Information Sciences xxx (xxxx) xxx

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Ordered fuzzy random variables: Definition and the concept of normality

Adam Marszałek [a,*], Tadeusz Burczyński [a,b]

[a] Institute of Computer Science, Computational Intelligence Department, Cracow University of Technology, Cracow, Poland
[b] Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland

ARTICLE INFO

Article history:
Received 31 March 2020
Received in revised form 7 July 2020
Accepted 31 August 2020
Available online xxxx

Keywords:
Ordered fuzzy numbers
Fuzzy random variables
Ordered fuzzy random variables
Normal ordered fuzzy random variable

ABSTRACT

The concept of fuzzy random variable combines two sources of uncertainty: randomness and fuzziness, whereas the model of ordered fuzzy numbers provides a representation of inaccurate quantitative data, and is an alternative to the standard fuzzy numbers model proposed by Zadeh. This paper develops the model of ordered fuzzy numbers by defining the concept of fuzzy random variables for these numbers, called further ordered fuzzy random variables. Thanks to the well-defined arithmetic of ordered fuzzy numbers (existence of neutral and opposite elements) and the introduced ordered fuzzy random variables; it becomes possible to construct fully fuzzy stochastic time series models such as e.g., the autoregressive model or the GARCH model in the form of classical equations, which can be estimated using the least-squares or the maximum likelihood method. Furthermore, the concept of normality of ordered fuzzy random variables and the method to generate pseudo-random ordered fuzzy variables with normal distribution are introduced.

1. Introduction

Many real-life models involve two important sources of uncertainty: randomness and fuzziness. Randomness relates to the stochastic variability of all possible outcomes of a situation. Fuzziness, on the other hand, can be traced to incomplete or imprecise knowledge regarding the situation. Accordingly, there are two prevailing theories, probability theory and possibility theory, to deal with them. Randomness and fuzziness can be merged to formulate a fuzzy random variable. Fuzzy random variable (FRV) initiated by Kwakernaak [17,18] is one of the appropriate ways to describe this type of uncertainty. Kwakernaak [17] conceptualized an FRV as a vague perception of a crisp but unobservable random variable that taking fuzzy value instead of real values. Further Puri and Ralescu [32] conceptualized the FRV as a fuzzification of a random set, whose values are fuzzy subsets of $\mathbb{R}^p$ or, more generally, of a Banach space. Later on and sometimes independently, other variants were proposed by Kruse and Meyer [16] and Diamond and Kloeden [4]. Krätschmer [15] surveyed all of these definitions and proposed a unified approach. Moreover, Liu and Liu [19,20] proposed definition of FRV with using different measure condition which based on a concept they called credibility measure. Summarizing, in all of these works, an FRV is defined as a function that assigns a fuzzy subset to each possible output of a random experiment that intends to model situations that combine fuzziness and randomness.

Despite excellences of fuzzy sets in describing imprecise data, fuzzy arithmetic is accompanying with difficulties and complexities that may decrease the utility of them. Computational efficiency is particularly important when a fuzzy set

* Corresponding author.
E-mail addresses: amarszalek@pk.edu.pl (A. Marszałek), tburczynski@ippt.pan.pl (T. Burczyński).

https://doi.org/10.1016/j.ins.2020.08.120
0020-0255/© 2020 Published by Elsevier Inc.

theory is used to solve real problems [37]. Concerning these problems, Dubois and Prade [5] introduced a flexible parametric family of membership functions which are called the LR family. By using this family, computational efficiency increases without limiting the generality beyond acceptable limits [37]. Moreover, the computational properties of classical fuzzy numbers (also known as convex fuzzy numbers) also have drawbacks due to rapidly growing imprecision after a sequence of operations [22]. To improve their arithmetic properties, several additional solutions were introduced. Usually, additional operations or constraints are needed (see [9,33,34]) to get around these drawbacks. An alternative solution is to use ordered fuzzy numbers (OFNs), formulated by Kosiński et al. [12–14,10]. This model takes into account the order of the individual parts of a fuzzy number. It creates an additional feature of the data – an orientation. By incorporating the orientation in the calculations, we get the opportunity to reduce the imprecision resulting from a series of operations. It leads to a situation where we can solve a basic equation of type where all elements can be OFNs. It can be done by the simple calculation of exactly as for real numbers. The computational model of the OFNs has many properties that were presented in the publications [14,10,28,30,29,24].

In this paper, the model of OFN is extended by defining the novel concept of fuzzy random variables for these numbers, called further ordered fuzzy random variables (OFRVs). Thanks to the well-defined arithmetic of OFNs (existence of neutral and opposite elements) and the introduced OFRVs, it becomes possible to construct fully fuzzy stochastic time series models such as e.g., the fuzzy autoregressive model in the form of classical equations, which can be estimation using the least-squares or the maximum likelihood method. In the wide context, the proposed concept of OFRVs can be considered as an additional computational model of uncertainties in granular computing paradigm [1].

This paper is organized as follows: Section 2 recalls three types of definitions of FRVs presented in the literature, recalls definitions of OFN, and some basics concepts of stochastic processes. Section 3 introduces a novel concept of fuzzy random variables for ordered fuzzy numbers and some basic properties such as fuzzy and crisp expected value and variance. Section 4 gives a novel concept of normality of OFRVs and introduces the method to generate pseudo-random ordered fuzzy variables with the normal distribution. Finally, concluding remarks make up the last section.

2. Preliminaries

Before proceeding to formal definitions of ordered fuzzy random variables, we first review briefly underlying concepts: fuzzy random variables and ordered fuzzy numbers. Readers familiar with these topics can skip this section.

2.1. Fuzzy random variables (FRVs)

The term *fuzzy random variable* was introduced by Kwakernaak in 1978, who descrided a fuzzy random variable as a vague perception of a crisp but unobservable random variable [17,18].

Let $(\Omega, \mathcal{A}, \Pr)$ be the classical probabilistic space, where Ω is the space of elementary events, $\mathcal{A}$is σ-algebra of subsets Ω, while Pris a probabilistic measure set at Ω. Let $\mathcal{F}(\mathbb{R})$denote the set of all convex fuzzy numbers specified in Euclidean space $\mathbb{R}$, of which α-cuts for each $\alpha \in [0,1]$are compact sets. Then *the fuzzy random variable* is a mapping $\tilde{\xi} : \Omega \to \mathcal{F}(\mathbb{R})$ such that for any $\alpha \in [0,1]$ and all $\omega \in \Omega$ the real valued mapping

$$\inf \tilde{\xi}_\alpha : \Omega \to \mathbb{R} \text{ such that } \inf \tilde{\xi}_\alpha(\omega) = \inf \left(\tilde{\xi}(\omega)\right)_\alpha, \tag{1}$$

$$\sup \tilde{\xi}_\alpha : \Omega \to \mathbb{R} \text{ such that } \sup \tilde{\xi}_\alpha(\omega) = \sup \left(\tilde{\xi}(\omega)\right)_\alpha, \tag{2}$$

are classical real valued random variables [17,18,16,7].

For such a definition of the fuzzy random variable, properties such as expected value and variance of the fuzzy random variable have been determined by applying the Zadeh's extension principle [27].

The concept of a fuzzy random variable introduced by Kwakernaak only applies to fuzzy numbers defined on $\mathbb{R}$. Dissatisfied with this fact, Puri and Ralescu proposed the concept of a fuzzy random variable whose values are fuzzy subsets of $\mathbb{R}^n$(or more generally, some Banach space) [32]. They defined the fuzzy random variable as a kind of fuzzification of a random set, thus combining fuzzy random variables with a well-developed theory of random sets [25].

Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of fuzzy subsets on $\mathbb{R}^n$ with membership functions $\mu : \mathbb{R}^n \to [0,1]$ and $\mathcal{K}(\mathbb{R}^n)$ denote the Borel subsets of $\mathbb{R}^n$. Then *the fuzzy random variable* is a function $\tilde{\xi} : \Omega \to \mathcal{F}(\mathbb{R}^n)$ such that for every $\alpha \in [0,1]$ mapping $\tilde{\xi}_\alpha : \Omega \to \mathcal{K}(\mathbb{R}^n)$ (where $\tilde{\xi}_\alpha(\omega) = \left(\tilde{\xi}(\omega)\right)_\alpha$ for each $\omega \in \Omega$) is a random set.

Puri and Ralescu defined the concept of expected value for a fuzzy random variable as a generalization of the classic expected value using the Aumann integral [21]. The variance, however, was defined similarly to the classic random variables as a measure of dispersion around the expected value and was defined as the crisp number [6].

In both approaches cited above, a fuzzy random variable is defined as a measurable function from a probabilistic space into a set of all fuzzy numbers. In their works, Liu and Liu, imposing a different condition on the measurability of functions, proposed a new definition of the fuzzy random variable [19,20].

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120

Let $\mathcal{F}_v$ be a set of all fuzzy variables [26] defined on the possibility space $(\Theta, \mathcal{P}(\Theta), Pos)$ [36]. Each element $\tilde{a}$ belonging to the set $\mathcal{F}_v$ is determined by the membership function $\mu_{\tilde{a}}$, also called the possibility distribution function. Then *the fuzzy random variable* is mapping $\tilde{\xi} : \Omega \to \mathcal{F}_v$ such that for each closed subset of $\mathbb{R}$

$$\tilde{\xi}^*(C)(\omega) = Pos\left\{\tilde{\xi}(\omega) \in C\right\} = \sup_{x \in C} \mu_{\tilde{\xi}(\omega)}(x) \tag{3}$$

is measurable function of ω, where $\mu_{\tilde{\xi}(\omega)}$ is membership function of fuzzy variable $\tilde{\xi}(\omega)$.

Unlike previous approaches, Liu and Liu determined the expected value and variance of the fuzzy random variable as a scalar value, not a fuzzy one. To defuzzifine the fuzzy variable, they proposed a defuzzification operator, called the expected value operator, as follows.

Definition 1. *The expected value operator* of fuzzy variable $\tilde{a}$ is a scalar number $E(\tilde{a})$ such that

$$E(\tilde{a}) = \int_0^{\infty} Cr\{\tilde{a} \geqslant r\}dr - \int_{-\infty}^{0} Cr\{\tilde{a} \leqslant r\}dr, \tag{4}$$

provided that at least one of the two integrals is finite, where $Cr\{\tilde{a} \leqslant r\}$ is a *credibility measure* such that

$$Cr\{\tilde{a} \leqslant r\} = 1 - Cr\{\tilde{a} \geqslant r\} = \frac{1}{2}\left(\sup_{x \leqslant r} \mu_{\tilde{a}}(x) + 1 - \sup_{x \geqslant r} \mu_{\tilde{a}}(x)\right). \tag{5}$$

Liu and Liu showed that if $\tilde{\xi}$ is a fuzzy random variable, then crisp value $E\left(\tilde{\xi}\right)$ is a random variable with real values [20]. This fact allows the determination of the expected value and variance of a fuzzy random variable as the expected value and variance of a classic real-valued random variable whose values are crisp values of a fuzzy random variable.

2.2. Ordered fuzzy numbers (OFNs)

Ordered fuzzy numbers (called also the Kosiński's Fuzzy Numbers) introduced by Kosiński et al. in series of papers [12–14,11,10] are defined by ordered pairs of continuous real functions defined on the interval [0, 1] i.e.

$$A = (f, g) \text{ with } f, g : [0, 1] \to \mathbb{R} \text{ as continuous functions.} \tag{6}$$

The ordered fuzzy number and ordered fuzzy number as a fuzzy number in classical meaning are presented in Fig. 1. Note that a pair of continuous functions (f, g) determines different ordered fuzzy numbers than the pair (g, f). In this way, an extra feature of this object named the orientation is appointed. To uniquely identify the orientation of ordered fuzzy numbers, the following order operator is proposed.

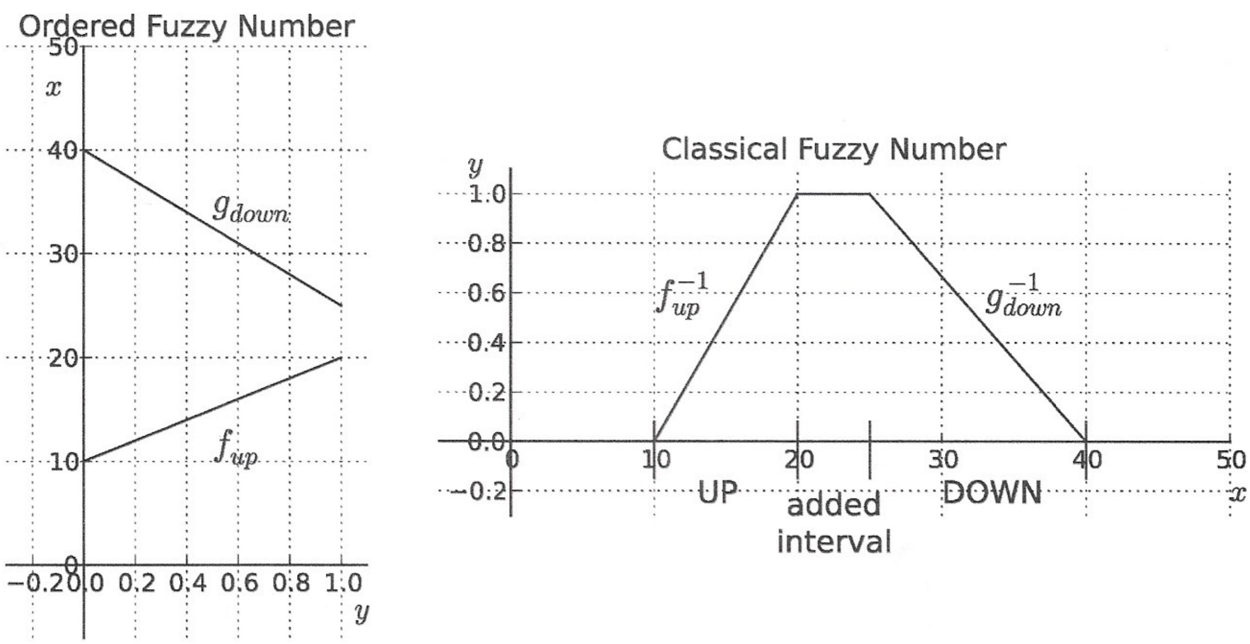

Fig. 1. Graphical interpretation of OFN and a OFN presented as fuzzy number in classical meaning.

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120

Definition 2. Let $\mathcal{O}$ be universe of all ordered fuzzy nymbers. *Order operator* is defined as mapping $Ord : \mathcal{O} \rightarrow \{-1, 0, 1\}$, such that

$$Ord(A) = \begin{cases} 1 & \text{if} \quad Defuzzy(A) > f_A(0), \\ 0 & \text{if} \quad Defuzzy(A) = f_A(0) \text{ or } Defuzzy(A) \text{ notexist}, \\ -1 & \text{if} \quad Defuzzy(A) < f_A(0), \end{cases} \tag{7}$$

where *Defuzzy(A)* is a fixed defuzzification operator for the ordered fuzzy number $A = (f_A, g_A)$. The numbers $-1, 1, 0$indicate the negative, positive and lack of orientation, respectively.

The basic arithmetic operations $\{+, -, \cdot, \div\}$ on ordered fuzzy numbers are defined as the pairwise operations of their elements. This definition leads to some useful properties. One of them is the existence of neutral elements of addition and multiplication. This fact causes that not always the result of an arithmetic operation is a fuzzy number with larger support. This allows us to build fuzzy models based on ordered fuzzy numbers in the form of the classical equations without losing accuracy. Similarly, basic math functions such as log, exp, sqrt can be defined (see [28]).

Sometimes there is also a need to determine the distance between two ordered fuzzy numbers. To do this, the following distance metric is proposed.

Definition 3. Distance between two ordered fuzzy numbers $A = (f_A, g_A)$ and $B = (f_B, g_B)$ is metric given by formula

$$d(A, B) = d((f_A, g_A), (f_B, g_B)) = \|f_A - f_B\|_2 + \|g_A - g_B\|_2, \tag{8}$$

where $\|\cdot\|_2$ is L_2-norm in space $\mathcal{C}([0, 1])$ (all real-valued continous functions defined on $[0, 1]$) given by formula

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{9}$$

3. Ordered fuzzy random variables (OFRVs)

Among the three cited definitions of fuzzy random variables in Section 2.1, the definition proposed by Kwakernaak seems to be the most natural and the simplest in the adaptation for ordered fuzzy numbers. In this definition, for each $\alpha \in [0, 1]$, the ends of α-cuts are identified with real-valued random variables. In case of ordered fuzzy number the ends of α-cuts correspond to the values of functions $f(x)$ and $g(x)$, for $x = \alpha$. Hence, the ordered fuzzy random variable could be defined as the mapping from the probabilistic space into a set of all ordered fuzzy numbers, such that for each $x \in [0, 1]$,$f(x)$ and $g(x)$ are real-valued random variables. However, such a definition is very general and does not specify any relations between random variables for different values of the x argument. For this reason, and because the ordered fuzzy number is an ordered pair of continuous functions, the first author in his dissertation thesis proposed a definition of an ordered fuzzy random variable as an ordered pair of continuous stochastic processes [23].

Let $(\Omega, \mathcal{A}, \Pr)$ be the specified probabilistic space and let $\mathcal{O}$ be the universe of all ordered fuzzy numbers.

Definition 4. *Ordered fuzzy random variable* is defined as mapping $\tilde{\xi} : \Omega \rightarrow \mathcal{O}$ such that $\tilde{\xi}(\omega) = \left(f_{\tilde{\xi}}(t, \omega), g_{\tilde{\xi}}(t, \omega)\right)$ is an ordered pair of continuous stochastic processes, where $f_{\tilde{\xi}}, g_{\tilde{\xi}} : \mathcal{T} \times \Omega \rightarrow \mathbb{R}, \mathcal{T} = [0, 1] \subset \mathbb{R}$. In addition, we assume that these processes are second order stochastic processes.

As can be notice, for every fixed $t \in [0, 1]$function $t \rightarrow (f_{\tilde{\xi}}(t, \omega), g_{\tilde{\xi}}(t, \omega))$is a random variable with values in $\mathbb{R}^2$(random vector), and for each fixed $\omega \in \Omega$values of function $\omega \rightarrow (f_{\tilde{\xi}}(t, \omega), g_{\tilde{\xi}}(t, \omega))$are ordered fuzzy numbers whose functions f and g are determined by the trajectories of stochastic processes.

For such defined ordered fuzzy random variables, terms such as expected value and variance can be defined as follows.

Definition 5. *The fuzzy expected value* of ordered fuzzy random variable $\tilde{\xi} : \Omega \rightarrow \mathcal{O}$ is defined as ordered fuzzy number $\mathbb{E}\left[\tilde{\xi}\right]$ such that

$$\mathbb{E}\left[\tilde{\xi}\right] = \left(f_{m_{\tilde{\xi}}}, g_{m_{\tilde{\xi}}}\right), \tag{10}$$

where functions $f_{m_{\tilde{\xi}}}$ and $g_{m_{\tilde{\xi}}}$ are the expected values of stochastic processes $f_{\tilde{\xi}}$ and $g_{\tilde{\xi}}$, respectively.

Definition 6. *The fuzzy variance* of ordered fuzzy random variable $\tilde{\xi} : \Omega \rightarrow \mathcal{O}$ is defined as ordered fuzzy number $\mathrm{Var}\left[\tilde{\xi}\right]$ such that

$$\mathrm{Var}\left[\tilde{\xi}\right] = \left(f_{var_{\tilde{\xi}}}, g_{var_{\tilde{\xi}}}\right), \tag{11}$$

where functions $f_{var_{\tilde{\xi}}}$ and $g_{var_{\tilde{\xi}}}$ are the variances of stochastic processes $f_{\tilde{\xi}}$ and $g_{\tilde{\xi}}$, respectively.

It is worth notice that thanks to such definitions, the basic properties of the expected value and variance are maintained.

Theorem 1. *Suppose that exist fuzzy expected values* $\mathbb{E}\left[\tilde{\xi}\right]$ and $\mathbb{E}[\tilde{\eta}]$ of ordered fuzzy random variables $\tilde{\xi}$ and $\tilde{\eta}$. Then

(i)If $\tilde{\xi} \geqslant 0$, *then* $\mathbb{E}\left[\tilde{\xi}\right] \geqslant 0$.

(ii)$|\mathbb{E}\left[\tilde{\xi}\right]| \leqslant \mathbb{E}\left[|\tilde{\xi}|\right]$.

(iii)For $a, b \in \mathbb{R}$ *fuzzy expected value of ordered fuzzy random variable* $a\tilde{\xi} + b\tilde{\eta}$ *exist and*

$$\mathbb{E}\left[a\tilde{\xi} + b\tilde{\eta}\right] = a\mathbb{E}\left[\tilde{\xi}\right] + b\mathbb{E}[\tilde{\eta}].$$

Theorem 2. *Let* $\tilde{\xi}$ *be ordered fuzzy random variable, for which exist fuzzy variance* $\mathrm{Var}\left[\tilde{\xi}\right]$. *Then*

(i) $\mathrm{Var}\left[\tilde{\xi}\right] \geqslant 0$.

(ii) For $c \in \mathbb{R}$ $\mathrm{Var}\left[c\tilde{\xi}\right] = c^2\mathrm{Var}\left[\tilde{\xi}\right]$.

(iii) For $c \in \mathbb{R}$ $\mathrm{Var}\left[\tilde{\xi} + c\right] = \mathrm{Var}\left[\tilde{\xi}\right]$.

Proofs of the above properties follow directly from the properties of the expected value and the variance of classical real-valued random variables because for each $t \in [0, 1]$ $f_{m_{\tilde{\xi}}}(t)$ and $g_{m_{\tilde{\xi}}}(t)$ are real-valued random variables. Proof of the above properties for real-valued random variables can be found in many books on the theory of probability, for example in [8].

As it was noticed by Liu and Liu, in practical issues, the concept of the expected value is often required as a crisp number rather than a fuzzy number, so that it will be possible to indicate the value based on which various decisions will be made. Similarly to the approach proposed by Liu and Liu, the definitions of the crisp expectation value and variance for ordered fuzzy random variables using the defuzzification operator, defined as the expected value operator, are introduced.

Definition 7. *Expected value operator* for the ordered fuzzy number $A = (f_A, g_A)$is defined as the defuzzification operator $E : \mathcal{O} \rightarrow \mathbb{R}$specified by the formula

$$E(A) = E(f_A, g_A) = \frac{1}{2}\int_0^1 [f_A(s) + g_A(s)]ds. \tag{12}$$

The expected value operator defined above is equivalent to the expected value operator defined by Liu and Liu (see Def. 1). This equivalence is determined by the following theorem.

Theorem 3. *Let* $A = (f, g)$ *be ordered fuzzy number such that exist inverse functions* f^{-1}, g^{-1} *and* $f(0) \leqslant f(1) \leqslant g(1) \leqslant g(0)$. *It means that the ordered fuzzy number Acan be represented as a classic convex fuzzy number* A^* *with membership function* $\mu_{A^*} : \mathbb{R} \rightarrow [0, 1]$ *as follows*

$$\mu_{A^*}(x) = \begin{cases} f^{-1}(x) & \text{if} \quad x \in [f(0), f(1)], \\ g^{-1}(x) & \text{if} \quad x \in [g(1), g(0)], \\ 1 & \text{if} \quad x \in [f(1), g(1)], \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Then the following equality is fulfilled

$$E(A^*) = E(A), \tag{14}$$

where E is the operator of the expected value according to the Definition 1.

Proof. According to the Eq. (5) the credibility measure for the fuzzy number A^* is given by the formula

$$Cr\{A^* \geqslant t\} = \begin{cases} 1 & \text{if} \quad t < f(0), \\ 1 - \frac{1}{2}f^{-1}(t) & \text{if} \quad t \in [f(0), f(1)], \\ \frac{1}{2} & \text{if} \quad t \in [f(1), g(1)], \\ \frac{1}{2}g^{-1}(t) & \text{if} \quad t \in [g(1), g(0)], \\ 0 & \text{if} \quad t > g(0). \end{cases} \tag{15}$$

By Definition 1, the expected value operator of the fuzzy number $A^*$is given by the formula

$$E(A^*) = \int_0^{\infty} Cr\{A^* \geqslant t\}dt - \int_{-\infty}^0 Cr\{A^* \leqslant t\}dt. \tag{16}$$

Using the (5) formula and the integral property we get

$$E(A^*) = \int_{-\infty}^{\infty} Cr\{A^* \geqslant t\}dt - \int_{-\infty}^{0} 1dt. \tag{17}$$

Then by inserting the formula (15) into the Eq. (17), using the linearity of the integral, and then grouping the appropriate integrals, we get

$$\begin{aligned} &E(A^*) = I_1 + I_2, \text{ where} \\ &I_1 = \textstyle\int_{-\infty}^{f(0)} 1dt + \int_{f(0)}^{f(1)} 1dt + \frac{1}{2}\int_{f(1)}^{g(1)} 1dt - \int_{-\infty}^{0} 1dt, \end{aligned} \tag{18}$$

$$I_2 = -\frac{1}{2}\left[\int_{f(0)}^{f(1)} f^{-1}(t)dt + \int_{g(0)}^{g(1)} g^{-1}(t)dt\right]. \tag{19}$$

The expression I_1, regardless of the location of the point 0 relative to the points $f(0)$,$f(1)$ and $g(1)$is

$$I_1 = \frac{1}{2}[f(1) + g(1)]. \tag{20}$$

The expression I_2, by substitution of appropriate integrals $s = f^{-1}(t)$ and $s = g^{-1}(t)$, and then applying the integration by parts for both integrals is reduced to following expression

$$I_2 = -\frac{1}{2}[f(1) + g(1)] + \frac{1}{2}\int_0^1 [f(s) + g(s)]ds. \tag{21}$$

Finally, adding up the terms I_1 and I_2 we get the right side of the Eq. (14). □

Definition 8. The *crisp expected value* of the ordered fuzzy random variable $\tilde{\xi} : \Omega \to \mathcal{O}$, for which existe a fuzzy expected value $\mathbb{E}\left[\tilde{\xi}\right]$, is defined as the real number $\mathcal{E}\left(\tilde{\xi}\right)$ such that

$$\mathcal{E}\left(\tilde{\xi}\right) = E\left(\mathbb{E}\left[\tilde{\xi}\right]\right) = \frac{1}{2}\int_0^1 \left[f_{m_{\tilde{\xi}}}(t) + g_{m_{\tilde{\xi}}}(t)\right]dt. \tag{22}$$

The crisp expected value of the ordered fuzzy random variable can be calculated in two ways. Simply by definition, i.e., calculate the fuzzy expected value, and then defuzzifine it with the expected value operator. We can also do the opposite, i.e., for fixed ω, calculate the crisp values of ordered fuzzy numbers $\tilde{\xi}(\omega)$. Then $E\left(\tilde{\xi}(\omega)\right)$ is a real-valued random variable, so we can count its expected value in the classic sense. The following theorem shows the equality of both methods.

Theorem 4. *Let $\tilde{\xi}$be an ordered fuzzy random variable specified on the probabilistic space $(\Omega, \mathcal{F}, \Pr)$. Then if exist $\mathbb{E}\left[\tilde{\xi}\right]$, then*

$$E\left(\mathbb{E}\left[\tilde{\xi}\right]\right) = \mathbb{E}\left[E\left(\tilde{\xi}\right)\right], \tag{23}$$

where $\mathbb{E}$ on the right side of the Eq. (23) means the expected value of a classic real-valued random variable.

Proof. According to the formula (22), using the definiton of expected value of the second order stochastic process, from the integral property and from the Fubini's theorem about iterated integrals we get

$$\begin{aligned} E\left(\mathbb{E}\left[\tilde{\xi}\right]\right) &= \tfrac{1}{2}\textstyle\int_0^1 \left[f_{m_{\tilde{\xi}}}(t) + g_{m_{\tilde{\xi}}}(t)\right]dt = \int_0^1 \frac{1}{2}f_{m_{\tilde{\xi}}}(t)dt + \int_0^1 \frac{1}{2}g_{m_{\tilde{\xi}}}(t)dt \\ &= \textstyle\int_0^1 \mathbb{E}\left[\frac{1}{2}f_{\tilde{\xi}}(t)\right]dt + \int_0^1 \mathbb{E}\left[\frac{1}{2}g_{\tilde{\xi}}(t)\right]dt \\ &= \textstyle\int_0^1 \left[\int_\Omega \frac{1}{2}f_{\tilde{\xi}}(t,\omega)\Pr(d\omega)\right]dt + \int_0^1 \left[\int_\Omega \frac{1}{2}g_{\tilde{\xi}}(t,\omega)\Pr(d\omega)\right]dt \\ &= \textstyle\int_\Omega \left[\frac{1}{2}\int_0^1 f_{\tilde{\xi}}(t,\omega)dt + \frac{1}{2}\int_0^1 g_{\tilde{\xi}}(t,\omega)dt\right]\Pr(d\omega) \\ &= \mathbb{E}\left[\tfrac{1}{2}\textstyle\int_0^1 \left[f_{\tilde{\xi}}(t) + g_{\tilde{\xi}}(t)\right]dt\right] = \mathbb{E}\left[E\left(\tilde{\xi}\right)\right]. \end{aligned}$$

□

The crisp variance of the ordered fuzzy random variable is defined as the variance of the classical real-valued random variable, which is created by defuzzifine ordered fuzzy random variable. Thus, it is the mean square of deviation of the defuzzifine values from the crisp expected value.

Definition 9. The *crisp variance* of the ordered fuzzy random variable $\tilde{\xi} : \Omega \to \mathcal{O}$, for which $\mathcal{E}\left[\tilde{\xi}\right]$ exists, is defined as a real number $\mathcal{V}\left(\tilde{\xi}\right)$ such that

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120

$$\mathcal{V}\left(\tilde{\xi}\right) = \mathrm{Var}\left[E\left(\tilde{\xi}\right)\right] = \mathbb{E}\left[\left(E\left(\tilde{\xi}\right) - \mathcal{E}\left(\tilde{\xi}\right)\right)^2\right], \tag{24}$$

where $\mathbb{E}$ and Var mean the expected value and the variance in the classic sense.

4. Concept of normality of OFRVs

For empirical issues, the most important issue is the ability to study and generate random numbers with given probability distributions. As the concept of a fuzzy random variable is defined by considering appropriately defined classic random variables, so defining the probability distribution of the fuzzy random variable is limited to determining the distributions of these random variables [31,35,7,3,2].

In this article, a definition of a normal probability distribution for an ordered fuzzy random variable is presented. However, unlike the classic convex fuzzy numbers, ordered fuzzy numbers have an additional property, namely the orientation, which should also be taken into account.

Definition 10. An ordered fuzzy random variable $\tilde{\xi}$ has *normal distrubution with parameters* $\tilde{\mu}, \tilde{\sigma}^2 \in \mathcal{O}, \sigma^2 \in \mathbb{R}_+$ *and* $p \in [0,1]$ such that $Ord(\tilde{\mu}) \geqslant 0, \tilde{\sigma}^2 > 0$ and $\sigma^2 > 0$, what is denoted as $\tilde{\xi} \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2, \sigma^2, p)$, if for each $t \in [0,1]$ real-valued random variables $f_{\tilde{\xi}}(t)$ and $g_{\tilde{\xi}}(t)$ have following mixed normal distrubutions

$$f_{\tilde{\xi}}(t) \sim p \cdot \mathcal{N}\left(f_{\tilde{\mu}}(t), f_{\tilde{\sigma}^2}(t) + \sigma^2\right) + (1-p) \cdot \mathcal{N}\left(g_{\tilde{\mu}}(t), g_{\tilde{\sigma}^2}(t) + \sigma^2\right)$$
$$g_{\tilde{\xi}}(t) \sim p \cdot \mathcal{N}\left(g_{\tilde{\mu}}(t), g_{\tilde{\sigma}^2}(t) + \sigma^2\right) + (1-p) \cdot \mathcal{N}\left(f_{\tilde{\mu}}(t), f_{\tilde{\sigma}^2}(t) + \sigma^2\right).$$

Furthermore, $\Pr\left(Ord\left(\tilde{\xi}\right) \geqslant 0\right) = p$and $\Pr\left(Ord\left(\tilde{\xi}\right) < 0\right) = 1 - p$, where $Ord$is the order operator defined using the expected value operator (see (7)).

It is worth notice that unlike the classical normal distribution, the parameters $\tilde{\mu}, \tilde{\sigma}^2$are not a expected value and a variance of $\tilde{\xi}$(neither fuzzy nor crisp). The fuzzy expected value and fuzzy variance of $\tilde{\xi}$are expressed by formulas:

$$\mathbb{E}\left[\tilde{\xi}\right] = p \cdot \left(f_{\tilde{\mu}}, g_{\tilde{\mu}}\right) + (1-p) \cdot \left(g_{\tilde{\mu}}, f_{\tilde{\mu}}\right), \tag{25}$$
$$\mathrm{Var}\left[\tilde{\xi}\right] = p(1-p) \cdot \left(\left(f_{\tilde{\mu}}, g_{\tilde{\mu}}\right) - \left(g_{\tilde{\mu}}, f_{\tilde{\mu}}\right)\right)^2 + p \cdot (f_{\tilde{\sigma}^2}, g_{\tilde{\sigma}^2}) + (1-p) \cdot (g_{\tilde{\sigma}^2}, f_{\tilde{\sigma}^2}) + \sigma^2. \tag{26}$$

In addition, estimators of parameters for the distribution defined in this way can be determined as follows.

Definition 11. Let $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n$ be a random sample of ordered fuzzy random variables with normal distribution with parameters $\tilde{\mu}, \tilde{\sigma}^2, \sigma^2$and p. Then

- Estimator of fuzzy expected value is defined as:

$$\hat{\mathbb{E}}\left[\tilde{\xi}\right] = \frac{1}{n}\sum_{i=1}^{n} \tilde{\xi}_i. \tag{27}$$

- Estimator of fuzzy variance is defined as:

$$\hat{\mathrm{Var}}\left[\tilde{\xi}\right] = \frac{1}{n-1}\sum_{i=1}^{n}\left(\tilde{\xi}_i - \hat{\mathbb{E}}\left[\tilde{\xi}\right]\right)^2. \tag{28}$$

- Estimator of parameter σ^2 is defined as:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(E\left(\tilde{\xi}_i\right) - E\left(\hat{\mathbb{E}}\left[\tilde{\xi}\right]\right)\right)^2. \tag{29}$$

- Estimator of probability p is defined as:

$$\hat{p} = \frac{\#\left\{i \in \{1, 2, \ldots, n\} : Ord\left(\tilde{\xi}_i\right) \geqslant 0\right\}}{n}. \tag{30}$$

- Estimators of parameter $\tilde{\mu}$ and $\tilde{\sigma}^2$ are defined as:

$$\hat{\tilde{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \tilde{\xi}_i^*, \tag{31}$$

$$\hat{\tilde{\sigma}}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\tilde{\xi}_i^* - \hat{\tilde{\mu}}\right)^2, \tag{32}$$

where

$$\tilde{\xi}_i^* = \begin{cases} \left(f_{\tilde{\xi}_i}, g_{\tilde{\xi}_i}\right), & \text{gdy } Ord\left(\tilde{\xi}_i\right) \geqslant 0, \\ \left(g_{\tilde{\xi}_i}, f_{\tilde{\xi}_i}\right), & \text{gdy } Ord\left(\tilde{\xi}_i\right) < 0. \end{cases} \tag{33}$$

To construct an ordered fuzzy random variable with a normal distribution with fuzzy parameters we use the following lemma.

Lemma 1. *Let $\tilde{\mu}, \tilde{\sigma}^2 \in \mathcal{O}, \sigma^2 \in \mathbb{R}_+$ be fixed parameters such that $\tilde{\mu} > 0, \tilde{\sigma}^2 > 0$ (for all $t \in [0,1]$) and $Ord(\tilde{\mu}) \geqslant 0$. Furthermore, let $\tilde{\eta}$ and $\tilde{\zeta}$ be ordered fuzzy random variables with stochastic processes $f_{\tilde{\eta}}, g_{\tilde{\eta}}$ and $f_{\tilde{\zeta}}, g_{\tilde{\zeta}}$ which are specified in following way*

$$f_{\tilde{\eta}}(t) = X_t \quad \wedge \quad g_{\tilde{\eta}}(t) = Y_t, \tag{34}$$

$$f_{\tilde{\zeta}}(t) = S \quad \wedge \quad g_{\tilde{\zeta}}(t) = S, \tag{35}$$

where $X_t \sim \mathcal{N}\left(1, \frac{f_{\tilde{\sigma}^2}(t)}{f^2_{\tilde{\mu}}(t)}\right), Y_t \sim \mathcal{N}\left(1, \frac{g_{\tilde{\sigma}^2}(t)}{g^2_{\tilde{\mu}}(t)}\right)$ for each $t \in [0,1]$ and $S \sim \mathcal{N}(0, \sigma^2)$. Then, the ordered fuzzy random variable $\tilde{\xi} = \tilde{\mu} \cdot \tilde{\eta} + \tilde{\zeta}$ has normal distrubution with parameters $\tilde{\mu}_{\tilde{\xi}} = \tilde{\mu}, \tilde{\sigma}^2_{\tilde{\xi}} = \tilde{\sigma}^2, \sigma^2_{\tilde{\xi}} = \sigma^2$ and $p = 1$.

The proof of the above lemma implies directly from the properties of the real-valued random variables with a normal distribution such as affine property and property of the sum of two normal random variables. Based on this lemma, an algorithm for generating ordered fuzzy pseudorandom numbers with a normal distribution with given parameters $\tilde{\mu}, \tilde{\sigma}^2 \in \mathcal{O}, \sigma^2 \in \mathbb{R}_+$ and $p \in [0,1]$ is presented below. For numerical calculations, it was assumed that all ordered fuzzy numbers are determined in a discrete manner on the interval $[0,1]$ using $T+1$ points, $t \in \{0, 1, \ldots, T\}$.

Algorithm:

Generate pseudorandom ordered fuzzy number with normal distribution.
Input: $\tilde{\mu}, \tilde{\sigma}^2 \in \mathcal{O}, \sigma^2 \in \mathbb{R}_+, p \in [0,1]$
Output: $\tilde{\xi} \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2, \sigma^2, p)$
Step 1. Calculate constant $C := |\min\left\{\min_{t\in\{0,1,\ldots,T\}}\left\{f_{\tilde{\mu}}(t)\right\}, \min_{t\in\{0,1,\ldots,T\}}\left\{g_{\tilde{\mu}}(t)\right\}, 0\right\}|$.
Step 2. Create ordered fuzzy number $\tilde{\eta} := \tilde{\mu} + C$.
Step 3. Generate pseudorandom real number with normal distribution $S \sim \mathcal{N}(0, \sigma^2)$.
Step 4. Generate two sequence X_t and Y_t ($t \in \{0, 1, \ldots, T\}$) of pseudorandom real numbers with following normal distributions

$$X_t \sim \mathcal{N}\left(1, \frac{f_{\tilde{\sigma}^2}(t)}{f^2_{\tilde{\eta}}(t)}\right),$$

$$Y_t \sim \mathcal{N}\left(1, \frac{g_{\tilde{\sigma}^2}(t)}{g^2_{\tilde{\eta}}(t)}\right).$$

Step 5. Generate pseudorandom real number with uniform distribution $r \sim \mathcal{U}[0,1]$.
Step 6.
IF $r < p$ **THEN**
Create ordered fuzzy number $\tilde{\xi}$ such that

$$\left(f_{\tilde{\xi}}(t), g_{\tilde{\xi}}(t)\right) = (X_t, Y_t) \cdot \left(f_{\tilde{\eta}}(t), g_{\tilde{\eta}}(t)\right) + S.$$

ELSE
Create ordered fuzzy number $\tilde{\xi}$ such that

$$\left(f_{\tilde{\xi}}(t), g_{\tilde{\xi}}(t)\right) = (X_t, Y_t) \cdot \left(g_{\tilde{\eta}}(t), f_{\tilde{\eta}}(t)\right) + S.$$

Step 7. Return ordered fuzzy number $\tilde{\xi} := \tilde{\xi} - C$.

Example Let $\tilde{\xi}$ be an ordered fuzzy random variable with normal distribution with parameters $p = 0.5, \sigma^2 = 1$ and $\tilde{\mu}, \tilde{\sigma}^2$ specified as in Fig. 2. The parameter $\tilde{\mu}$is defined as the Gaussian OFN with parameters $\mu_f = \mu_g = 5$ and $\sigma_f = -\sigma_g = -0.5$. The $\tilde{\sigma}^2$is a Triangular OFN with the parameters $f(0) = 0.005, f(1) = g(1) = 0.01, g(0) = 0.02$. Figs. 3 and 4 show random samples of ordered fuzzy variable ξ generated according to the proposed algorithm for $n = 20$ and 100, respectively (charts on the left). Comparison between the fuzzy expected value/variance (fat line) and the fuzzy sample mean/variance (dash line) are shown also in Fig. 3 and Fig. 4 (charts on the right).

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120

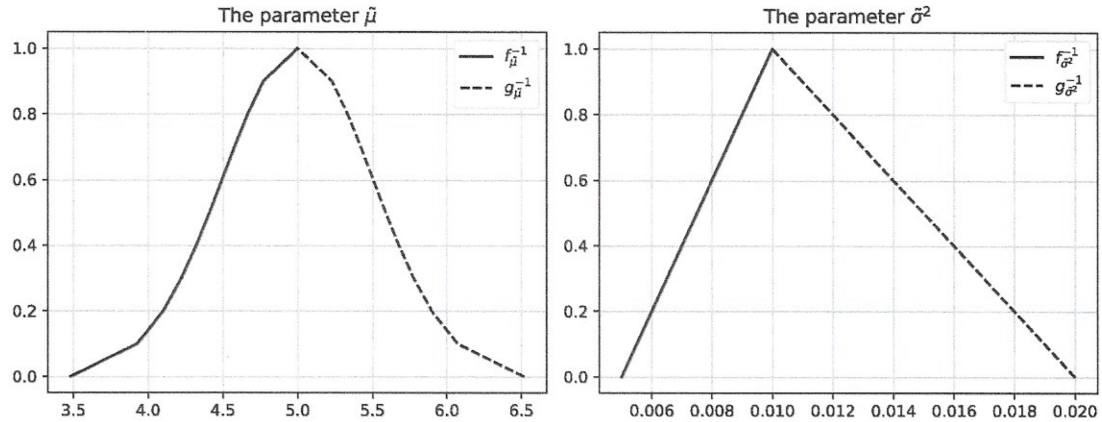

Fig. 2. Fuzzy parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$.

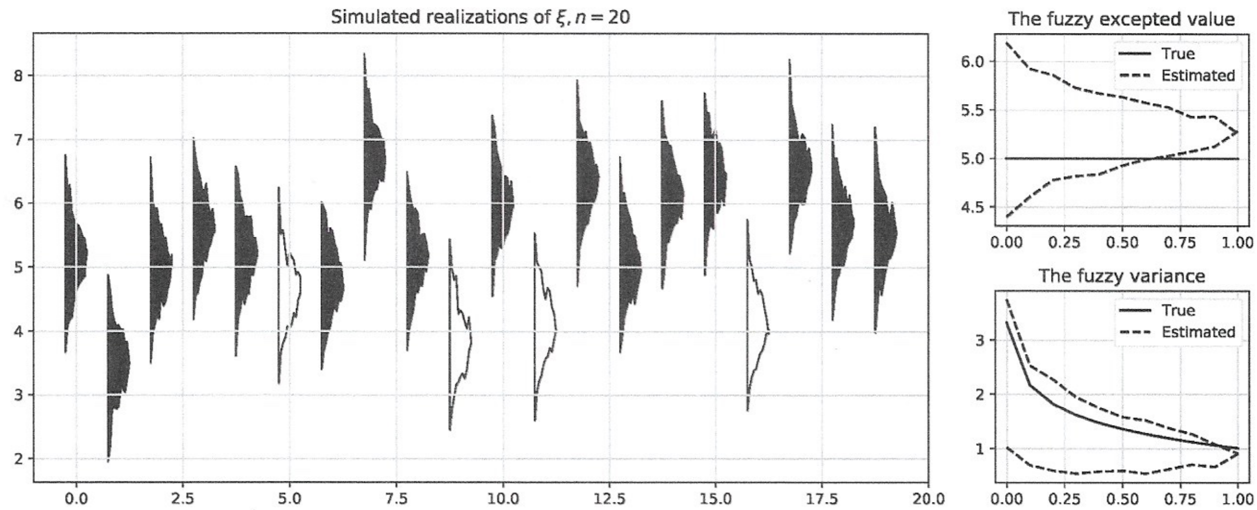

Fig. 3. Random sample of $\tilde{\xi}$ and their sample means and variance for $n = 20$.

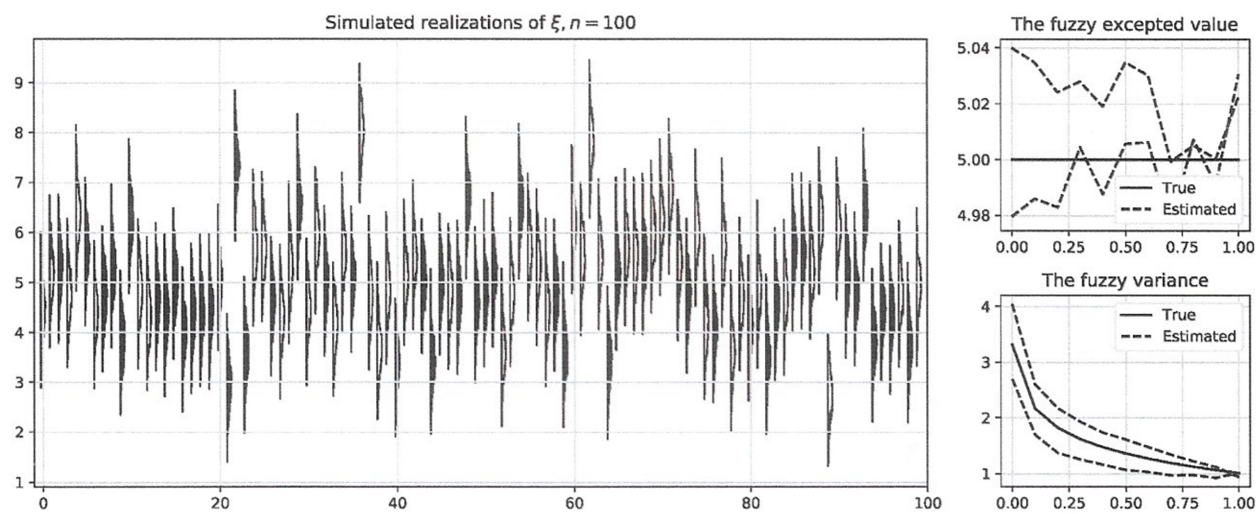

Fig. 4. Random sample of $\tilde{\xi}$ and their sample means and variance for $n = 100$.

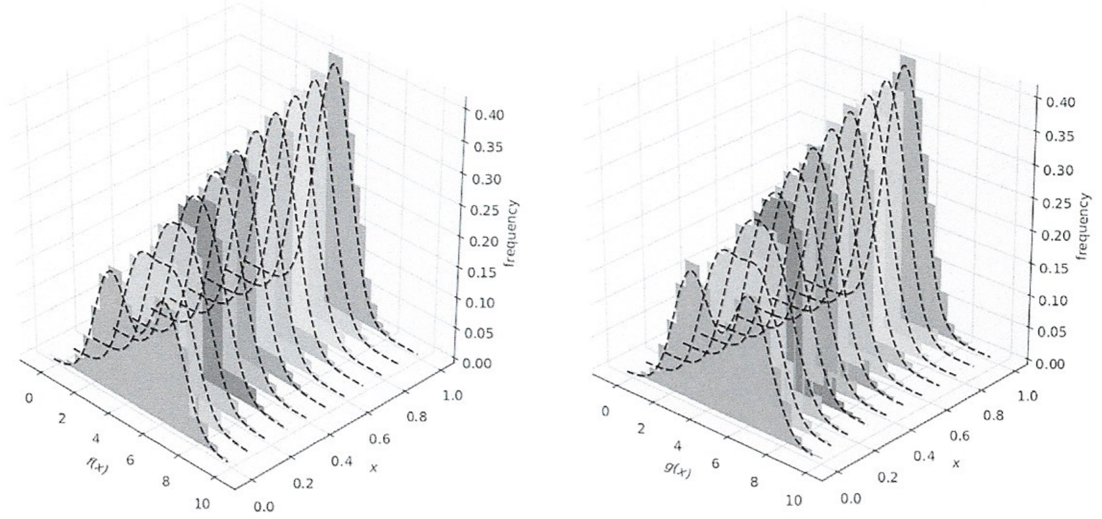

Fig. 5. Normalized histograms and theoretical density functions of fuzzy random variables $\tilde{\xi}$ for several values of $x \in [0, 1]$.

Table 1
Average distances between the theoretical values of the parameters $\mathbb{E}, Var, \tilde{\mu}, \tilde{\sigma}^2, p$, and their estimators depending on the sample size n.

n	10	50	100	1000	10,000	100,000	1,000,000
$\lvert p - \hat{p} \rvert$	0.1800	0.0420	0.0420	0.0096	0.0022	0.0015	0.0003
$d(\mathbb{E}, \hat{\mathbb{E}})$	0.6775	0.3348	0.2273	0.0567	0.0145	0.0063	0.0018
$d\left(Var, \hat{Var}\right)$	0.9243	0.4884	0.3955	0.0901	0.0385	0.0099	0.0034
$\lvert \sigma^2 - \hat{\sigma}^2 \rvert$	0.2735	0.1382	0.1227	0.0213	0.0084	0.0034	0.0009
$d\left(\tilde{\mu}, \hat{\tilde{\mu}}\right)$	0.4079	0.2933	0.1661	0.0496	0.0133	0.0049	0.0016
$d\left(\tilde{\sigma}^2, \hat{\tilde{\sigma}}^2\right)$	0.0815	0.0400	0.0295	0.0118	0.0047	0.0029	0.0027

Fig. 5 shows normalized histograms and theoretical density functions of fuzzy random variables $\tilde{\xi}$ for several values of $x \in [0, 1]$ and $n = 10,000$. However, the Table 1 shows the average distances between the theoretical values of the parameters $\mathbb{E}, Var, \tilde{\mu}, \tilde{\sigma}^2, p$, and their estimators depending on the sample size n (10 simulations for each n). The distance between scalars was calculated as the absolute value of the difference, while between the ordered fuzzy numbers using the metric d given by formula (8). The obtained results allow us to suppose that the proposed algorithm for generating ordered fuzzy pseudorandom numbers works correctly, and the adopted estimators of normal distribution parameters are consistent estimators.

5. Conclusion

In this paper the model of ordered fuzzy numbers was extended by introducing the definition of ordered fuzzy random variables. Moreover, the concept of normal distribution and method to simulate pseudorandom ordered fuzzy numbers with this distribution were also presented. Simulation of OFRV with normal distribution was studied because of their natural applications in econometric models but the methods presented can be extended to simulate other distribution.

Some numerical examples were included to indicate the treatment of this approach. Further to study the accuracy and validity of the method, theoretical fuzzy and crisp parameters of normal distribution were compared with their estimated counterparts in some different sample size. The obtained results were shown that estimated parameters move towards theoretical parameters with growing the sample size. It means that the simulation method operates accurately and generates valid data according to the target.

Now, thanks to our work it becomes possible to construct fully fuzzy stochastic time series models such as e.g., the autoregressive model or the GARCH model in the form of classical equations with error term as ordered fuzzy random variables. In general, such a model can be represented as follows

$$\tilde{Y}_t = F\left(\tilde{A}_i; \tilde{X}_{ti}\right) + \tilde{\varepsilon}_t, \tag{36}$$

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120

where $\tilde{Y}_t$ is explained variable, OFN $(t = 1, \ldots, T)$; $\tilde{X}_{ti}$ are explanatory variables OFN, $(i = 1, \ldots, n)$; $\tilde{A}_i$ are model coefficients, OFN; $\tilde{\varepsilon}_t$ is error term represented by OFRV. The F function is any functional dependence specified on ordered fuzzy numbers.

Assuming a fixed distribution of a random component $\tilde{\varepsilon}_t$ (e.g. normal distribution, introduced in Definition 10), we can estimate coefficients using the least-squares or the maximum likelihood method and then analyze the residuals of the model. Moreover, thanks to the ability to generate pseudo-random ordered fuzzy numbers, we can simulate many different realizations of the process and using such a model to solve problems where multiple simulations of a given process are required. For example, in our future work we are going to use the Ordered Fuzzy GARCH model in the problem of option pricing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Adam Marszałek: Conceptualization, Methodology, Software, Validation, Validation, Formal analysis, Investigation, Writing - original draft, Visualization. **Tadeusz Burczyński:** Writing - review & editing, Supervision.

References

[1] A. Bargiela, W. Pedrycz, Granular Computing: An Introduction, Kluwer Academic, Dordrecht, 2003, https://doi.org/10.1007/978-1-4615-1033-8.
[2] A. Colubi, R. Coppi, P. D'Urso, M.A. Gil, Statistics with fuzzy random variables. Metron - International Journal of, Statistics LXV (2007) 277–303.
[3] A. Colubi, C.F. García, M.A. Gil, Simulation of random fuzzy variables: an empirical approach to statistical/probabilistic studies with fuzzy experimental data, IEEE Trans. Fuzzy Syst. 10 (2002) 384–390.
[4] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets: Theory and Applications, World Scientific, Singapur, 1994.
[5] D. Dubois, H. Prade, Fuzzy real algebra: some results, Fuzzy Sets Syst. 2 (1979) 327–348, https://doi.org/10.1016/0165-0114(79)90005-8, URL: http://www.sciencedirect.com/science/article/pii/0165011479900058.
[6] Y. Feng, L. Hu, H. Shu, The variance and covariance of fuzzy random variables and their applications, Fuzzy Sets Syst. 120 (2001) 487–497, https://doi.org/10.1016/S0165-0114(99)00060-3, URL: http://www.sciencedirect.com/science/article/pii/S0165011499000603.
[7] M.A. Gil, M. López-Díaz, D.A. Ralescu, Overview on the development of fuzzy random variables, Fuzzy Sets Syst. 157 (2006) 2546–2557, URL: http://www.sciencedirect.com/science/article/pii/S016501140600203X. Fuzzy Sets and Probability/Statistics Theories.
[8] C.M. Grinstead, J.L. Snell, Introduction to Probability. AMS (2003). URL: http://www.dartmouth.edu/chance/teaching_aids/books_articles/probability_book/book.html. .
[9] G.J. Klir, Fuzzy arithmetic with requisite constraints, Fuzzy Sets Syst. 91 (1997) 165–175, https://doi.org/10.1016/S0165-0114(97)00138-3.
[10] W. Kosiński, On soft computing and modelling, Image Process. Commun. 11 (2006) 71–82.
[11] W. Kosiński, P. Prokopowicz, Algebra of fuzzy numbers, Mathematica Applicanda J. Pol. Math. Soc. 32 (2004) 37–63.
[12] W. Kosiński, P. Prokopowicz, D. Ślęzak, Drawback of fuzzy arithmetic – new intuitions and propositions, in: T. Burczyński, W. Cholewa, W. Moczulski (Eds.), Methods of Artificial Intelligence, PACM, Gliwice, 2002, pp. 231–237.
[13] W. Kosiński, P. Prokopowicz, D. Ślęzak, On algebraic operations on fuzzy numbers, in: M.A. Kłopotek, S.T. Wierzchoń, & K. Trojanowski (Eds.), Intelligent Information Processing and Web Mining, 2003, Springer, Berlin Heidelberg, Berlin, Heidelberg, 2003, pp. 353–362.https://doi.org/10.1007/978-3-540-36562-4_37. .
[14] W. Kosiński, P. Prokopowicz, D. Ślęzak, Ordered fuzzy numbers, Bull. Pol. Acad. Sci. Ser. Sci. Math. 51 (2003) 327–338.
[15] V. Krätschmer, A unified approach to fuzzy random variables, Fuzzy Sets Syst. 123 (2001) 1–9, https://doi.org/10.1016/S0165-0114(00)00038-5, URL: http://www.sciencedirect.com/science/article/pii/S0165011400000385.
[16] R. Kruse, K.D. Meyer, Statistics with Vague Data, Reidle Publication, Dordrecht, 1987.
[17] H. Kwakernaak, Fuzzy random variables–I. Definitions and theorems, Inf. Sci. 15 (1978) 1–29, https://doi.org/10.1016/0020-0255(78)90019-1.
[18] H. Kwakernaak, Fuzzy random variables–II. Algorithms and examples for the discrete case, Inf. Sci. 17 (1979) 253–278, https://doi.org/10.1016/0020-0255(79)90020-3.
[19] B. Liu, Y. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE Trans. Fuzzy Systems 10 (2002) 445–450, https://doi.org/10.1109/TFUZZ.2002.800692.
[20] Y. Liu, B. Liu, Fuzzy random variables: a scalar expected value operator, FO & DM 2 (2003) 143–160, https://doi.org/10.1023/A:1023447217758.
[21] M.A. Lubiano, M.A. Gil, M. Lpez-Daz, M.T. Lpez, The λ-mean squared dispersion associated with a fuzzy random variable, Fuzzy Sets Syst. 111 (2000) 307–317, https://doi.org/10.1016/S0165-0114(97)00389-8, URL: http://www.sciencedirect.com/science/article/pii/S0165011497003898.
[22] M. Mareš, Weak arithmetics of fuzzy numbers, Fuzzy Sets Syst. 91 (1997) 143–153.https://doi.org/10.1016/S0165-0114(97)00136-X. URL: http://www.sciencedirect.com/science/article/pii/S016501149700136X. Fuzzy Arithmetic. .
[23] A. Marszałek, Ordered fuzzy numbers in modelling and simulation of financial time series (in Polish). Ph.D. thesis IPPT PAN, 2017. .
[24] A. Marszałek, T. Burczyński, Modeling and forecasting financial time series with ordered fuzzy candlesticks, Inf. Sci. 273 (2014) 144–155, https://doi.org/10.1016/j.ins.2014.03.026, URL: http://www.sciencedirect.com/science/article/pii/S0020025514003107.
[25] G. Matheron, Random sets and integral geometry. Wiley series in probability and mathematical statistics: Probability and mathematical statistics, Wiley, 1975. URL: https://books.google.pl/books?id=bgzvAAAAMAAJ. .
[26] S. Nahmias, Fuzzy variables, Fuzzy Sets Syst. 1 (1978) 97–110, https://doi.org/10.1016/0165-0114(78)90011-8, URL: http://www.sciencedirect.com/science/article/pii/0165011478900118.
[27] W. Näther, R. Körner, Linear regression with random fuzzy numbers, in: Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics, and Neural Network Approach, Springer, US, Boston, MA, 1998, pp. 193–212.https://doi.org/10.1007/978-1-4615-5473-8_13. .
[28] P. Prokopowicz, Flexible and simple methods of calculations on fuzzy numbers with the ordered fuzzy numbers model, in: L. Rutkowski, M. Korytkowski, R. Scherer, R. Tadeusiewicz, L.A. Zadeh, & J.M. Zurada (Eds.), Artificial Intelligence and Soft Computing, Springer, Berlin Heidelberg, Berlin, Heidelberg, 2013, pp. 365–375.https://doi.org/10.1007/978-3-642-38658-9_33. .
[29] P. Prokopowicz, The use of ordered fuzzy numbers for modelling changes in dynamic processes, Inf. Sci. 470 (2019) 1–14, https://doi.org/10.1016/j.ins.2018.08.045, URL: http://www.sciencedirect.com/science/article/pii/S0020025517303705.

[30] P. Prokopowicz, D. Ślęzak, Ordered fuzzy numbers: Definitions and operations, in: P. Prokopowicz, J. Czerniak, D. Mikołajewski, Ł. Apiecionek, & D. Śl?zak (Eds.), Theory and Applications of Ordered Fuzzy Numbers: A Tribute to Professor Witold Kosiński, Springer International Publishing, Cham, 2017, pp. 57–79.https://doi.org/10.1007/978-3-319-59614-3_4..
[31] M.L. Puri, D.A. Ralescu, The concept of normality for fuzzy random variables, Ann. Probab. 13 (1985) 1373–1379, https://doi.org/10.1214/aop/1176992822.
[32] M.L. Puri, D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422, https://doi.org/10.1016/0022-247X(86)90093-4, URL: http://www.sciencedirect.com/science/article/pii/0022247X86900934.
[33] E. Sanchez, Solution of fuzzy equations with extended operations, Fuzzy Sets Syst. 12 (1984) 237–248.
[34] M. Wagenknecht, R. Hampel, V. Schneider, Computational aspects of fuzzy arithmetics based on archimedean t-norms, Fuzzy Sets Syst. 123 (2001) 49–62.
[35] H. Wu, Statistical confidence intervals for fuzzy data, Expert Syst. Appl. 36 (2009) 2670–2676.
[36] L. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems, Fuzzy Sets Syst. 11 (1983) 199–227, https://doi.org/10.1016/S0165-0114(83)80081-5, URL: http://www.sciencedirect.com/science/article/pii/S0165011483800815.
[37] H.-J. Zimmermann, Fuzzy set theory, WIREs Comput. Stat. 2 (2010) 317–332, https://doi.org/10.1002/wics.82.

Please cite this article as: A. Marszałek and T. Burczyński, Ordered fuzzy random variables: Definition and the concept of normality, Information Sciences, https://doi.org/10.1016/j.ins.2020.08.120