finding six unknown stiffness parameters s_i based on three experimental data (126), namely: $D_{11} = u_{11}$, $D_{12} = u_{12} = u_{21}, D_{22} = u_{22}$. Our first guess is a structure model assumed as a beam of uniform cross-section represented by (127)

$$\mathbf{s}^{(2)} = [1;\ 1;\ 1;\ 1;\ 1;\ 1] \times 10^3 \, \mathrm{kN\,m^2} \,. \tag{127}$$

This structure model will be improved in the step-by-step procedure.

In this example, for brevity, we denote by $.^{(1)}$ all quantities that refer to the actual structure, whereas the respective quantities referring to structure model are denoted as $.^{(2)}$. In Sect. 3 subscripts $._1$ and $._2$ were used, respectively. However, there is a substantial difference between compliance matrices (96) in Sect. 3 and the matrices (126), (128). The former matrices had dimensions $n \times n$ resulting from FEM discretization, and referred to the general case, when the load was specified as a vector in the space R^n. In this section the load is specified by two components f_1 and f_2. The reduction of space dimensions is similar to that discussed in Sect. 2.3. In fact we use three Euclidean spaces in this example, namely R^n for discrete mathematical model FEM, R^2 for optimization of load $\mathbf{f}$ and R^6 for structural identification of $\mathbf{s}$.

Step 1. Optimization of loading.
Using the parameters of the structure model $\mathbf{s}^{(2)}$ we compute displacements induced by unit forces f_1 and f_2. Thus, the computed compliance matrix $\mathbf{D}^{(2)}$ is

$$\mathbf{D}^{(2)} = \begin{bmatrix} 10.864 & 11.358 \\ 11.358 & 19.753 \end{bmatrix} \times 10^{-4} \, \frac{\mathrm{m}}{\mathrm{kN}} \,. \tag{128}$$

Let us determine the loading that maximizes the discrepancy between the model and the actual structure. We introduce $\rho_0 = 1\,\mathrm{kN}$ to the constraint (90) and maximize the quadratic distance measure (97) setting $\mathbf{A} = \mathbf{I}$. Hence, the distance measure is

$$I_1 = \frac{1}{2} \left(u^{(2)} - u^{(1)} \right)^T \left(u^{(2)} - u^{(1)} \right) =$$

$$\frac{1}{2} \left[\left(u_1^{(2)} - u_1^{(1)} \right)^2 + \left(u_2^{(2)} - u_2^{(1)} \right)^2 \right] \tag{129}$$

and the optimal loading is specified by the eigenvalue problem (99). The respective matrices are

$$\left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)} \right) = \begin{bmatrix} 5.8710 & 6.4209 \\ 6.4209 & 10.3989 \end{bmatrix} \times 10^{-4} \, \frac{\mathrm{m}}{\mathrm{kN}} \,, \tag{130}$$

and

$$\left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)} \right)^T \left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)} \right) =$$

$$\begin{bmatrix} 0.75697 & 1.04468 \\ 1.04468 & 1.49366 \end{bmatrix} \times 10^{-6} \, \frac{\mathrm{m}}{\mathrm{kN}} \,. \tag{131}$$

Solving the eigenvalue problem (99), the following eigenvalues μ and eigenvectors $\mathbf{f}$ are obtained

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1.7599 \\ 223.303 \end{bmatrix} \times 10^{-8} \, \mathrm{m}^2 \,,$$

$$\mathbf{f} = \begin{bmatrix} f_1 & f_1 \\ f_2 & f_2 \end{bmatrix} = \begin{bmatrix} 0.81625 & 0.5777 \\ -0.5777 & 0.8163 \end{bmatrix} \mathrm{kN} \,. \tag{132}$$

The extremal values of distance measure I_1 can be evaluated from (129) for the loads $(132)^2$, or using compliance matrices and solution (132) we transform (129) to

$$I_1 = \frac{1}{2} \mathbf{f}^T \left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)} \right)^T \left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)} \right) \mathbf{f} =$$

$$\frac{1}{2} \mu \mathbf{f}^T \mathbf{f} \,. \tag{133}$$

Since we have set $\rho_0 = 1$ in (90) the eigenvalues μ_1 and μ_2 provide the extremal values of distance function (129) multiplied by two. Hence, the minimum and maximum of distance measure, $\min(2I_1) = \mu_1$ and $\max(2I_1) = \mu_2$, are associated with the loads specified by the first and second columns of $(132)^2$, respectively. Note the extremely large value of the ratio max I_1/min $I_1) = (\mu_2/\mu_1) = 126.9$, which demonstrates the importance of load optimization. The eigenvector $\mathbf{f} = [f_1, f_2]$ shown in the second column specifies the forces acting in the same direction.

Step 2. Optimization of structure model parameters $\mathbf{s}$
Let the structure be subjected to extremal loads

$$f_1 = 0.5777\,\mathrm{kN}\,, \quad f_2 = 0.8163\,\mathrm{kN} \tag{134}$$

Since the displacement $\mathbf{u}^{(2)}$ of the structure model depends on the control vector $\mathbf{s}$, the response function (129) is also a function of $\mathbf{s}$. We can find the sensitivity gradient $\nabla I_1 = \partial I_1(\mathbf{s})/\partial \mathbf{s}$ and next minimize I_1 with respect to $\mathbf{s}$. We use the adjoint variable method. The primary structure is loaded by forces (134). The loads of the adjoint structure are

$$f_1^a = \frac{\partial I_1}{\partial u_1^{(2)}} = u_1^{(2)} - u_1^{(1)}\,,$$

$$f_2^a = \frac{\partial I_1}{\partial u_2^{(2)}} = u_2^{(2)} - u_2^{(1)}\,, \tag{135}$$

or briefly

$$\mathbf{f}^a = \left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)}\right)\mathbf{f}\,. \tag{136}$$

The sensitivity gradient is

$$\nabla I_1 = -\mathbf{u}^a \frac{\partial \mathbf{K}^{(2)}}{\partial \mathbf{s}} \mathbf{u}^{(2)}\,, \tag{137}$$

where $\mathbf{u}^a$, $\mathbf{u}^{(2)}$ and $\mathbf{K}^{(2)}$ are the displacement vectors and the stiffness matrix of the structure model in n-dimensional space. In this way we arrived at the sensitivity gradient

$$\nabla I_1 = -\frac{1}{2}[2.727;\ 0.198;\ 0.940;$$

$$2.228;\ 1.784;\ 0.255] \times 10^{-1}\,\mathrm{kN}^{-1}\,. \tag{138}$$

Our model structure can be improved in the one-dimensional search in the direction opposite to ∇I_1, namely

$$\mathbf{s}_k^{(2)} = \mathbf{s}^{(2)} - \alpha_k 2\nabla I_1\,. \tag{139}$$

Employing a typical gradient algorithm one proceeds in the direction (139) until $\min I_1$ is reached. Actually, increasing α_k resulted in reduction of the distance function I_1 measured for the fixed load (134). However, simultaneously the distance function I_1 measured for the other load increased. Therefore, in a one-dimensional search in direction (139), at each step k we computed eigenvalues and eigenvectors (132) for the actual $\mathbf{s}_k$ and we evaluated the maximum of distance function using the optimal load. In fact, we repeated the procedure of Step 1 for each increment of α_k in (139). In this way an optimum $\alpha_{opt} = 6.27781657$ has been reached in a one-directional search. Thus, the obtained structure model was represented by the vector $\mathbf{s}_{opt}$ and distance function I_1

$$\mathbf{s}_{opt}^{(2)} = [2.711;\ 1.124;\ 1.590;\ 2.399;\ 2.120;\ 1.160] \times 10^3\,\mathrm{kN\,m^2}$$

$$2I_1 = 2.573 \times 10^{-10}\,\mathrm{m}^2\,. \tag{140}$$

The solution (140) is definitely better than the first model (127) and the corresponding distance function $2I_1 = 2.233 \times 10^{-6}\,\mathrm{m}^2$ specified by $(132)^1$. The result of identification (140) can be considered satisfactory since the maximum error in the components of $\mathbf{s}^{(2)}$ is 16% in the last section of the beam. However, in the identification procedure, components of the vector $\mathbf{s}^{(1)}$ are not known, and we should try to further reduce the distance measure I_1. In common gradient algorithms one computes a new sensitivity gradient ∇I_1 and repeats the one-dimensional search. We keep in mind what Cherkaev (1999a) observed that in problems of load optimization for structural identification purposes, multiple eigenvalues may appear. We encountered a much more interesting and difficult situation that at the optimum solution (140) multiple and equal eigenvalues appear, since the matrix $[\mathbf{D}^{(2)} - \mathbf{D}^{(1)}]^2$ is isotropic, while $\mathbf{D}^{(2)} - \mathbf{D}^{(1)}$ is deviatoric, namely

$$\left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)}\right) = \begin{bmatrix} 1.2052 & 1.0586 \\ 1.0586 & -1.22052 \end{bmatrix} \times 10^{-5}\,\frac{\mathrm{m}}{\mathrm{kN}}\,, \tag{141}$$

$$\left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)}\right)^T \left(\mathbf{D}^{(2)} - \mathbf{D}^{(1)}\right) =$$

$$\begin{bmatrix} 2.5732 & 0 \\ 0 & 2.5732 \end{bmatrix} \times 10^{-10}\,\frac{\mathrm{m}}{\mathrm{kN}}\,. \tag{142}$$

All vectors $\mathbf{f}$ generate the same distance measure $2I_1 = 2.5732 \times 10^{-10}\,\mathrm{m}^2$. The sensitivity gradients for different loading $\mathbf{f}$ all have components of the same sign and value and do not induce reduction of the distance measure I_1.

In this situation we decided to change the distance measure. Introduction of I_2 specified by (101) brought no improvement in view of (103), (141) and (142). So we employed a new distance measure namely the square of the

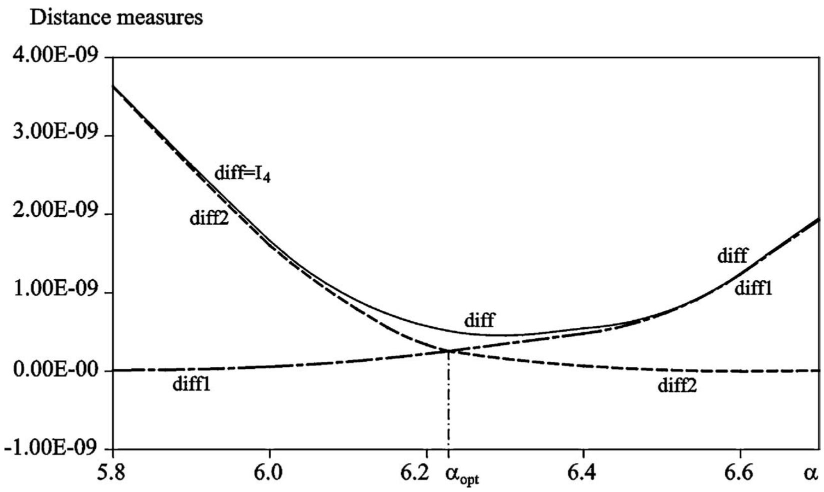

Fig. 15 Variation of distance measures in the one-dimensional search (139) and the optimal solution (140) for $\alpha_{\mathrm{opt}} = 6.27781657$

Euclidean norm of the matrix $\mathbf{D}^{(2)} - \mathbf{D}^{(1)}$.

$$\mathrm{I}_4 = \left(\left\|\mathbf{D}^{(2)} - \mathbf{D}^{(1)}\right\|_{\mathrm{Eucl}}\right)^2 = \left(\mathrm{D}_{11}^{(2)} - \mathrm{D}_{11}^{(1)}\right)^2 + 2\left(\mathrm{D}_{12}^{(2)} - \mathrm{D}_{12}^{(1)}\right)^2 + \left(\mathrm{D}_{22}^{(2)} - \mathrm{D}_{22}^{(1)}\right)^2 . \tag{143}$$

Note that I_4 does not depend on loads since all terms represent displacements induced by unit forces f_1 or f_2. For better insight into the problem of interrelation of the discussed distance measures the distance function I_4 for all steps k in the one-dimensional search (139) has been computed. The results are shown in Fig. 15.

The following notation was used in Fig. 15: diff2 $= \mu_2$ denotes the doubled distance function I_1 evaluated for the load as the eigenvector $\mathbf{f}$ associated with μ_2 providing maximum distance at the beginning of the search, diff1 $= \mu_1$ denotes the doubled distance for the other eigenvector, whereas diff $= \mathrm{I}_4$ denotes the square of Euclidean norm. The "bimodal" optimum at the crossing of diff1 and diff2 at optimum α is illustrated. Interesting is the fact of very flat sub-optimum branches of diff1 and diff2 in Fig. 15. This fact clearly demonstrates the significance of load optimization. Evaluation of distance function for non-optimal load can be ineffective in structural identification. It is interesting to note that quadratic Euclidean norm is so close to both branches diff1 and diff2 in their effective regions. Note that the distance function max(diff1, diff2) is not differentiable at the optimal point, whereas the function I_4 is differentiable.

The starting point for optimization with the use of quadratic Euclidean norm (143) is $\mathbf{s}^{(2)}$ and distance measures diff1, diff2 shown in (140), and $\mathrm{I}_4 = \mathrm{diff} = 5.14642433 \times 10^{-10}$. The sensitivity gradient was computed using the perturbation method. All components of $\mathbf{s}^{(2)}$ were sequentially perturbed by 1×10^{-6} kN m^2, i.e., less than 0.1%. The sensitivity gradient was

$$\nabla\mathrm{I}_4 = [-3.8853;\ -0.9868;\ -2.7832;\ 0.6355;\ 1.8966;\ 0.8498] \times 10^{-4}\ \mathrm{m}^2\,\mathrm{kN}^{-2} . \tag{144}$$

One-dimensional search

$$\mathbf{s}_{\mathrm{k}}^{(2)} = \mathbf{s}^{(2)} - \alpha_{\mathrm{k}} \nabla \mathrm{I}_4 \tag{145}$$

brought us to optimal distance functions $\mathrm{I}_4 = \mathrm{diff} = 1.6460 \times 10^{-10}$, $\min 2\mathrm{I}_1 = \mathrm{diff1} = 5.0355 \times 10^{-12}$, $\max 2\mathrm{I}_1 = \mathrm{diff2} = 1.5957 \times 10^{-10}$.

The optimal structure model now is

$$\mathbf{s}_{\mathrm{opt}}^{(2)} = [2.808;\ 1.149;\ 1.659;\ 2.383;\ 2.073;\ 1.139] \times 10^3\ \mathrm{kN\,m}^2$$

$$\mathrm{I}_4 = 1.6460 \times 10^{-10},\ 2\mathrm{I}_1 = 1.5957 \times 10^{-10} . \tag{146}$$

Comparing (146) with (140) we note that minimization of I_4 also resulted in reduction of I_1, however, the accuracy of $\mathbf{s}^{(2)}$ in relation to actual parameters $\mathrm{s}^{(1)}$ from (125) has not been considerably improved.

5 Concluding remarks

The sensitivity derivatives with respect to load parameters and respective optimality conditions were derived in this paper using the adjoint variable approach. The main result of the analysis is that the coaxiality of load and adjoint displacement vectors occurs at the stationary condition. These stationary solutions are obtained from the respective eigenvalue problems and may correspond to a minimum or maximum of the response functional, thus specifying the best or the worst loading conditions. In the case of identification, the coaxiality occurs between the load vector and the displacement differences of experimental and predicted values. The results obtained generalize previous studies and provide uniform treatment of load control problems in static conditions.

Acknowledgements This work was supported by grant 5T70E-045 22 from the State Committee for Scientific Research.

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