HOMOGENIZATION OF PIEZOELECTRIC SOLID AND THERMODYNAMICS

R. WOJNAR

Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warszawa, Poland
(e-mail: rwojnar@ippt.gov.pl)

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The problem of homogenization of a piezoelectric periodic composite in which thermal effects are taken into account is treated by a method of 2-scale asymptotic expansions. Albeit a nonlinear entropy-temperature relation is used, the local problems are formulated in a way similar to that of a linear theory. Also, effective material coefficients are in majority the same as those obtained in a linearized theory. The difference appears in the homogenized thermoelastic, thermoelectric and specific heat coefficients only. A Francfort type result on shift of the initial conditions for a homogenized problem is also obtained.

1. Introduction

There are two ways of every man's activity, especially in physics; analytical one in which a general idea is used to understand inner structure of matter, and a synthetical one when knowledge of the properties of a number of elements leads to a general result. The atomic paradigm is the most known example of such conceptual approach and statistical physics is one of the ways of its realization. Another program, associated with some of the most famous names in science, is to determine the properties of a composite material from those of its components considered at the level not as deep as atomic or molecular. J. C. Maxwell (1873) and Lord Rayleigh (1892) computed the effective conductivity of composites consisting of a matrix with spherical and cylindrical inclusions, respectively, [1, 2]. Maxwell's result was verified experimentally by Lise Meitner in 1906 [3]. In the same year Albert Einstein proposed a method of calculating the effective viscosity of a dilute suspension of rigid spherical particles [4]. In 1910, assuming uniform strain in a polycrystalline aggregate, W. Voigt found the lower bounds on its elastic properties [5]. The elastic properties of such an aggregate were calculated by I. M. Lifshitz and L. N. Rozenzweig in 1946, [6]. The main idea of these works, especially visible in the Einstein's dissertation was that the inclusions resulted in the modified equations of motion for a matrix with "corrected" material coefficients.

A homogenization procedure is one of the many mathematical methods that lead to finding the properties of composite materials. Its idea is based on a small parameter series expansion, similar to that proposed by G. Sandri in the 60's [7], and, in particular, on so called two-scale asymptotic expansions. For example, to study wave propagation in a periodically nonhomogeneous elastic body of period Y one introduces Y-scale of the heterogeneity for material coefficients, and one looks for a homogenized wave of length much greater then Y. The two-scale homogenization methods were discussed by G. Duvaut [8], A. Bensoussan, J. L. Lions and G. Papanicolau [9], and E. Sanchez-Palencia [10].

Thus, the homogenization is a mathematical procedure in which a heterogeneous body (elastic solid) is replaced by a homogeneous one, the physical properties of which are to some extent equivalent to those of the original body. As a result, an initial boundary value problem for a nonhomogeneous elastic solid is replaced by an initial boundary value problem for a homogeneous body. In [11] a homogenization procedure was described for a thermopiezoelectric composite. The case was studied by a 2-scale asymptotic expansions method and the results were obtained under a linear entropy--temperature relationship. Such a hypothesis inherent for a linear thermoelasticity was used earlier in G. A. Francfort's works on the homogenization of a linear thermoelastic composite [12, 13].

It appears however that linear thermoelasticity in which the entropy s is a linear function of temperature T (cf. D. E. Carlson [14], W. Nowacki [15]), and the nonlinear term $T\dot{s}$ in the energy equation is replaced by the linear terms $T_0\dot{s}$ (T_0 = reference temperature), is overlinearized as far as a homogenization procedure is concerned.

In the present paper we are to outline a homogenization procedure for a periodic piezoelectric composite using a quasi-linear thermoelasticity in which the law of elasticity and the law of heat conduction are linear, but the entropy is a nonlinear function of temperature, and there is no need to linearize the term $T\dot{s}$ in the energy equation. Such an approach to other problems of thermomechanics has been proposed before by J. Ignaczak [16]. The paper extends the author's results on the homogenization of a quasi-linear thermoelastic body [17] to a quasi-linear piezoelectric thermoelastic body. It is shown that *local* problems are the same as in a linear theory. Also, effective material coefficients are shown to be, in general, the same as those obtained in a linearized theory. The difference appears only for the homogenized thermoelastic, thermoelectric and specific heat coefficients.

To obtain the homogenized material constants, a two scale expansion method (cf. [18–24]) is used.

2. Piezoelectric body

We consider a piezoelectric body occupying a volume Ω and made of the identical elementary cells such that physical properties of the body change periodically and the period is equal to the dimension of an elementary cell.

The relation between strain tensor ϵ_{ij} and displacement u_i is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{2.1}$$

Electric field E_i is given by the gradient of electric potential ϕ

$$E_i = -\frac{\partial \phi}{\partial x_i} . \tag{2.2}$$

The relation between strain ε_{ij}, stress σ_{ij}, absolute temperature T and electrical field E_i is assumed to be linear, cf. Landau and Lifshitz [25], Nowacki [26]

$$\varepsilon_{ij} = a_{ijmn}\sigma_{mn} + \alpha_{ij}(T - T_0) + A_{kij}E_k ,$$

where a_{ijmn}, α_{ij} and A_{kij} denote the compliance, thermal expansion and strain-electrical tensors, respectively. T_0 is the temperature of reference at which the strain, stress and entropy vanish.

If we define the stress-temperature tensor γ_{ij} by

$$\gamma_{ij} = c_{ijmn}\alpha_{mn}$$

and the piezoelectric tensor π_{kij} by

$$\pi_{kij} = c_{ijmn}A_{kmn} ,$$

we obtain the inverse relation

$$\sigma_{ij} = c_{ijmn}\epsilon_{mn} - \gamma_{ij}(T - T_0) - \pi_{kij}E_k , \tag{2.3}$$

where c_{ijmn} are elasticities. The formula for the electric induction D_i is postulated also in the linear form

$$D_i = \pi_{imn}\epsilon_{mn} + \lambda_i(T - T_0) + \epsilon_{ik}E_k , \tag{2.4}$$

where λ_i and ϵ_{ik} are the induction-temperature and dielectric coefficients.

A nonlinear temperature–entropy relation is assumed in the form

$$T = T_0 e^{(s - \gamma_{ij}\varepsilon_{ij} - \lambda_i E_i)/C_e} , \tag{2.5}$$

where $C_e \equiv C_{e,E,\rho}$ denotes the specific heat at a constant triple $(\varepsilon_{ij}, E_i, \rho)$.

The body is to obey the following balance laws: the equation of motion

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j}\sigma_{ij} , \tag{2.6}$$

Gauss' law

$$\frac{\partial}{\partial x_i}D_i = 0 , \tag{2.7}$$

and the conservation of energy

$$T\dot{s} = -\frac{\partial}{\partial x_i}q_i$$

with the heat flux q_i given by Fourier law

$$q_i = -K_{ij}\frac{\partial}{\partial x_j}T\,.$$

Hence

$$T\dot{s} = \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial}{\partial x_j}T\right) \tag{2.8}$$

or equivalently

$$\dot{s} = K_{ij}\frac{\partial \ln T}{\partial x_i}\frac{\partial \ln T}{\partial x_j} + \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \ln T}{\partial x_j}\right)\,. \tag{2.8'}$$

Here ρ is the density of solid and K_{ij} is the heat conductivity tensor. The coefficients $\rho, c_{ijmn}, \gamma_{ij}$ and K_{ij} satisfy the inequalities

$$\rho > 0\,, \tag{2.9}$$

$$c_{ijmn}\xi_{ij}\xi_{mn} \geq 0\,, \qquad \forall \xi_{ij} \in \mathbb{E}^3_s\,, \tag{2.10}$$

$$\epsilon_{ij}\eta_i\eta_j \geq 0\,, \qquad \forall \eta_i \in \mathbb{R}^3\,, \tag{2.11}$$

$$K_{ij}\eta_i\eta_j \geq 0\,, \qquad \forall \eta_i \in \mathbb{R}^3\,. \tag{2.12}$$

3. Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded, sufficiently regular domain and $(0,\tau)$, $\tau > 0$, a time interval. We identify $\bar{\Omega}$ with the underformed state of the thermopiezoelectric composite with a microperiodic structure. Thus, for $\varepsilon > 0$ the material functions just introduced are εY-periodic, where $Y = (0, Y_1) \times (0, Y_2) \times (0, Y_3)$ is the so-called basic cell, cf. [10]. More precisely, we write

$$c^{\varepsilon}_{ijkl}(x) = c_{ijkl}\left(\frac{x}{\varepsilon}\right)\,, \qquad \pi^{\varepsilon}_{ijk}(x) = \pi_{ijk}\left(\frac{x}{\varepsilon}\right)\,, \quad \text{etc.},$$

where $x \in \Omega$ and the functions c^{ε}_{ijkl}, π^{ε}_{ijk}, etc. are εY-periodic, where $\varepsilon > 0$ is a small parameter.

For a fixed $\varepsilon > 0$ the basic relations describing a linear, thermopiezoelectric solid with the microperiodic structure are then given by

$$\rho\ddot{u}^{\varepsilon}_i = \frac{\partial}{\partial x_j}\left\{c^{\varepsilon}_{ijmn}\frac{\partial u^{\varepsilon}_m}{\partial x_n} - \gamma^{\varepsilon}_{ij}T_0[e^{S^{\varepsilon}} - 1]\right\} + \pi^{\varepsilon}_{kij}\frac{\partial \phi^{\varepsilon}}{\partial x_k}\,, \tag{3.1}$$

$$\frac{\partial}{\partial x_i}\left\{\pi^{\varepsilon}_{imn}\frac{\partial u^{\varepsilon}_m}{\partial x_n} + \lambda^{\varepsilon}_i T_0[e^{S^{\varepsilon}} - 1] - \epsilon^{\varepsilon}_{ij}\frac{\partial}{\partial x_j}\phi^{\varepsilon}\right\} = 0\,, \tag{3.2}$$

$$\dot{s}^{\varepsilon} = K^{\varepsilon}_{ij}\frac{\partial S^{\varepsilon}}{\partial x_i}\frac{\partial S^{\varepsilon}}{\partial x_j} + \frac{\partial}{\partial x_i}\left[K^{\varepsilon}_{ij}\frac{\partial S^{\varepsilon}}{\partial x_j}\right]\,, \tag{3.3}$$

where

$$S^{\varepsilon} = \frac{1}{C^{\varepsilon}_e}\left(s^{\varepsilon} - \gamma^{\varepsilon}_{mn}\frac{\partial u^{\varepsilon}_m}{\partial x_n} + \lambda^{\varepsilon}_n\frac{\partial \phi^{\varepsilon}}{\partial x_n}\right)\,.$$

These are the three equations for fields $\varphi^{\varepsilon}, u^{\varepsilon}_i$ and s^{ε}, and they will be discussed below.

4. Homogenization—an outline of the procedure

The 2-scale asymptotic expansions for $u_i^\varepsilon, \phi^\varepsilon$ and s^ε are postulated (for $\varepsilon > 0, \varepsilon \to 0$),

$$u_i^\varepsilon = u_i^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon u_i^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 u_i^{(2)}(\mathbf{x},\mathbf{y}) + \cdots , \tag{4.1}$$

$$\phi^\varepsilon = \phi^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon \phi^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 \phi^{(2)}(\mathbf{x},\mathbf{y}) + \cdots , \tag{4.2}$$

$$s^\varepsilon = s^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon s^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 s^{(2)}(\mathbf{x},\mathbf{y}) + \cdots . \tag{4.3}$$

The functions $u_i^\varepsilon, \phi^\varepsilon$ and s^ε are εY periodic with respect to $\mathbf{y} = \mathbf{x}/\varepsilon$. Substituting (4.1–4.3) into (3.1–3.3) and taking into account the relation $\partial/\partial x_i f(\mathbf{x},\mathbf{y}) = (\partial/\partial x_i + \varepsilon^{-1}\partial/\partial y_i) f(\mathbf{x},\mathbf{y})$ we get

$$\rho \ddot{u}_i^\varepsilon = \left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left\{c_{ijmn}\left(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)u_m^\varepsilon - \gamma_{ij}T_0[e^{S^\varepsilon} - 1] + \pi_{kij}\left(\frac{\partial}{\partial x_k} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_k}\right)\phi^\varepsilon\right\}, \tag{4.4}$$

$$0 = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\left\{\pi_{imn}\left(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)u_m^\varepsilon + \lambda_i T_0[e^{S^\varepsilon} - 1] - \epsilon_{ij}\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\phi\right\} \tag{4.5}$$

and

$$\dot{s}^\varepsilon = K_{ij}\left[\left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)S^\varepsilon\right]\cdot\left[\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)S^\varepsilon\right] + \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\left\{K_{ij}\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)S^\varepsilon\right\}, \tag{4.6}$$

where

$$S^\varepsilon = \frac{1}{C_e}\left[s^\varepsilon - \gamma_{ab}\left(\frac{\partial}{\partial x_b} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_b}\right)u_a^\varepsilon + \lambda_a\left(\frac{\partial}{\partial x_a} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_a}\right)\phi^\varepsilon\right].$$

The material coefficients $\epsilon_{ij}, \rho, c_{ijmn}, K_{ij}$ etc. in (4.4–4.6) are assumed to be the εY -periodic functions of $\mathbf{y}$ coordinate only.

5. Homogenization of the energy equation

(i) Equating to zero the coefficient at ε^{-4} in (4.6) and denoting

$$f = \frac{1}{C_e}\left[-\gamma_{ab}\frac{\partial}{\partial y_b}u_a^{(0)} + \lambda_a\frac{\partial}{\partial y_a}\phi^{(0)}\right], \tag{5.1}$$

$$f_i = \frac{\partial f}{\partial y_i}, \tag{5.2}$$

we obtain

$$K_{ij} f_i f_j = 0 \,. \tag{5.3}$$

By the positive definiteness of K_{ij}, (5.3) yields $f_i = 0$ or

$$\frac{\partial}{\partial y_i} \left\{ \frac{1}{C_e} \left[-\gamma_{ab} \frac{\partial}{\partial y_b} u_a^{(0)} + \lambda_a \frac{\partial}{\partial y_a} \phi^{(0)} \right] \right\} = 0 \,. \tag{5.4}$$

Integrating over the cell Y we find

$$\frac{1}{C_e} \left[-\gamma_{ab} \frac{\partial}{\partial y_b} u_a^{(0)} + \lambda_a \frac{\partial}{\partial y_a} \phi^{(0)} \right] = \omega(\mathbf{x}, t) \,, \tag{5.5}$$

where the function $\omega(\mathbf{x}, t)$ plays the role of the integration *constant* (for variable $\mathbf{y}$) to be determined.

(ii) Keeping in mind (5.5), the coefficient at ε^{-3} vanishes.

(iii) Denoting

$$S^{(0)} = \frac{1}{C_e} \left[s^{(0)} - \gamma_{ab} \left(\frac{\partial u_a^{(0)}}{\partial x_b} + \frac{\partial u_a^{(1)}}{\partial y_b y} \right) + \lambda_a \left(\frac{\partial \phi^{(0)}}{\partial x_a} + \frac{\partial \phi^{(1)}}{\partial y_a} \right) \right] , \tag{5.6}$$

and equating the coefficient at ε^{-2} to zero, we have

$$0 = K_{ij} \frac{\partial S^{(0)}}{\partial y_i} \frac{\partial S^{(0)}}{\partial y_j} + \frac{\partial}{\partial x_i} \left\{ K_{ij} \frac{\partial f}{\partial y_j} \right\} + \frac{\partial}{\partial y_i} \left\{ K_{ij} \frac{\partial S^{(0)}}{\partial y_j} \right\} , \tag{5.7}$$

and after using (5.5) we arrive at

$$\frac{\partial}{\partial y_i} \left\{ K_{ij} \frac{\partial}{\partial y_j} \exp \left\{ \frac{1}{C_e} \left[s^{(0)} - \gamma_{pq} \left(\frac{\partial u_p^{(0)}}{\partial x_q} + \frac{\partial u_p^{(1)}}{\partial y_q} \right) + \lambda_q \left(\frac{\partial \phi^{(0)}}{\partial x_q} + \frac{\partial \phi^{(1)}}{\partial y_q} \right) \right] \right\} \right\} = 0 \,. \tag{5.7'}$$

Bearing in mind (2.12), the above equation implies

$$\exp \left\{ \frac{1}{C_e} \left[s^{(0)} - \gamma_{pq} \left(\frac{\partial u_p^{(0)}}{\partial x_q} + \frac{\partial u_p^{(1)}}{\partial y_q} \right) + \lambda_q \left(\frac{\partial \phi^{(0)}}{\partial x_q} + \frac{\partial \phi^{(1)}}{\partial y_q} \right) \right] \right\} = C_T(\mathbf{x}, t) \,, \tag{5.8}$$

where $C_T(\mathbf{x}, t)$ is an arbitrary function and the index T indicates a relation of $C_T(\mathbf{x}, t)$ with a homogenized temperature T^H. In the following we show that $C_T(\mathbf{x}, t) = T^H / T_0$ ($C_T > 0$); cf. also (2.5). Taking logarithms on both sides of (5.8) we get

$$\frac{1}{C_e} \left[s^{(0)} - \gamma_{pq} \left(\frac{\partial u_p^{(0)}}{\partial x_q} + \frac{\partial u_p^{(1)}}{\partial y_q} \right) + \lambda_q \left(\frac{\partial \phi^{(0)}}{\partial x_q} + \frac{\partial \phi^{(1)}}{\partial y_q} \right) \right] = \ln C_T(\mathbf{x}, t) \,. \tag{5.9}$$

(iv) Equating the coefficient at ε^{-1} to zero, after some transformations one gets

$$\begin{aligned} 0 \;=\; & K_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \{\omega(\mathbf{x}, t)\} + \frac{\partial}{\partial y_i} \left\{ K_{ij} \frac{\partial}{\partial x_j} \{\ln C_T(\mathbf{x}, t)\} \right\} \qquad (5.10) \\ & + \frac{\partial}{\partial y_i} \left\{ K_{ij} \frac{\partial}{\partial y_j} \left\{ \frac{1}{C_e} \left[s^{(1)} - \gamma_{pq} \left(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q} \right) + \lambda_q \left(\frac{\partial \phi^{(1)}}{\partial x_q} + \frac{\partial \phi^{(2)}}{\partial y_q} \right) \right] \right\} \right\} . \end{aligned}$$

If we introduce a function $\theta_k(\mathbf{y})$ that satisfies the following *local equation* on Y

$$\frac{\partial}{\partial y_i}\left(K_{ik} + K_{ij}\frac{\partial \vartheta_k(\mathbf{y})}{\partial y_j}\right) = 0\,, \tag{5.11}$$

then (5.10) takes the form

$$\begin{aligned} 0 \;=\; & K_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\{\omega(\mathbf{x},\mathbf{t})\} + \frac{\partial}{\partial y_i}\left\{K_{ij}\frac{\partial}{\partial y_j}\left[-\vartheta_k \frac{\partial}{\partial x_k}\{\ln C_T(\mathbf{x},t)\}\right.\right. \\ & \left.\left. + \frac{1}{C_e}\left[s^{(1)} - \gamma_{pq}\left(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q}\right) + \lambda_q\left(\frac{\partial \phi^{(1)}}{\partial x_q} + \frac{\partial \phi^{(2)}}{\partial y_q}\right)\right]\right]\right\}. \end{aligned} \tag{5.12}$$

In the subsequent section it will be shown that $\omega(\mathbf{x},t) = 0$, cf. (6.2). The positive definiteness of K_{ij} implies that

$$\begin{aligned} & -\vartheta_k \frac{\partial}{\partial x_k}\{\ln C_T(\mathbf{x},t)\} \\ & + \frac{1}{C_e}\left[s^{(1)} - \gamma_{pq}\left(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q}\right) + \lambda_q\left(\frac{\partial \phi^{(1)}}{\partial x_q} + \frac{\partial \phi^{(2)}}{\partial y_q}\right)\right] = k(\mathbf{x},t)\,, \end{aligned} \tag{5.13}$$

where k is an arbitrary function of $\mathbf{x}$ and t.

(v) Now, we are ready to analyse the last term of the *energy equation*, by equating coefficient at ε^0 in (4.6) to zero. The result is given by

$$\begin{aligned} \dot{s}^{(0)} = K_{ij}&\left[L_iL_j + \frac{\partial \vartheta_k}{\partial y_i}L_kL_j + L_i\frac{\partial \vartheta_q}{\partial y_j}L_q + \frac{\partial(-\vartheta_k)}{\partial y_i}L_k\frac{\partial \vartheta_q}{\partial y_j}L_q\right] \\ & + \frac{\partial}{\partial x_i}\left\{K_{ij}\left[L_j + \frac{\partial(-\vartheta_k)}{\partial y_j}L_k\right]\right\} + \frac{\partial}{\partial y_i}\left\{\quad\right\}, \end{aligned} \tag{5.14}$$

where

$$L_i = \frac{\partial \ln C_T(\mathbf{x},t)}{\partial x_i}$$

and the last term $\partial/\partial y_i\{\,\}$ is inessential because its contribution vanishes after cell-averaging.

We introduce an operation of averaging over elementary cell Y

$$\langle(\cdots)\rangle = \frac{1}{|\,Y\,|}\int_Y (\cdots)\,dy\,. \tag{5.15}$$

Averaging (5.14) yields

$$\begin{aligned} \dot{s}^{(0)} = \langle K_{ij}\rangle L_iL_j &+ \left\langle K_{ij}\frac{\partial \vartheta_k}{\partial y_i}\right\rangle L_kL_j + L_i\left\langle K_{ij}\frac{\partial \vartheta_k}{\partial y_j}\right\rangle L_k \\ &+ \left\langle K_{ij}\frac{\partial \vartheta_k}{\partial y_i}\frac{\partial \vartheta_q}{\partial y_j}\right\rangle L_kL_q + \frac{\partial}{\partial x_i}\left\langle\left\{K_{ik} + K_{ij}\frac{\partial \vartheta_k}{\partial y_j}\right\}\right\rangle L_k\,. \end{aligned} \tag{5.16}$$

Integrating by parts and using periodic boundary conditions imposed on ϑ_k we get

$$\left\langle K_{ij} \frac{\partial \vartheta_k}{\partial y_i} \frac{\partial \vartheta_q}{\partial y_j} \right\rangle = \left\langle -K_{iq} \frac{\partial \vartheta_k}{\partial y_i} \right\rangle .$$

Reducing other terms in (5.16) in a similar way we arrive at the result

$$\dot{s}^{(0)} = K^H_{ik} \frac{\partial \ln C_T(\mathbf{x},t)}{\partial x_i} \frac{\partial \ln C_T(\mathbf{x},t)}{\partial x_k} + K^H_{ik} \frac{\partial^2 \ln C_T(\mathbf{x},t)}{\partial x_i \partial x_k} , \tag{5.17}$$

where

$$K^H_{ik} = \left\langle K_{ik} + K_{ij} \frac{\partial \vartheta_k}{\partial y_j} \right\rangle . \tag{5.18}$$

6. Homogenization of the equation of motion and Gauss' law

The equation of motion (4.4) and Gauss' law (4.5), similar in structure, are to be homogenized parallely.

First, we note that the highest singularity on right-hand side of these equations is due to the *exponent* factor at the temperature term

$$\exp\left\{ \frac{1}{\varepsilon C_e} \left[-\gamma_{ab} \frac{\partial}{\partial y_b} u^{(0)}_a + \lambda_a \frac{\partial}{\partial y_a} \phi^{(0)} \right] \right\} = \exp\left\{ \frac{1}{\varepsilon} \omega(\mathbf{x},t) \right\} , \tag{6.1}$$

cf. (5.5). So, in order to remove this singularity to comply with left-hand side of (4.3) and (4.4), we let

$$\omega(\mathbf{x},t) = 0 . \tag{6.2}$$

Next, equating coefficients of ε^{-2} in (4.4) and (4.5) to zero, we get respectively

$$\frac{\partial}{\partial y_j} \left\{ c_{ijmn} \frac{\partial u^{(0)}_m}{\partial y_n} + \pi_{kij} \frac{\partial \phi^{(0)}}{\partial y_k} \right\} = 0 \tag{6.3}$$

and

$$\frac{\partial}{\partial y_i} \left\{ \pi_{imn} \frac{\partial u^{(0)}_m}{\partial y_n} - \epsilon_{ij} \frac{\partial \phi^{(0)}}{\partial y_j} \right\} = 0 . \tag{6.4}$$

Consider now the average, cf. (2.10),

$$I = \left\langle c_{ijmn} \frac{\partial u^{(0)}_i}{\partial y_j} \frac{\partial u^{(0)}_m}{\partial y_n} \right\rangle ; \qquad I \geq 0 . \tag{6.5}$$

By (6.3–6.4) we find that $I = 0$, and $u^{(0)}_a$ and $\phi^{(0)}$ depend on $\mathbf{x}$ and t only,

$$u^{(0)}_a = u^{(0)}_a(\mathbf{x},t) , \tag{6.6}$$

$$\phi^{(0)} = \phi^{(0)}(\mathbf{x},t) . \tag{6.7}$$

In order to find coefficients at ε^{-1} and ε^0 in (4.4) and (4.5) let us begin with an analysis of ε-order of terms produced by the exponential component on RHS of (4.4) and (4.5), which we denote by g_i and l, respectively. Denoting

$$E = \varepsilon[k(\mathbf{x},\mathbf{t}) + \vartheta_k \frac{\partial \ln C_T(\mathbf{x},t)}{\partial x_k}] + \varepsilon^2[\cdots] + \cdots$$

we get

$$g_i = -\frac{1}{\varepsilon} T_0 \left\{ \frac{\partial \gamma_{ij}}{\partial y_j} \left[C_T(\mathbf{x},t)e^E - 1\right] + \gamma_{ij} C_T(\mathbf{x},\mathbf{t}) e^E \frac{\partial E}{\partial y_j} \right\} \tag{6.8}$$

and observe that g_i produces terms of order ε^{-1} (the first member) and ε^0 (the second member) as $\varepsilon \to 0$. In a similar way we also find

$$l = \frac{1}{\varepsilon} T_0 \left\{ \frac{\partial \lambda_i}{\partial y_i} \left[C_T(\mathbf{x},t)e^E - 1\right] + \lambda_i C_T(\mathbf{x},\mathbf{t}) e^E \frac{\partial E}{\partial y_i} \right\} \tag{6.9}$$

and make the similar observation: l produces terms of order ε^{-1} (the first member) and ε^0 (the second member) as $\varepsilon \to 0$.

Using this result we find:

(i) Equating the coefficient at ε^{-1} to zero, by virtue of (6.6–6.7) one has

$$0 = \frac{\partial}{\partial y_j} \left\{ c_{ijmn} \left(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n} \right) - \gamma_{ij}\Theta + \pi_{kij} \left(\frac{\partial \phi^{(0)}}{\partial x_k} + \frac{\partial \phi^{(1)}}{\partial y_k} \right) \right\}, \tag{6.10}$$

$$0 = \frac{\partial}{\partial y_i} \left\{ \pi_{imn} \left(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n} \right) - \lambda_i\Theta + \epsilon_{ij} \left(\frac{\partial \phi^{(0)}}{\partial x_j} + \frac{\partial \phi^{(1)}}{\partial y_j} \right) \right\}, \tag{6.11}$$

where

$$\Theta = \Theta(\mathbf{x},\mathbf{t}) = T_0[C_T(\mathbf{x},\mathbf{t}) - 1]. \tag{6.12}$$

Thus we let

$$u_m^{(1)} = \chi_{mpq}(\mathbf{y}) \frac{\partial u_p^{(0)}}{\partial x_q} + F_{mq}(\mathbf{y}) \frac{\partial \phi^{(0)}}{\partial x_q} + G_m(\mathbf{y})\Theta, \tag{6.13}$$

$$\phi^{(1)} = \psi_{pq}(\mathbf{y}) \frac{\partial u_p^{(0)}}{\partial x_q} + P_q(\mathbf{y}) \frac{\partial \phi^{(0)}}{\partial x_q} + Q(\mathbf{y})\Theta, \tag{6.14}$$

and observe that (6.10–6.11) are satisfied if the *local functions* $\chi_{mpq}, F_{mq}, G_q, \psi_{pq}, P_q$ and Q satisfy the equations

$$\frac{\partial}{\partial y_j} \left\{ c_{ijpq} + c_{ijmn} \frac{\partial \chi_{mpq}}{\partial y_n} + \pi_{kij} \frac{\partial \psi_{pq}}{\partial y_k} \right\} = 0, \tag{6.15}$$

$$\frac{\partial}{\partial y_j} \left\{ \pi_{qij} + c_{ijmn} \frac{\partial F_{mq}}{\partial y_n} + \pi_{kij} \frac{\partial P_q}{\partial y_k} \right\} = 0, \tag{6.16}$$

$$\frac{\partial}{\partial y_j}\left\{-\gamma_{ij}+c_{ijmn}\frac{\partial G_m}{\partial y_n}+\pi_{kij}\frac{\partial Q}{\partial y_k}\right\} = 0\,, \tag{6.17}$$

$$\frac{\partial}{\partial y_i}\left\{\pi_{ipq}+\pi_{imn}\frac{\partial \chi_{mpq}}{\partial y_n}+\epsilon_{ij}\frac{\partial \psi_{pq}}{\partial y_j}\right\} = 0\,, \tag{6.18}$$

$$\frac{\partial}{\partial y_i}\left\{-\epsilon_{iq}+\pi_{imn}\frac{\partial F_{mq}}{\partial y_n}-\epsilon_{ij}\frac{\partial P_q}{\partial y_j}\right\} = 0\,, \tag{6.19}$$

$$\frac{\partial}{\partial y_i}\left\{\lambda_i+\pi_{imn}\frac{\partial G_m}{\partial y_n}-\epsilon_{ij}\frac{\partial Q}{\partial y_j}\right\} = 0\,. \tag{6.20}$$

Eqs. (6.15)–(6.20) are identical with those resulting from a homogenization of linear thermoelasticity of piezoelectrics (cf. [11, 18]).

(ii) Equating to zero the coefficient at ε^0 (as $\varepsilon \to 0$), using (6.10) we obtain

$$\begin{aligned}
\rho\ddot{u}_i^{(0)} = {} & \frac{\partial}{\partial x_j}\left\{c_{ijmn}\left(\frac{\partial u_m^{(0)}}{\partial x_n}+\frac{\partial u_m^{(1)}}{\partial y_n}\right)-\gamma_{ij}T_0[C_T(\mathbf{x},\mathbf{t})-1]\right\}\\
& -\gamma_{ij}T_0\frac{\partial \vartheta_k}{\partial y_j}\frac{\partial C_T(\mathbf{x},\mathbf{t})}{\partial x_k}+\frac{\partial}{\partial x_j}\left\{\pi_{kij}\left(\frac{\partial \varphi^{(0)}}{\partial x_k}+\frac{\partial \varphi^{(1)}}{\partial y_k}\right)\right\}\\
& +\frac{\partial}{\partial y_j}\left\{c_{ijmn}\left(\frac{\partial}{\partial x_n}u_m^{(1)}+\frac{\partial}{\partial y_n}u_m^{(2)}\right)+\pi_{kij}\left(\frac{\partial \varphi^{(1)}}{\partial x_k}+\frac{\partial \varphi^{(2)}}{\partial y_k}\right)\right\}.
\end{aligned} \tag{6.21}$$

Averaging over the cell Y yields

$$\begin{aligned}
\langle\rho\rangle\ddot{u}_i^{(0)} = \frac{\partial}{\partial x_j}\Bigg\{ & \left\langle c_{ijmn}\left(\frac{\partial u_m^{(0)}}{\partial x_n}+\frac{\partial u_m^{(1)}}{\partial y_n}\right)\right\rangle+\langle\gamma_{ij}\rangle T_0[C_T(\mathbf{x},\mathbf{t})-1]\\
& +\left\langle\gamma_{iq}\frac{\partial \vartheta_j}{\partial y_q}\right\rangle T_0C_T(\mathbf{x},\mathbf{t})+\left\langle\pi_{kij}\left(\frac{\partial \varphi^{(0)}}{\partial x_k}+\frac{\partial \varphi^{(1)}}{\partial y_k}\right)\right\rangle\Bigg\}.
\end{aligned} \tag{6.22}$$

If we compare (6.22) with (2.6) and (2.3) we see that the term $T_0[C_T(\mathbf{x},\mathbf{t})-1]$ in (6.22) is equivalent to the temperature difference (T^H-T_0) for a homogenized body, with T^H being a temperature of such a body; therefore

$$\Theta = T_0[C_T(\mathbf{x},\mathbf{t})-1] = (T^H-T_0)\,, \tag{6.23}$$

$$C_T(\mathbf{x},t) = \frac{T^H}{T_0}\,. \tag{6.24}$$

The same result is obtained by comparison of (5.17) with (3.3) and (2.5). By virtue of (6.13) and (6.14) we obtain

$$\langle\rho\rangle\ddot{u}_i^{(0)} = c_{ijpq}^H\frac{\partial^2 u_p^{(0)}}{\partial x_j\partial x_q}-\gamma_{ij}^H\frac{\partial \Theta}{\partial x_j}+\pi_{kij}^H\frac{\partial^2\varphi^{(0)}}{\partial x_j\partial x_k}\,. \tag{6.25}$$

In a similar way we arrive at the homogenized Gauss law

$$\pi^H_{imn}\frac{\partial^2 u^{(0)}_m}{\partial x_i \partial x_n} + \lambda^H_i \frac{\partial \Theta}{\partial x_i} - \epsilon^H_{ij}\frac{\partial^2 \varphi^{(0)}}{\partial x_i \partial x_j} = 0 , \tag{6.26}$$

where

$$c^H_{ijpq} = \left\langle c_{ijpq} + c_{ijmn}\frac{\partial \chi_{mpq}}{\partial y_n} + \pi_{kij}\frac{\partial \psi_{pq}}{\partial y_k}\right\rangle , \tag{6.27}$$

$$\pi^H_{kij} = \left\langle \pi_{kij} + c_{ijmn}\frac{\partial F_{mk}}{\partial y_n} + \pi_{mij}\frac{\partial P_k}{\partial y_m}\right\rangle , \tag{6.28}$$

$$\epsilon^H_{ij} = \left\langle \epsilon_{ij} - \pi_{imn}\frac{\partial F_{mj}}{\partial y_n} + \epsilon_{im}\frac{\partial P_j}{\partial y_m}\right\rangle , \tag{6.29}$$

$$\gamma^H_{ij} = \left\langle \gamma_{ij} - c_{ijmn}\frac{\partial G_m}{\partial y_n} - \pi_{mij}\frac{\partial Q}{\partial y_m} + \gamma_{iq}\frac{\partial \vartheta_j}{\partial y_q}\right\rangle , \tag{6.30}$$

$$\lambda^H_i = \left\langle \lambda_i - \epsilon_{im}\frac{\partial Q}{\partial y_m} + \pi_{ijm}\frac{\partial G_j}{\partial y_m} + \lambda_i\frac{\partial \vartheta_k}{\partial y_k}\right\rangle . \tag{6.31}$$

Also, we obtain

$$K^H_{ik} = \left\langle K_{ik} + K_{ij}\frac{\partial \vartheta_k}{\partial y_j}\right\rangle . \tag{5.18}$$

Clearly, the tensor c^H_{ijpq} given by (6.27) is identical with that derived in a linear theory while γ^H_{ij} given by (6.30) is inherent for a quasi-linear theory: we observe that γ^H_{ij} is composed of the two terms:

$$\gamma^H_{ij} = \gamma^{HL}_{ij} + \left\langle \gamma_{iq}\frac{\partial \vartheta_j}{\partial y_q}\right\rangle \tag{6.30'}$$

with

$$\gamma^{HL}_{ij} = \left\langle \gamma_{ij} - c_{ijmn}\frac{\partial G_m}{\partial y_n} - \pi_{mij}\frac{\partial Q}{\partial y_m}\right\rangle \tag{6.32}$$

being the homogenized γ_{ij} coefficient of the linear theory, cf. [11]; similarly,

$$\lambda^H_i = \lambda^{HL}_i + \left\langle \lambda_i\frac{\partial \vartheta_q}{\partial y_q}\right\rangle \tag{6.31'}$$

with

$$\lambda^{HL}_i = \left\langle \lambda_i - \epsilon_{im}\frac{\partial Q}{\partial y_m} + \pi_{ijm}\frac{\partial G_j}{\partial y_m}\right\rangle . \tag{6.32'}$$

Substituting (6.13) and (6.14) into (5.9) and taking the mean value of this result and using (6.17), we obtain

$$\left\langle s^{(0)}\right\rangle - \gamma^{HL}_{mn}\frac{\partial u^{(0)}_m}{\partial x_n} + \lambda^{HL}_m\frac{\partial \phi^{(0)}}{\partial x_m} - \sigma(T^H - T_0) = \langle C_\varepsilon\rangle \ln(T^H/T_0) , \tag{6.33}$$

where

$$\sigma = \left\langle \gamma_{mn} \frac{\partial G_m}{\partial y_n} + \lambda_m \frac{\partial Q}{\partial y_m} \right\rangle , \tag{6.34}$$

and γ_{pq}^{HL} and λ_m^{HL} are given by (6.32). Eq. (6.33) is a transcendental equation for an unknown T^H. It can be cast into

$$\left\langle s^{(0)} \right\rangle - \gamma_{mn}^{HL} \frac{\partial u_m^{(0)}}{\partial x_n} + \lambda_m^{HL} \frac{\partial \phi^{(0)}}{\partial x_m} = C_e^H \ln(T^H/T_0) \tag{6.35}$$

with the following "homogenized" specific heat

$$C_\varepsilon^H = \langle C_e \rangle + \sigma \frac{T^H - T_0}{\ln(T^H/T_0)} . \tag{6.36}$$

Also, by virtue of (6.24), the averaged entropy production equation (5.17) takes the form

$$\left\langle \dot{s}^{(0)} \right\rangle = K_{ik}^H \frac{\partial \ln T^H(\mathbf{x}, \mathbf{t})}{\partial x_i} \frac{\partial \ln T^H(\mathbf{x}, \mathbf{t})}{\partial x_k} + K_{ik}^H \frac{\partial^2 \ln T^H(\mathbf{x}, \mathbf{t})}{\partial x_i \partial x_k} \tag{6.37}$$

or

$$T^H(\mathbf{x}, t) \left\langle \dot{s}^{(0)} \right\rangle = K_{ik}^H \frac{\partial^2 T^H(\mathbf{x}, \mathbf{t})}{\partial x_i \partial x_k} . \tag{6.38}$$

This equation is similar to (2.8).

7. Shift of the initial condition for temperature

A shift of the initial condition for the temperature appears to be a paradox in the result obtained by Francfort [12]. To see this take the initial conditions in the form

$$u_i(\mathbf{x}, 0) = U_i(\mathbf{x}); \quad \dot{u}_i(\mathbf{x}, 0) = \dot{U}_i(\mathbf{x}); \quad T(\mathbf{x}, 0) = \mathcal{T}(\mathbf{x}); \quad \varphi(\mathbf{x}, 0) = f(\mathbf{x}) . \tag{7.1}$$

The equivalent form of (2.5)

$$s - \gamma_{ij} \frac{\partial u_i}{\partial x_j} + \lambda_i \frac{\partial \varphi}{\partial x_i} = C_\varepsilon \ln(T/T_0) \tag{7.2}$$

taken at $t = 0$ reads

$$s^\varepsilon - \gamma_{ij} \left(\frac{\mathbf{x}}{\varepsilon}\right) \frac{\partial U_i}{\partial x_j} + \lambda_i \left(\frac{\mathbf{x}}{\varepsilon}\right) \frac{\partial f}{\partial x_i} = C_e \left(\frac{\mathbf{x}}{\varepsilon}\right) \ln(\mathcal{T}/T_0) , \tag{7.3}$$

and after averaging we obtain

$$\lim_{\varepsilon \to 0} \langle s^\varepsilon(\mathbf{x}, 0) \rangle = \langle \gamma_{ij} \rangle \frac{\partial U_i}{\partial x_j} + \langle C_\varepsilon \rangle \ln(\mathcal{T}/T_0) - \langle \lambda_i \rangle \frac{\partial f}{\partial x_i} . \tag{7.4}$$

On the other hand, from (6.35) taken at $t = 0$ one gets

$$\left\langle s^{(0)}(\mathbf{x}, \mathbf{y}, 0) \right\rangle = \gamma_{ij}^{HL} \frac{\partial U_i}{\partial x_j} + C_\varepsilon^H \ln(T^H(\mathbf{x}, 0)/T_0) - \lambda_m^{HL} \frac{\partial f}{\partial x_m} . \tag{7.5}$$

A comparison of (7.4) and (7.5) yields

$$\begin{aligned}\langle\gamma_{ij}\rangle \frac{\partial U_i}{\partial x_j} + \langle C_e\rangle \ln(\mathcal{T}/T_0) - \langle\lambda_i\rangle \frac{\partial f}{\partial x_i} \\ = \gamma_{ij}^{HL}\frac{\partial U_i}{\partial x_j} + C_\varepsilon^H \ln(T^H(\mathbf{x},0)/T_0) - \lambda_m^{HL}\frac{\partial f}{\partial x_m}.\end{aligned} \tag{7.6}$$

Hence

$$\begin{aligned}\frac{1}{C_\varepsilon^H}\left[\langle\gamma_{ij}\rangle - \gamma_{ij}^{HL}\right]\frac{\partial U_i}{\partial x_j} - \frac{1}{C_\varepsilon^H}\left[\langle\lambda_i\rangle - \lambda_m^{HL}\right]\frac{\partial f}{\partial x_m} \\ = \ln(T^H(\mathbf{x},0)/T_0) - (\langle C_e\rangle / C_\varepsilon^H)\ln(\mathcal{T}/T_0)\end{aligned}$$

or

$$T^H(\mathbf{x},0) = T_0\left(\frac{\mathcal{T}}{T_0}\right)^n \exp\left\{\frac{1}{C_\varepsilon^H}\left[\left(\langle\gamma_{ij}\rangle - \gamma_{ij}^H\right)\frac{\partial U_i}{\partial x_j} - \left(\langle\lambda_i\rangle - \lambda_m^{HL}\right)\frac{\partial f}{\partial x_m}\right]\right\} \tag{7.7}$$

with

$$n = \langle C_\varepsilon\rangle / C_\varepsilon^H .$$

Thus, in general $T^H(\mathbf{x},0) \neq \mathcal{T}(\mathbf{x})$. If the exponent is expanded into a series, and because of

$$\frac{\mathcal{T}}{T_0} = 1 + \frac{\mathcal{T} - T_0}{T_0},$$

we get after linearization

$$T^H(\mathbf{x},0) - T_0 = \frac{1}{C_\varepsilon^H}\left[\langle C_e\rangle(\mathcal{T} - T_0) + T_0\left[\left(\langle\gamma_{ij}\rangle - \gamma_{ij}^H\right)\frac{\partial U_i}{\partial x_j} - \left(\langle\lambda_i\rangle - \lambda_m^{HL}\right)\frac{\partial f}{\partial x_m}\right]\right. \tag{7.8}$$

This expression is identical with the result in [11], p. 323.

8. Conclusions

The homogenized field equations and the effective coefficients for a homogenized quasi-linear thermo-piezoelectric body may be obtained if, similarly to the linear case *seven local problems* are solved, Eqs. (5.11), (6.15)–(6.20). The homogenized coefficients $c_{ijpq}^H, \pi_{kij}^H, \epsilon_{ij}^H$ and K_{ij}^H are the same as in the linear theory while γ_{ij}^H and λ_i^H are different. The homogenized nonlinear energy equation takes the form (6.38). A shift of the initial value of temperature after linearization transforms into one of the Francfort type given in [11].

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