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Research Paper

Parametrization of Cauchy Stress Tensor Treated as Autonomous Object Using Isotropy Angle and Skewness Angle

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Intrinsic features (eigenproperties) of the Cauchy stress tensor are discussed. Novelty notions of isotropy and skewness mode angles are introduced for the improved parametric description of spherical (isotropic) and deviatoric (anisotropic) components of stress tensor. The skewness angle is defined with pure shear employed as a comparison reference mode upon observing that pure shear states can be interpreted as elementary (atomic) blocks of any macroscopic deviatoric stress state. An original statistical-physical interpretation of the stress tensor orthogonal invariants is provided. A micromechanical explanation for observed decrease of the stress tensor anisotropy factor values, measured in terms of the tensor orbit diameter, with stress deviator diverging from pure shear mode, is proposed. Explicit reasons explaining why biaxial experimental layouts (simple shear and/or planar shear) are insufficient for the comprehensive characterization of materials properties submitted to complex stress states loadings are presented. New explicit formulas for the triaxiality factor valid for biaxial stress states are delivered.

Key words: Cauchy stress; oriented geometrical object; isotropy angle; skewness angle; isomorphic cylindrical coordinates; pure shear, comparison reference state; anisotropy factor; biaxial tests; simple shear; planar shear; triaxiality factor.

1. Introduction

Three-dimensional second order tensors were introduced into science and engineering with a landmark lecture of Cauchy in 1822, and in written form in 1823, cf. [4]. The brilliant idea of Cauchy remained in unchanged form till nowadays, as his argument on the tetrahedron element and the forces acting on it can be found in practically unchanged form in any respectable textbook devoted to continuum mechanics, technical physics, the strength of materials and many more engineering and science fields. In his original presentation, Cauchy was talking about *pressures* and not *stress tensor* simply because, at that time, the tensor notion did not exist yet. From a philosophical point of view, Cauchy's introduction of the stress tensor concept can be treated as a step toward developing a *constitutive theory of forces*. The Cauchy idea of describing force interactions through surfaces can be understood as a certain (averaged) continuum model description of the force interactions between molecules – transition from molecular interactions toward continuum media interactions. Interesting discussion of physical foundations underlying the stress tensor concept from the above-mentioned perspective can be found in Chap. 3 of MAUGIN's book [13], and stimulating historical notes on the creation and unfolding of the stress tensor idea can be found in Chap. 4 of TRUESDELL's book [30]. The constitutive theory of forces is practically undeveloped in comparison to the maturity of the *constitutive theory of materials*. EUGSTER and GLOCKER have recently published a work remaining within the stream of development of the constitutive theory of forces, cf. [6]. The mathematical grounds of *tensor analysis* with all the fundamental underlying formal apparatus were originally developed by Ricci-Curbastro in 1888–1892, cf. Ref. (19) in TONOLO [29]. The motivation behind his work was completely different from that of Cauchy. Namely, it was an investigation on *invariance of quadratic forms* and Ricci-Curbastro called the technique *absolute differential calculus*. Ricci-Curbastro can be regarded as the father of tensor analysis as it is an *invariance feature* with respect to coordinate frames, which is the essence and profound sense of tensorial objects. It is this feature that determines the versatile usefulness of tensors, and their becoming the language of all advanced technical sciences nowadays, see, e.g., ITSKOV [9] or SPENCER [28]. The term *tensor*, in its contemporary meaning, was coined by VOIGT in his work from 1898, cf., e.g., pages v–vi in [31].

Tensors gained their today's omnipresence in science and technology – attained the status of a common language, because they proved to be excellent modeling objects. They enable the reliable description of many features of real physical objects: *state* (e.g., temperature, velocity, stress, strain), as well as their *properties* (e.g., thermal expansion, piezoelectricity, elastic behavior – with the aid of second, third and fourth order tensors, respectively), and many other features (e.g., mechanical, thermal and/or electromagnetic *loadings*). Tensors and their *eigenproperties* – represented by their various *invariants*, by embodying modeling idealization of physical reality indirectly enable its deeper understanding. When the tensor of a specific type is found to reliably describe and/or predict a specific physical phenomenon, then it is reasonable to infer that the specific eigenproperties of the tensor deliver a somewhat clarified picture of specific characteristics of a real physical situation, not blurred by disturbing secondary effects. The feedback in the process of model development and its experimental validation can bring about more precise identification of key characteristic features of the phenomenon itself and, in return, also its model.

The present work focuses on the Cauchy stress tensor treated as a generic instance of any three-dimensional second order symmetric Eulerian tensor, more precisely, Cartesian one because in the prevailing part of the present work Cartesian orthonormal bases are used. In fixed tensorial basis generated by three versors of a basis of three-dimensional Eulerian space – usually accepted as coordinates frame, the second order symmetric tensor is fully characterized by its representation, i.e., components of 3×3 symmetric matrix. These components, in an involved manner, contain information on six linearly independent parameters, out of which there can be constructed a set of three linearly independent invariants – with respect to change (rotation) of coordinates frame, and complementary set of three parameters being Euler angles describing the orientation of the tensor object – its triad of principal directions, with respect to coordinates frame. The Euler angles are not invariants as they do change with alteration (rotation) of coordinates frame. Actually, infinitely many invariants can be constructed for the second order symmetric tensor, but only sets of maximum *three* of them can be linearly independent. Actual physical interpretations of intrinsic features (invariants) of the second order symmetric tensor can be quite diverse. For example, when the tensor is used as modeling abstraction of a parcel, these could be interpreted as "length", "width", "height", when on the other hand, the tensor is used as modeling abstraction of an animal, these can find interpretation as "hue", "brightness", "saturation" of its fur. Higher order symmetric tensors can deliver a more precise modeling description of real-life objects. For example, the fourth order tensor with internal symmetries characteristic of Hooke's tensor is characterized by 18 invariants (eigenproperties).

In mechanical engineering, a trace of stress tensor has a very important physical interpretation of representing pressure ($p = -\mathrm{tr}(\boldsymbol{\sigma})$). In the case of small strains tensor, its trace only approximately describes volumetric changes of the material ($\mathrm{tr}(\boldsymbol{\varepsilon}) \approx \mathrm{d}V/\mathrm{d}V_0$), while the precise characterization of material volumetric changes is assured by the determinant of a tensor of deformation gradient ($\mathrm{d}V/\mathrm{d}V_0 = \det(\mathbf{F})$). Qualitative decomposition of parameters fully characterizing the second order symmetric tensor, in a fixed basis, into two triple sets composed of invariants and Euler angles delivers grounds for the idea that it is natural and convenient to interpret second order symmetric tensor – in particular stress tensor, as an *oriented geometrical object* (entity), which besides orientation in space is characterized by some number of intrinsic features – eigenproperties. These intrinsic features are described by some conveniently selected set of tensor invariants. The proposed approach to treat tensors as *autonomous directed entities* seems much more intuitive and useful in dealing with tensors as modeling objects representing real physical objects than a standard mathematical understanding of the tensor defined as some kind of *algebraic structure*.

Frequently, the impression that eigenproperties (invariants) of the stress tensor are the most important features can be acquired. For example, this premise finds reflection in the presently deep-rooted concept of *effective (equivalent) stress* notion $\sigma_{ef}(\boldsymbol{\sigma})$, which is a function of stress tensor only – the quantity universally at present used in engineering sciences, cf., e.g., Leckie and Dal Bello [10]. Nevertheless, the interaction of stress tensor object with other objects creates qualitatively different situations. For example, upon the interaction of the stress tensor (loading) with Hooke's tensor representing elastic material (external system), the three Euler angles of the stress tensor (not invariant) and three Euler angles of Hooke's tensor (not invariant) transform together into definite three invariants. The effect can be better intuitively understood when one realizes that the stress tensor orientation with respect to Hooke's tensor orientation does not depend on these objects' individual orientations with respect to some freely adopted common coordinates frame. Thus, *in interaction with external system*, *six parameters* (components) fully characterizing any second order symmetric tensor can be uniquely transformed into *six invariants*.

This observation gives grounds for assessment that much more versatile and perhaps an accurate general measure of some *limit condition* – start of plastic yield flow, cracking, damage or phase transition, is not equivalent stress but some expression involving a combination of stress tensor and some fourth order tensor ($\mathbf{M}$) characterizing material under stress action, for example, quadratic form $\sigma_M(\boldsymbol{\sigma}, \mathbf{M}) = \boldsymbol{\sigma}\mathbf{M}\boldsymbol{\sigma}$. The expression of this type was introduced by Misses already in 1928 – cf. formula (1) in [15], but it receives relatively little attention. The σ_M might be called *weighted equivalent stress* or *Misses stress intensity*, since the stress tensor does not appear in it as a stand-alone, sovereign entity but instead is pondered in interaction with external surroundings represented by tensor $\mathbf{M}$ similarly, as in the case of investigating stability problems.

Here, a new generic parametrization of stress tensor invariants is introduced, composed of stress modulus, isotropic angle and skewness angle ($||\boldsymbol{\sigma}||$, θ_{iso}, θ_{sk}). It enables the simplification of some useful formulas for example the one for the anisotropy factor. A new statistical interpretation of stress tensor invariants enabled introduction of the concept of internal entropy of stress tensor and revealed existence of connections between the internal entropy and the anisotropy degree of stress tensor.

Discussion on practical applications of newly introduced concepts and interpretations in experimental mechanics is also delivered. In particular, new and very simple explicit formulas are presented for the triaxiality factor expressed in terms of skewness angle.

2. Tensors – definitions, important bases and representations

Tensors proved to be very convenient modeling objects of reality. Due to that, it is important to understand what they actually are and what are their properties because then real physical situation modeled with the use of tensors can be better understood. There exist at least several definitions of the tensor notion.

2.1. Definitions and understanding of a tensor notion

2.1.1. Algebraic definition of a tensor. The most developed and mathematically precise definition of a tensor is the algebraic definition. In engineering sciences, so-called Euclidean tensors are used most frequently. Definition of Euclidean tensor can be formulated as follows.

A *Euclidean tensor* of order q and dimension n is an *algebraic structure*, an *element of linear tensorial space* $\mathcal{T}_q$, which is generated by the q-fold tensorial product of n-dimensional Euclidean vector spaces E_n. When the same basis is accepted in all spaces E_n then any tensor $\mathbf{T}$ belonging to $\mathcal{T}_q$ can be presented in the following form:

$$\mathbf{T} = \underbrace{T_{ij...m}}_{\text{components}} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m}_{\text{basis}} \quad i,j,...,m = 1,...,n, \quad \mathbf{T} \in \mathcal{T}_q, \quad \mathbf{e}_i \in E_n, \tag{2.1}$$

where a set of n versors $\mathbf{e}_i$ is a basis of Euclidean vector space E_n, a set of n^q simple tensors $\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m$ (q-fold tensorial products of versors $\mathbf{e}_i$) is a basis of space $\mathcal{T}_q$, and n^q numbers $T_{ij...m}$ are called components of tensor $\mathbf{T}$ – its *representation* in basis $\{\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_m\}$. In the case of second order tensors examined here, $q = 2$ and $n = 3$, as we are primarily interested in modeling real physical space ($\mathcal{T}_2 \equiv E_3 \otimes E_3$). A pair composed of a point O belonging to *Euclidean point space* ($O \in \mathcal{E}_3$) and a set of basis versors $\{\mathbf{e}_i\}$ of associated with it Euclidean vector space ($e_i \in E_3$) is called *coordinates frame* (*coordinates system*) $(O, \{\mathbf{e}_i\})$, $i = 1, 2, 3$. In shortcut, the coordinates frame is frequently noted by the set of basis vectors $\{\mathbf{e}_i\}$ only. It is a *three-dimensional point Euclidean space* $\mathcal{E}_3$ with coordinates frame $(O, \{\mathbf{e}_i\})$, which is accepted as a convenient *model of real physical space* in continuum mechanics[1)]. When the basis $\{\mathbf{e}_i\}$ composed of mutually orthonormal versors is adopted, then customarily *Eulerian tensors* are called *Cartesian tensors.*

It is impossible to introduce here explicitly and in detail all the necessary apparatus of tensorial calculus due to space limitations. The precise mathematical definitions can be found in many popular textbooks on continuum mechanics and/or tensorial calculus, see, e.g., Chap. 1 in OGDEN [20] for a compact introduction to the tensor theory, and, for example, ITSKOV [9] or OSTROWSKA-MACIEJEWSKA [21] for comprehensive and mathematically precise expositions.

The most important information to be acquired from the *algebraic definition* of tensors (elements of a linear space) is that *tensor makes the integrity of*

[1)] Algebraic structure $\{\mathcal{E}_3, E_3, \oplus\}$ composed of $\mathcal{E}_3$ treated as a set of points, associated with this set vector space E_3, and operation $\oplus$ defining adding of vectors to points makes an *affine space.*

basis and components (representation of the tensor in the basis). When the basis is fixed, isomorphism exists between the tensor and its representation (components) in this basis, i.e., the tensor can be equated with its components. The *tensor components transform* in a specific (linear) manner *with linear change (rotation) of the basis.*

Actually, the algebraic definition of a tensor is not very transparent nor appealing as it requires extensive preliminary knowledge from linear algebra. It is very difficult for conceptual, intuitive understanding, and thus using it in modeling real-life phenomena. Its biggest advantage is the quantitative precision necessary and required for obtaining any predictive numerical results when using tensors for modeling purposes.

Let us point out some other definitions of tensor notion, which are more convenient conceptually and, as it seems, more attractive for modeling purposes.

2.1.2. Operational definition of a tensor. A tensor $\mathbf{L} \in \mathcal{T}_{p+q}$ is a *linear operator* transforming tensor $\mathbf{P}$ of order p ($\mathbf{P} \in \mathcal{T}_p$) into a tensor $\mathbf{Q}$ of order q ($\mathbf{Q} \in \mathcal{T}_q$). It is realized by full contraction of tensors $\mathbf{L}$ and $\mathbf{P}$ ($\mathbf{Q} = \mathbf{L} \cdot \mathbf{P}$).

For example, fourth order tensor $\mathbf{L} \in \mathcal{T}_4$ upon its multiplication (contraction) with second order tensor $\boldsymbol{\varepsilon} \in \mathcal{T}_2$ transforms this tensor into some other second order tensor ($\mathbf{L} \cdot \boldsymbol{\varepsilon} \to \boldsymbol{\sigma} \in \mathcal{T}_2$). It is clear that correct methodological execution of whatever linear transformation anyway requires all the algebraic apparatus of tensor calculus. However, in certain tasks, it is much easier to perceive the tensor conceptually as a linear operator with specific properties than algebraic structure.

Here, still another approach (interpretation) of a tensor, which somehow returns to the roots of the tensor concept, is advocated.

2.1.3. Geometric definition of a tensor. A tensor is an *oriented geometrical object*, i.e., the geometrical object for which orientation can be identified with respect to some fixed *reference frame*[2)], and simultaneously characterized by a certain number of features described by its invariants, i.e., properties not changing with the rotation of the reference frame. The mathematically precise definition and accessible elucidation of the concept of a *geometrical object* can be found, for example, in Chap. II of Gołąb's book [7].

[2)] *Coordinates frame* and *reference frame* "physically" are both sets composed of some anchoring point and a set of basis vectors $(O, \{\mathbf{e}_i\})$. The difference is rather in their functionality. The coordinates frame makes a reference for the determination of vector (tensor) components, while the reference frame makes a reference for the examination of, e.g., motions (kinematics). Naturally, the same pair $(O, \{\mathbf{e}_i\})$ can be adopted to be simultaneously coordinates frame and reference frame or different pairs can be adopted depending on the need and convenience in examining specific problem.

In this approach, a second order symmetric tensor can be envisioned as an autonomous entity whose direction in space is prescribed by the orientation of a triad of its principal directions with respect to the prescribed *laboratory (reference) frame* and possessing features described by three linearly independent invariants. Again for actual quantitative calculations, all the algebraic tensorial apparatus is still necessary when conceptually treating tensors as oriented geometrical objects. This definition, at first sight, seems to be quite indistinct, but in essence, it well reflects natural characteristics possessed by many real physical objects. The situation is similar to the one from classical geometry, in which *point*, *line* and *plane* are actually undefined prime notions, but everybody knows what they are and/or how to understand them.

The usefulness of a specific definition and/or conceptual understanding of tensors depend on their uses's actual area, tasks and targets. However, in order to grasp an intuitive and profound understanding of the tensor object as a modeling tool and in this manner, e.g., indirectly comprehend some physical rule/law expressed by the tensorial relation, it is argued that *geometrical definition/interpretation* of the tensor is very convenient. First of all, this is so because one does not have to concentrate on technical (secondary) issues such as numerous components, their indices and contexts resulting from an adopted specific basis (coordinates frame).

2.2. Instances of convenient representations (bases and coordinate frames) and reference frames of second order symmetric tensors

It is important to carefully distinguish different notations used for tensors as information about them is differently distributed between the basis and the components depending on the notation. The basis of second order tensors space $\mathcal{T}_2$ can be constructed from nine simple tensors – dyads $\{\mathbf{i}_i \otimes \mathbf{i}_j\}$, $i, j = 1, ..., 3$. Frequently, for the vectors $\mathbf{i}_i$ elements of the orthonormal basis of three-dimensional Euclidean vector space E_3 are selected $\mathbf{i}_i = \mathbf{e}_i$; $\mathbf{i}_i \otimes \mathbf{i}_j \to \mathbf{e}_i \otimes \mathbf{e}_j$. In the case of symmetric second order tensors, its matrix representation components fulfill the condition of symmetry $\sigma_{ij} = \sigma_{ji}$, cf. $(2.2)_1$. The stress tensor is a symmetric second order tensor ($\boldsymbol{\sigma} \in \mathcal{T}_2^s$). The following notations are very commonly used to express stress tensor:

$$
\begin{aligned}
&\boldsymbol{\sigma} = \sum_{i,j=1,3} \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j, &&\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \\
&\boldsymbol{\sigma} = \sum_{K=1,6} \sigma_K \mathbf{a}_K, &&[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6], \\
&\boldsymbol{\sigma} = \sum_{J=\mathrm{I,II,III}} \sigma_J \mathbf{N}_J, \quad \mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J, &&[\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}],
\end{aligned}
\tag{2.2}
$$

where σ_{I}, σ_{II}, σ_{III} denote principal values and $\mathbf{n}_{\mathrm{I}}$, $\mathbf{n}_{\mathrm{II}}$, $\mathbf{n}_{\mathrm{III}}$ are principal directions (eigenvectors) of the tensor $\boldsymbol{\sigma}$ ($\mathbf{n}_J = \mathbf{n}_J(\mathbf{e}_i)$). The following denotations are used in $(2.2)_3$:

$$\begin{gathered}
\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sqrt{2}\sigma_{23},\ \sigma_5 = \sqrt{2}\sigma_{13},\ \sigma_6 = \sqrt{2}\sigma_{12},\\
a_1 = \mathbf{e}_1 \otimes \mathbf{e}_1,\ a_2 = \mathbf{e}_2 \otimes \mathbf{e}_2,\ a_3 = \mathbf{e}_3 \otimes \mathbf{e}_3,\ a_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2),\\
a_5 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1),\ a_6 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),\\
\mathbf{a}_K \cdot \mathbf{a}_L = \delta_{KL} \quad (K, L = 1, ..., 6),\\
\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\mathbf{a}_1},\ \underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\mathbf{a}_2},\ \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\mathbf{a}_3},\ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\mathbf{a}_4},\ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\mathbf{a}_5},\ \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\mathbf{a}_6}.
\end{gathered} \tag{2.3}$$

In the last row of (2.3), the structure of tensors $\mathbf{a}_K$ in a basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is shown graphically.

The top notation in Eq. (2.2) is a standard notation for three-dimensional second order tensors written in a nine-dimensional orthonormal basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. The middle notation in (2.2), also known as the Kelvin notation, results from writing a second order symmetric tensor in a six-dimensional orthonormal symmetric basis $\{\mathbf{a}_K\}$. In the bottom notation, tensor $\boldsymbol{\sigma}$ is written in a very special manner, in which all not diagonal terms of tensor representation are zero, i.e., $\boldsymbol{\sigma} = \sigma_{\mathrm{I}}\mathbf{n}_{\mathrm{I}} \otimes \mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\mathbf{n}_{\mathrm{II}} \otimes \mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\mathbf{n}_{\mathrm{III}} \otimes \mathbf{n}_{\mathrm{III}}$. Information about the tensor object is differently distributed between the basis and components in different notations. In notation $(2.2)_1$, the basis can be completely freely selected, so all information about the specific tensor is contained in its components. In notation $(2.2)_3$ in view of the known symmetry of the tensor $\boldsymbol{\sigma}$, the size of the basis was reduced from nine to six elements and vector (simpler) recording of components was introduced. However, again a symmetrical tensorial base can be freely selected; thus, all the information about the tensor is contained in its components. In the notation $(2.2)_5$, the components do not bear all the information about the tensor, as part of it is contained in a specially selected basis composed of principal directions $\mathbf{n}_J$ – always orthogonal, cf., e.g., Ogden [20]). These directions cannot be freely selected but are specific functions of the general basis $\{\mathbf{e}_i\}$ ($\mathbf{n}_J = \mathbf{n}_J(\mathbf{e}_i)$). The triple of principal axes $\{\mathbf{n}_{\mathrm{I}}, \mathbf{n}_{\mathrm{II}}, \mathbf{n}_{\mathrm{III}}\}$ is rotated with respect to triple $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by three (Euler) angles $\boldsymbol{\theta}_n = (\theta_{\mathrm{I}}, \theta_{\mathrm{II}}, \theta_{\mathrm{III}})$. All notations (2.2) assure that the value of the norm of tensor $\boldsymbol{\sigma}$ $\left(||\boldsymbol{\sigma}||^2 \equiv \operatorname{tr}\left(\boldsymbol{\sigma}^2\right)\right)$ is preserved when computed using a standard for the specific notation operational rules. This property is not preserved in the case of Voigt notation frequently used in numerical computations, for which ($\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$, $\sigma_3 = \sigma_{33}$, $\sigma_4 = \sigma_{23}$, $\sigma_5 = \sigma_{13}$, $\sigma_6 = \sigma_{12}$). Notations (2.2) are valid regardless of the tensor $\boldsymbol{\sigma}$ interaction with some external settings.

Let us also present one more notation (decomposition) of second order tensor when its interaction with some external object is taken into account and which seems not to be widely known nor sufficiently exploited. Let us assume that the fourth order symmetric tensor is known $\mathbf{S} \in \mathcal{T}_4^s$ that may be recognized as physically representing properties of some linear elastic material, i.e., Hooke's tensor. Then, the problem for eigenvalues (eigenstresses) of this tensor can be solved, i.e., roots of the so-called *characteristic equation* for $\mathbf{S}$ can be found. The characteristic equation of the fourth order symmetric tensor takes the form of sixth order polynomial equation with real coefficients. In the most general case, the solution of the characteristic equation is composed of six different eigenvalues (λ_K) and, corresponding to them, six different eigentensors ($\boldsymbol{\omega}_K \in \mathcal{T}_2^s$). It has been proved by Rychlewski, cf., for example, [23], that eigentensors $\boldsymbol{\omega}_K$ of $\mathbf{S}$ corresponding to different in value eigenvalues λ_K are mutually orthogonal and generate the whole space of second order symmetric tensors, i.e., the set $\boldsymbol{\omega}_K$, $K = 1, 6$, make a basis of this space, and the set $\{\boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_L\}$ can be adopted as the basis of $\mathcal{T}_4^s$:

$$
\begin{aligned}
&\mathbf{S} \cdot \boldsymbol{\omega} = \lambda \boldsymbol{\omega} \rightarrow \det(\mathbf{S} - \lambda \mathbf{I}^{(4s)}) = 0\\
&\rightarrow \lambda_K, \ \ \omega_K \in \mathcal{T}_2^s, \ \ \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}, \ \ K, L = 1, ..., 6,\\
&\mathbf{S} = \lambda_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + + \lambda_{\mathrm{VI}} \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}},\\
&\mathbf{I}^{(4s)} = \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + ... + \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}},
\end{aligned}
\tag{2.4}
$$

where λ_K are so-called *Kelvin moduli* ($\lambda_K \geq 0$ in view of the physical requirement of semipositiveness of Hooke's tensor), $\boldsymbol{\omega}_K$ are corresponding to Kelvin moduli eigentensors, $\mathbf{I}^{(4s)}$ denotes unit tensor in the space of fourth order symmetric tensors ($I_{KL}^{(4s)} = \delta_{KL}$, $K, L = 1, ..., 6$, $\sim \mathrm{diag}[1, 1, 1, 1, 1, 1]$). The decomposition $(2.4)_3$ of tensor $\mathbf{S}$ – with only diagonal components having nonzero values, is called the *spectral decomposition* of Hooke's tensor.

Taking advantage of (2.4), the following decomposition of stress tensor can be obtained:

$$
\begin{aligned}
&\boldsymbol{\sigma} = \sigma_{(1)} \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \sigma_{(6)} \boldsymbol{\omega}_{\mathrm{VI}} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4 + \boldsymbol{\sigma}_5 + \boldsymbol{\sigma}_6,\\
&\sigma_{(K)} \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_K, \quad \boldsymbol{\sigma}_K \cdot \mathbf{S} \boldsymbol{\sigma}_L = \delta_{KL}, \quad K, L = 1, ..., 6,\\
&\boldsymbol{\sigma} = \mathbf{I}^{(4s)} \cdot \boldsymbol{\sigma} = (\boldsymbol{\omega}_{\mathrm{I}} \cdot \boldsymbol{\sigma})\, \boldsymbol{\omega}_I + ... + (\boldsymbol{\omega}_{\mathrm{VI}} \cdot \boldsymbol{\sigma})\, \boldsymbol{\omega}_{\mathrm{VI}}.
\end{aligned}
\tag{2.5}
$$

Notation $(2.5)_1$ is also called the *energy-orthogonal decomposition* of stress space for a *given elastic body* $\mathbf{S}$, in view of $(2.5)_4$. This decomposition is *unique* when all the Kelvin moduli λ_K have different values. For fixed tensor $\mathbf{S}$, the six components $\sigma_{(K)}$ are scalar invariants of stress tensor $\boldsymbol{\sigma}$, in the standard sense that they do not change when the reference frame $\{\mathbf{e}_i\}$ is changed (rotated). It may also be said that $\sigma_{(K)}$ are invariants of $\boldsymbol{\sigma}$ in interaction with $\mathbf{S}$ $(\sigma_{(K)}^{\sigma\leftrightarrow S})$. The parts $\boldsymbol{\sigma}_K$ can be classified as tensorial invariants of $\boldsymbol{\sigma}$. The decomposition $(2.5)_1$ changes when tensor $\mathbf{S}$ changes. A detailed presentation of this issue can be found in the original papers of Rychlewski, see, e.g., [23, 24].

It was already mentioned that the basis of tensor space can be freely selected. A natural desire exists to select such a basis of tensor space $\mathcal{T}_2^s$ to make analytical and/or numerical computations executed on components of tensors, possibly simple and/or effective. Let us list some convenient tensorial bases – coordinates frames, useful in further discussion.

2.2.1. Laboratory basis. Such a basis set of dyads $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$, and corresponding to it coordinates frame – set of versors $\{\mathbf{e}_i\}$, is selected for example in view of convenience in expressing imposed boundary conditions, prescribing loadings or constraints.

2.2.2. Symmetry basis. Such a basis and corresponding to it coordinates frame is selected to be collinear with axes of some kind of material symmetry or geometrical shape/layout of examined engineering structure/device. For example, it is preset to be collinear, consistent with natural axes of symmetry of (anisotropic) material of which the engineering device is made.

2.2.3. Principal axes basis. Such a basis set of dyads $\{\mathbf{n}_I \otimes \mathbf{n}_J\}$, and corresponding to it coordinates frame – set of principal directions vectors $\{\mathbf{n}_J\}$, is selected when it is a subject matter justified or convenient to work with principal values of second order symmetric tensor. In 1920 Haigh [8] and independently Westergaard [32], in search of the best manner to describe the strength of materials, intuitively assumed that for isotropic, linear elastic materials Euler angles of stress tensor (loading) should not influence material strength and can be neglected. Based on this assumption, they proposed to introduce three-dimensional *principal values vector space* with orthogonal coordinates frame composed of principal directions of the stress tensor, cf., notation $(2.2)_5$. The principal values of the stress tensor σ_{I}, σ_{II}, σ_{III} are Cartesian coordinates of points in this space. The *principal values space* was given the name *Haigh-Westergaard space*, see, e.g., p. 14 in Maugin [14].

2.2.4. Eigentensors (eigenstresses) basis. Orthonormal, eigenstresses bases composed of sets $\{\boldsymbol{\omega}_K\}$ – cf. (2.4), are convenient, for example, in constructing models of materials effort (plastic yield, cracking, etc.). Let us now recall a definition of *isometric* bases. Two orthonormal bases are called *isometric* with respect to a group of *proper orthogonal rotations*, if such an orthogonal tensor $\mathbf{Q}$ exists ($\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$) – cf., e.g., [21], that

$$\mathbf{p}_\alpha = \delta^i_\alpha \mathbf{Q}\mathbf{e}_i, \quad \mathbf{p}_\alpha \otimes \mathbf{p}_\beta = \delta^i_\alpha \mathbf{Q}\mathbf{e}_i \otimes \delta^j_\beta \mathbf{Q}\mathbf{e}_j. \tag{2.6}$$

It is worth noting that *all* orthonormal bases in three-dimensional Euclidean space are isometric.

An important property of orthonormal bases $\{\boldsymbol{\omega}_K\}$ is that, in general, they are *not isometric* with orthonormal basis $\{\mathbf{a}_K\}$ – cf. (2.3), i.e., there does not exist such an orthogonal tensor $\mathbf{Q}$ that will transform basis $\{\mathbf{a}_K\}$ into basis $\{\boldsymbol{\omega}_K\}$. Let us take, for example, a set of second order tensors:

(2.7)

$$\mathbf{h}_1 \equiv \frac{1}{\sqrt{3}}\mathbf{1}, \qquad \mathbf{h}_2 \equiv \frac{1}{\sqrt{6}}\left[2\mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3\right],$$

$$\mathbf{h}_3 \equiv \frac{1}{\sqrt{2}}\left[\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3\right], \qquad \mathbf{h}_4 \equiv \frac{1}{\sqrt{2}}\left[\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2\right] = \mathbf{a}_4,$$

$$\mathbf{h}_5 \equiv \frac{1}{\sqrt{2}}\left[\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1\right] = \mathbf{a}_5, \qquad \mathbf{h}_6 \equiv \frac{1}{\sqrt{2}}\left[\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1\right] = \mathbf{a}_6,$$

$$\mathbf{1} = \mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3, \qquad \mathbf{h}_K\cdot\mathbf{h}_L = \delta_{KL}, \quad \mathbf{h}_K \in \mathcal{T}^s_{2(n=3)}, \quad K, L = 1, ..., 6.$$

It is easy to verify by making direct calculations that tensors $\mathbf{h}_K$ are *the eigenstates of the isotropic Hooke tensor* ($\mathbf{C}^{iso} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}^{(4s)}$, $\sim C^{iso}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$), i.e., they are the solution of the eigenvalue problem $\mathbf{C}^{iso}\mathbf{h}_\alpha = \lambda_\alpha\mathbf{h}_\alpha$, cf. (2.4), with eigenvalues $\lambda_1 = 3K = 3\lambda + 2\mu$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 2\mu$. The set $\{\mathbf{h}_K\}$ makes an orthonormal basis of symmetric second order tensors space $\mathbf{h}_K \in \mathcal{T}^s_2$ and it is *not isometric* with orthonormal basis $\mathbf{a}_K \in \mathcal{T}^s_2$, cf. (2.3). The tensor $\mathbf{h}_1$ is a spherical tensor and the other tensors $\mathbf{h}_\alpha$ ($\alpha = 2, ..., 6$) are deviators.

The *isotropic tensors*[3)] *have identical representation in all isometric orthonormal bases*, but they *do not have the same representation in all orthonormal bases.*

[3)] Definition of *isotropic tensors* is recalled in the Supplement at the end of the present work.

3. Various sets of useful second order symmetric tensor invariants (eigenproperties)

3.1. Characteristics (eigenproperties) of second order symmetric tensor and their physical interpretations

The second order symmetric tensor is fully characterized by six parameters – its components in some basis, as it was recalled in the previous section. Usually, the values of components are known from experimental or numerical tests. An infinite number of invariants can be constructed out of them, but maximum three are always linearly independent. The common set of second order tensor invariants most frequently encountered in mathematical studies are so-called *basic invariants* $\{I_{b1}, I_{b2}, I_{b3}\}$, see, e.g., Spencer [28]:

$$
\begin{aligned}
I_{b1} &\equiv \mathrm{tr}\,(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\cdot\mathbf{1},\\
I_{b2} &\equiv \mathrm{tr}(\boldsymbol{\sigma}^2) = ||\boldsymbol{\sigma}||^2 = \boldsymbol{\sigma}^2\cdot\mathbf{1},\\
I_{b3} &\equiv \mathrm{tr}(\boldsymbol{\sigma}^3) = \boldsymbol{\sigma}^3\cdot\mathbf{1},
\end{aligned}
\tag{3.1}
$$

where $||\boldsymbol{\sigma}|| = (\sigma_{ij}\sigma_{ij})^{1/2}$ denotes *modulus (norm) of a tensor*, $\mathbf{1}$ (δ_{ij}) denotes unit tensor (also called identity tensor) in second order tensors space – in notation $(2.2)_3$ [1, 1, 1, 0, 0, 0] or in notation $(2.2)_5$ [1, 1, 1]. The dot symbol denotes full (double) contraction of second order tensors $\mathbf{a}\cdot\mathbf{b}$ $(a_{ij}b_{ij})$.

The popularity of basic invariants called *main invariants* in some publications comes from the *computational (numerical) effectiveness* of their determination, which requires only multiplication of tensor matrix representation for which very effective numerical algorithms exist.

In continuum mechanics, an alternative set of linearly independent invariants, so-called *principal values* $\{\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}\}$ of second order symmetric tensor, gained popularity and is widely used. The reason for that is their physical interpretation, e.g., in the case of the stress tensor, they proved to be handy in the relatively simple description of the *effort state* of engineering material submitted to a specific stress loading. The principal values are determined as roots of the so-called *characteristic equation* for principal values:

$$
\begin{gathered}
\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n} \to (\boldsymbol{\sigma}-\sigma\mathbf{1})\mathbf{n} = 0 \to \det(\boldsymbol{\sigma}-\sigma\mathbf{1}) = 0 \to \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0\\
\to \sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}, \quad \boldsymbol{\sigma} = \sigma_{\mathrm{I}}\mathbf{n}_{\mathrm{I}}\otimes\mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\mathbf{n}_{\mathrm{II}}\otimes\mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\mathbf{n}_{\mathrm{III}}\otimes\mathbf{n}_{\mathrm{III}},\\
\boldsymbol{\sigma}\mathbf{n}_J = \sigma_J\mathbf{n}_J\,(!J) \to \boldsymbol{\sigma}^n\mathbf{n}_J = \sigma_J{}^n\mathbf{n}_J\,(!J)\,, \quad J = \mathrm{I, II, III},
\end{gathered}
\tag{3.2}
$$

where $\mathbf{n}_{\mathrm{I}}$, $\mathbf{n}_{\mathrm{II}}$, $\mathbf{n}_{\mathrm{III}}$ are principal directions (eigenvectors) of a tensor $\boldsymbol{\sigma}$, the second order unit tensor in notation $(2.2)_1$ takes the form $\mathbf{1} = \mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3$, and $(2.2)_5$ $\mathbf{1} = \mathbf{n}_{\mathrm{I}} \otimes \mathbf{n}_{\mathrm{I}} + \mathbf{n}_{\mathrm{II}} \otimes \mathbf{n}_{\mathrm{II}} + \mathbf{n}_{\mathrm{III}} \otimes \mathbf{n}_{\mathrm{III}} = \mathbf{N}_{\mathrm{I}} + \mathbf{N}_{\mathrm{II}} + \mathbf{N}_{\mathrm{III}}$. Exclamation symbol $(!J)$ means no summation over index J. The naming convention that $\sigma_{\mathrm{I}} \geq \sigma_{\mathrm{II}} \geq \sigma_{\mathrm{III}}$ is adopted here. The following denotations were introduced in (3.2):

$$
\begin{aligned}
&I_1 \equiv \mathrm{tr}\,(\boldsymbol{\sigma}) = 3\sigma_m && (\sigma_{ii})\,,\\
&I_2 \equiv \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\left(\boldsymbol{\sigma}^2\right)\right] && \left(\frac{1}{2}\left[{\sigma_{nn}}^2 - \sigma_{ij}\sigma_{ij}\right]\right),\\
&I_3 \equiv \det(\boldsymbol{\sigma}) && \left(\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right),
\end{aligned}
\tag{3.3}
$$

symbol det(...) denotes determinant operation, σ_m is the mean value of principal values, and ε_{ijk} is the permutation symbol. Upon substituting principal values σ_J consecutively into characteristic Eq. $(3.2)_2$, multiplying it by the corresponding eigenvector $\mathbf{n}_J$, taking advantage of the relations $(3.2)_3$ for powers of $\boldsymbol{\sigma}^n$, and summing up three such obtained identities, the well-known Cayley-Hamilton equation can be recovered:

$$
\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} - I_3\mathbf{1} = 0. \tag{3.4}
$$

The set of three coefficients appearing in the characteristic equation for determining the principal values of second order tensor $\{I_1, I_2, I_3\}$ are called *principal invariants*. In the case of symmetric second order tensor, all principal values are real, and when they are unique, then three principal directions make an orthogonal triad, cf., e.g., Ogden [20].

3.2. Decomposition of a second order tensor into spherical (isotropic) and deviatoric (anisotropic) parts and their attributes

Any second order tensor $\boldsymbol{\sigma}$ can be decomposed into *spherical* and *deviatoric* parts – they are mutually orthogonal, cf. also Fig. 1:

$$
\begin{gathered}
\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sph} + \mathbf{s}, \quad \boldsymbol{\sigma}_{sph} \equiv \sigma_m\mathbf{1}, \quad \sigma_m \equiv \frac{1}{3}\sigma_{ii} = \frac{1}{3}I_1, \quad \mathrm{dev}\,(\boldsymbol{\sigma}_{sph}) = 0,\\
\mathbf{s} \equiv \boldsymbol{\sigma} - \sigma_m\mathbf{1}, \quad s_{ij} \equiv \sigma_{ij} - \sigma_m, \qquad \mathrm{tr}\,(\mathbf{s}) = 0,\\
\boldsymbol{\sigma}_{sph}\cdot\mathbf{s} = 0, \quad ||\boldsymbol{\sigma}_{sph}|| \equiv \sqrt{\boldsymbol{\sigma}_{sph}^2\cdot\mathbf{1}} = \sqrt{3}\,|\sigma_m| = \frac{|I_1|}{\sqrt{3}},
\end{gathered}
\tag{3.5}
$$

where $\boldsymbol{\sigma}_{sph}$ denotes the spherical part and $\mathbf{s}$ denotes the deviatoric part of a tensor $\boldsymbol{\sigma}$.

The decomposition means that the sum of any two spherical tensors ($a\mathbf{1}$, $b\mathbf{1}$) always gives a spherical tensor, and the sum of any two deviatoric tensors ($\mathbf{s}_a$, $\mathbf{s}_b$) always gives a deviatoric tensor. Thus, the space of second order symmetric tensors can be divided into two separate (orthogonal) subspaces. Decomposition $(3.5)_1$ at the same time divides tensor $\boldsymbol{\sigma}$ into *isotropic part* ($\sigma_m\mathbf{1}$) and *anisotropic part* ($\mathbf{s}$). The spherical part of the tensor is isotropic in conventional sense, i.e., it does not change under applying any orthogonal tensor $\mathbf{Q} \in \mathcal{O}$, where $\mathcal{O}$ is a group of all orthogonal tensors, cf. Supplement at the end of the present work. A set $\{\boldsymbol{\sigma}^Q\}$ of all tensors that can be obtained by transformation of $\boldsymbol{\sigma}$ with any orthogonal tensor $\mathbf{Q} \in \mathcal{O}$ is called an *orbit* of tensor $\boldsymbol{\sigma}$:

$$\begin{gathered}\boldsymbol{\sigma}^Q = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} = \sigma_m\mathbf{1} + \mathbf{s}^Q,\\ \mathbf{s}^Q = \mathbf{Q}\mathbf{s}\mathbf{Q}^{\mathrm{T}}; \quad \mathbf{Q} \in \mathcal{O} \quad \left(\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1},\ \det\mathbf{Q} = \pm 1\right),\\ \{\boldsymbol{\sigma}^Q\} = \{\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T |\ \mathbf{Q} \in \mathcal{O}\} \text{ – orbit of a tensor } \boldsymbol{\sigma}.\end{gathered} \tag{3.6}$$

In analogy to the characteristic equation for principal values of complete stress tensor, cf. $(3.2)_2$, there can be formulated a characteristic equation for eigenvalues of the tensor deviator $\mathbf{s}$ only, coefficients of which make a set of *principal invariants of tensor deviator* $\{J_1, J_2, J_3\}$ defined as follows:

$$\begin{gathered}s^3 - J_1 s^2 - J_2 s - J_3 = 0 \to s_{\mathrm{I}}, s_{\mathrm{II}}, s_{\mathrm{III}} \to \mathbf{s}^3 - J_2\mathbf{s} - J_3\mathbf{1} = 0,\\ J_1 \equiv \mathrm{tr}\,(\mathbf{s}) = 0, \quad J_2 \equiv \left(\frac{1}{2}\right)\mathrm{tr}\,(\mathbf{s}^2) \geq 0 \quad \left(\frac{1}{2}s_{ij}s_{ij}\right),\\ J_3 \equiv \det(\mathbf{s}) = \frac{1}{3}\mathrm{tr}(\mathbf{s}^3) \qquad\qquad \left(\frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}s_{ip}s_{jq}s_{kr}\right),\end{gathered} \tag{3.7}$$

where $\mathbf{s}$ denotes the tensor deviator. The opposite sign in the definition of the second invariant of deviator (J_2) in comparison to the definition of the second invariant of full tensor (I_2) assures that it is always nonnegative. The J_2 invariant gained widespread use due to its physical interpretation of the *shear stress intensity* measure.

Two tensors $\mathbf{a}$, $\mathbf{b}$ are *coaxial* when they have the *same principal directions*. It can be shown that the necessary and sufficient condition for coaxiality of two tensors is that their *single contraction products commute* ($\mathbf{ab} = \mathbf{ba}$). It can be shown by direct calculation that the tensor and its deviator products commute $\boldsymbol{\sigma}\mathbf{s} = \mathbf{s}\boldsymbol{\sigma}$. Hence, they have the same principal directions $\{\mathbf{n}_J\}$.

The characteristic equation for principal values of deviator $(3.7)_1$ can be solved upon substitution for $s = 2\left(\frac{1}{3}J_2\right)^{1/2}\cos(\theta)$ and taking advantage of the following trigonometric identity $4\cos^3(\theta) - 3\cdot\cos(\theta) - \cos(3\theta) = 0$ to obtain explicit formulas for stress deviator principal values $\{s_{\mathrm{I}}, s_{\mathrm{II}}, s_{\mathrm{III}}\}$, cf. also p. 92 in Malvern [12]. Then, the stress tensor and its principal values can be expressed in the following form:

(3.8)

$$s_{\mathrm{I}}=\frac{2}{3}\sigma_{ef}\cos(\theta_L),\quad s_{\mathrm{II}}=\frac{2}{3}\sigma_{ef}\cos(120^\circ-\theta_L),\quad s_{\mathrm{III}}=\frac{2}{3}\sigma_{ef}\cos(120^\circ+\theta_L),$$

$$\sigma_J=\sigma_m+s_J,\quad J=\mathrm{I, II, III},$$

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}\left(\sigma_m,\sigma_{ef},\theta_L\right)=\sigma_m\mathbf{1}+s_{\mathrm{I}}\mathbf{n}_{\mathrm{I}}\otimes\mathbf{n}_{\mathrm{I}}+s_{\mathrm{II}}\mathbf{n}_{\mathrm{II}}\otimes\mathbf{n}_{\mathrm{II}}+s_{\mathrm{III}}\mathbf{n}_{\mathrm{III}}\otimes\mathbf{n}_{\mathrm{III}},$$

$$||\boldsymbol{\sigma}||^2=||\sigma_m\mathbf{1}||^2+||\mathbf{s}||^2=3\sigma_m^2+2J_2,\quad \sigma_{ef}\equiv\sqrt{3J_2}=\sqrt{\frac{3}{2}s_{ij}s_{ij}},$$

$$\cos(3\theta_L)\equiv\overline{J}_3,\quad \theta_L\in\langle 0,60^\circ\rangle,\quad \overline{J}_3\equiv\frac{3\sqrt{3}}{2}\frac{J_3}{J_2{}^{3/2}}=\frac{3\sqrt{6}J_3}{\sqrt{(2J_2)^3}}\in\langle -1,1\rangle,$$

where σ_{ef} denotes so-called *effective stress*, θ_L is called the *Lode angle*, and the term $\overline{J}_3$ is the *normalized third invariant of deviator* $(4\cos(\theta)\cos(\theta+120^\circ)\cdot\cos(\theta-120^\circ)=\cos(3\theta))$.

Let us return to the stress tensor anisotropy feature finding a source in its deviatoric part only. Rychlewski in [25, 26] proposed that in order to quantitatively evaluate magnitude of the tensor anisotropy, it is appropriate and convenient to employ the concept of the tensor orbit diameter, cf. $(3.5)_3$. He defined the *tensor orbit diameter* as the maximum distance between any two members in the orbit of tensor $\boldsymbol{\sigma}$ – in the sense of tensorial norm, cf. $(3.1)_2$. It can be expressed as follows:

$$d(\boldsymbol{\sigma})\equiv\max_{\alpha,\beta\in\{\boldsymbol{\sigma}^Q\}}\rho(\boldsymbol{\alpha},\boldsymbol{\beta})=\max_{Q\in\mathcal{R}}\rho\left(\boldsymbol{\sigma},\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\right),\quad \rho(\boldsymbol{\alpha},\boldsymbol{\beta})\equiv||\boldsymbol{\alpha}-\boldsymbol{\beta}||, \tag{3.9}$$

where d denotes the diameter of the tensor orbit, ρ denotes distance generated by the usual tensorial norm, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ denote any two tensors in the tensor orbit, and $\mathbf{Q}$ is any orthogonal tensor.

Rychlewski proposed the measure of tensor anisotropy, which he called *degree of anisotropy*, to be expressed with the following formula, cf. relation (6) in [26]:

$$\eta_{ani}(\boldsymbol{\sigma})\equiv\frac{d(\boldsymbol{\sigma})}{2||\boldsymbol{\sigma}||},\quad \boldsymbol{\sigma}\neq 0,\quad \eta_{ani}(\boldsymbol{\sigma})\in\langle 0,1\rangle. \tag{3.10}$$

Here, Rychlewski's degree of anisotropy is called the *anisotropy factor* and is denoted by η_{ani}. The definition of anisotropy factor, Eq. (3.10), is actually applicable to tensors of any degree.

Rychlewski has proved that the diameter of the orbit of the second order symmetric tensor is equal to $d = \sqrt{2}\,(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}})$ and next taking advantage of this, by far not obvious result, he showed that the anisotropy factor (3.10) could be expressed in the following form, cf. also formulas (32), (37) in [26]:

$$\eta_{ani} = \frac{\sqrt{2}\tau_{\max}}{||\boldsymbol{\sigma}||} = \frac{||\mathbf{s}||}{||\boldsymbol{\sigma}||}\sin(\theta_L + 60^\circ),$$

$$d(\boldsymbol{\sigma}) = \sqrt{2}\cdot(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}) = 2\sqrt{2}\cdot\tau_{\max}, \tag{3.11}$$

$$\tau_{\max} = \frac{1}{2}(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}) = \frac{1}{\sqrt{2}}||\mathbf{s}||\sin(\theta_L + 60^\circ),$$

where $\tau_{\max}$ denotes the maximum shear stress of the tensor $\boldsymbol{\sigma}$. It is clear from the above that the anisotropy factor η_{ani} is still another *invariant* of tensor $\boldsymbol{\sigma}$, and taking it formally makes a fundamental measure of the sensitivity of the tensor $\boldsymbol{\sigma}$ to rotations.

When passing from vector space – natural model of real three-dimensional physical space, to spaces of higher order tensors, many "obviously true" intuitive feelings from a standard vector analysis fail. For example, while the full tensor and its deviator are collinear – in a tensorial sense, their vector representations utterly correctly obtained from a methodological standpoint are not parallel, cf. Fig. 1. The collinearity of tensors *does not* translate into the parallelism of vectors. The same failure of intuitive feelings one experiences when dealing with the measure of the tensor orbit diameter, Eq. (3.9). The origins of this second

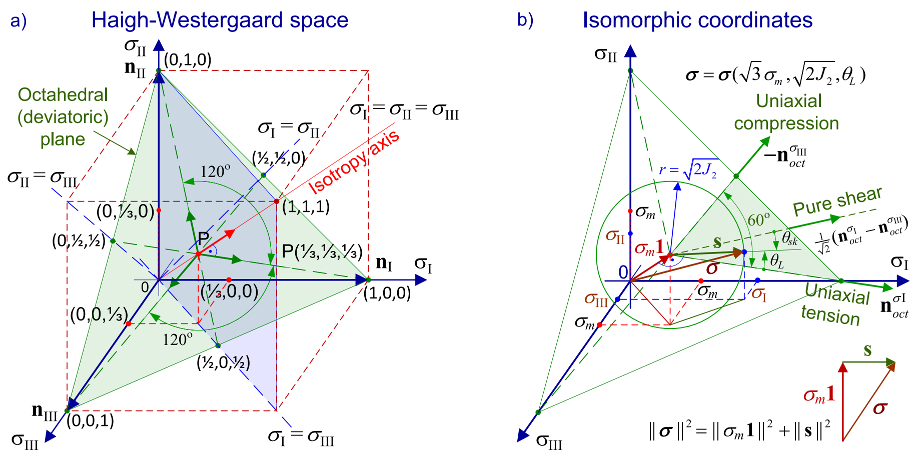

FIG. 1. a) Graphical illustration of structure of direct sum (orthogonal) decomposition of a second order symmetric tensor into spherical (isotropic) part and deviatoric (anisotropic) part ($\boldsymbol{\sigma} = \sigma_m\mathbf{1} + \mathbf{s}$, $\sigma_m\mathbf{1}\perp\mathbf{s}$) in the Haigh-Westergaard (H-W) principal values space, b) graphical illustration of elements involved in isomorphic – Murzewski, cylindrical coordinates, cf. Subsec. 3.3.

difficulty can be at least partially, intuitively grasped when one realizes that, e.g., the distance between the unit tensor and any rotated unit tensor is zero – in the sense of tensorial norm, more generally between any isotropic tensor and its orthogonally rotated instance. The direct cause of this is that in any rotated coordinates frame, a unit tensor **1** has the same values of components (representation), namely δ_{ij}. This effect is difficult to agree on with our everyday experience because there *do not exist* isotropic vectors; actually, the only vector whose components do not change under rotation of basis is the zero vector.

The invariants I_α, J_α, $\alpha = 1, 2, 3$, can be expressed in terms of general components of the stress tensor and in terms of its principal values as follows – cf. also (2.2):

$$
\begin{aligned}
& I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}}, \qquad J_1 = s_{\mathrm{I}} + s_{\mathrm{II}} + s_{\mathrm{III}} = 0,\\
& I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}\sigma_{21} - \sigma_{23}\sigma_{32} - \sigma_{13}\sigma_{31}\\
& \quad = \sigma_{\mathrm{I}}\sigma_{\mathrm{II}} + \sigma_{\mathrm{I}}\sigma_{\mathrm{III}} + \sigma_{\mathrm{II}}\sigma_{\mathrm{III}},\\
& J_2 = \frac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2,\\
& J_2 = -\left(s_{\mathrm{I}}s_{\mathrm{II}} + s_{\mathrm{II}}s_{\mathrm{III}} + s_{\mathrm{I}}s_{\mathrm{III}}\right) = \frac{1}{2}\left(s_{\mathrm{I}}^2 + s_{\mathrm{II}}^2 + s_{\mathrm{III}}^2\right) \geq 0,\\
& I_3 = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\\
& \quad = \sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{13}\sigma_{32}\sigma_{21} - \sigma_{11}\sigma_{23}\sigma_{32} - \sigma_{22}\sigma_{13}\sigma_{31}\\
& \qquad - \sigma_{33}\sigma_{12}\sigma_{21}, \quad I_3 = \sigma_{\mathrm{I}}\sigma_{\mathrm{II}}\sigma_{\mathrm{III}},\\
& J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki} = \frac{1}{3}(s_{\mathrm{I}}^3 + s_{\mathrm{II}}^3 + s_{\mathrm{III}}^3) = s_{\mathrm{I}}s_{\mathrm{II}}s_{\mathrm{III}}\\
& \quad = (\sigma_{\mathrm{I}} - \sigma_m)(\sigma_{\mathrm{II}} - \sigma_m)(\sigma_{\mathrm{III}} - \sigma_m).
\end{aligned} \tag{3.12}
$$

The following relations are valid for *basic*, *principal* and *deviatoric* invariants $I_{b\alpha}$, I_α, J_α:

$$
\begin{aligned}
& J_2(\boldsymbol{\sigma}) = -I_2(\mathbf{s}) = \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}^2) - \frac{1}{6}(\mathrm{tr}\boldsymbol{\sigma})^2 = 3\sigma_m^2 - I_2(\boldsymbol{\sigma}) = \frac{1}{3}I_1^2(\boldsymbol{\sigma}) - I_2(\boldsymbol{\sigma}),\\
& J_3(\boldsymbol{\sigma}) = I_3(\mathbf{s}) = \det(\mathbf{s}) = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^3) - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\sigma}^2) + \frac{2}{27}(\mathrm{tr}\boldsymbol{\sigma})^3\\
& \qquad = I_3(\boldsymbol{\sigma}) - \frac{1}{3}I_1(\boldsymbol{\sigma})I_2(\boldsymbol{\sigma}) + \frac{2}{27}I_1^3(\boldsymbol{\sigma}),\\
& I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma}) = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^3) - \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\mathrm{tr}(\boldsymbol{\sigma}^2) + \frac{1}{6}(\mathrm{tr}\boldsymbol{\sigma})^3\\
& \qquad = J_3 - J_2 \cdot \sigma_m + \sigma_m^3.
\end{aligned} \tag{3.13}
$$

Relations $(3.13)_{2,3}$ can be straightforwardly obtained upon double (full) contraction of the Cayley-Hamilton equation (3.4) with the unit tensor – taking the trace of it. The Cayley-Hamilton equation written down for deviator leads directly to the relation $\mathbf{s}^3 = J_3\mathbf{1} + J_2\mathbf{s}$ $(3J_3 = \mathrm{tr}(\mathbf{s}^3))$.

3.3. Sets of coordinates based on stress tensor invariants

Every set of three linearly independent invariants may be adopted as a system of coordinates in three-dimensional stress principal values vector space – *Haigh-Westergaard (H-W) space*, in place of the Cartesian system of principal values coordinates. Such action actually involves modification of not only coordinates but also the basis vectors (curvilinear in general), which after that are usually no longer stress principal directions. A very popular set of coordinates system in H-W space are (cylindrical) coordinates $(p, \sigma_{ef}, \theta_L)$ – pressure, effective stress, and Lode angle. They are used to present plastic flow yield, damage, failure or phase transition critical surfaces for different materials in octahedral $(p = \mathrm{const}, \sigma_{ef}, \theta_L)$ and/or meridional $(p, \sigma_{ef}, \theta_L = \mathrm{const})$ cross-sections – frequently $(p, r = \sqrt{2J_2})$ coordinates are also encountered. The problem with the mentioned coordinates is that they distort two-dimensional projection shapes of the actual three-dimensional shape of the critical surfaces.

According to the present author's literature survey, MURZEWSKI was the first researcher who in his work from 1960 – cf. [16], consciously introduced *isomorphic cylindrical coordinates* into H-W space, i.e., coordinates preserving correct shapes (distances and angles) of critical surfaces in H-W space cross-sections. This system of coordinates is as follows – cf. Fig. 1:

(3.14)

$\left(z=\sqrt{3}\sigma_m,\ r=\sqrt{2J_2},\ \theta_L\right)$ *isomorphic* (Murzewski's) cylindrical coordinates,

$\left(p=-\sigma_m,\ \sigma_{ef}=\sqrt{3J_2},\ \theta_L\right)$ *non-isomorphic* cylindrical coordinates,

$$z \equiv \frac{I_1}{\sqrt{3}} = \sqrt{3}\sigma_m, \quad \sigma_m = -p = \frac{I_1}{3}, \quad |z| \equiv ||\sigma_m\mathbf{1}||\,,$$

$$r \equiv ||\mathbf{s}|| = \sqrt{2J_2}, \quad ||\boldsymbol{\sigma}|| = \sqrt{3\sigma_m^2 + 2J_2}.$$

MURZEWSKI, in his work [16], used the following denotations ($z \leftrightarrow \sigma_A$, $r \leftrightarrow \sigma_D$, $\theta_L \leftrightarrow \omega_\sigma$).

The stress tensor and its principal values expressed in Murzewski's coordinates take the following form:

$$\sigma_{\mathrm{I}} = \frac{1}{\sqrt{3}}z + \sqrt{\frac{2}{3}}r\cos(\theta_L), \quad \sigma_{\mathrm{II}} = \frac{1}{\sqrt{3}}z + \sqrt{\frac{2}{3}}r\cos(\theta_L - 120^\circ),$$

$$\sigma_{\mathrm{III}} = \frac{1}{\sqrt{3}}z + \sqrt{\frac{2}{3}}r\cos(\theta_L - 240^\circ),$$

$$\text{(3.15)} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}(z, r, \theta_L) = z\mathbf{N}_{oct} + r\cos\theta_L \mathbf{N}_{oct}^{\sigma_{\mathrm{I}}} + r\sin\theta_L \mathbf{N}_{oct}^{\sigma_{\mathrm{I}}\perp}, \quad \boldsymbol{\sigma}_{sph} = z\mathbf{N}_{oct},$$

$$\mathbf{N}_{oct} = \frac{1}{\sqrt{3}}(\mathbf{N}_{\mathrm{I}} + \mathbf{N}_{\mathrm{II}} + \mathbf{N}_{\mathrm{III}}) = \frac{1}{\sqrt{3}}\mathbf{1}, \quad ||\mathbf{N}_{oct}|| = ||\mathbf{N}_{oct}^{\sigma_{\mathrm{I}}}|| = \left|\left|\mathbf{N}_{oct}^{\sigma_{\mathrm{I}}\perp}\right|\right| = 1,$$

$$\mathbf{N}_{oct}^{\sigma_{\mathrm{I}}} = \frac{1}{\sqrt{6}}(2\mathbf{N}_{\mathrm{I}} - \mathbf{N}_{\mathrm{II}} - \mathbf{N}_{\mathrm{III}}), \quad \mathbf{N}_{oct}^{\sigma_{\mathrm{I}}\perp} = \frac{1}{\sqrt{2}}(\mathbf{N}_{\mathrm{II}} - \mathbf{N}_{\mathrm{III}}),$$

where the second order tensors $\mathbf{N}_{\mathrm{I}}$, $\mathbf{N}_{\mathrm{II}}$, $\mathbf{N}_{\mathrm{III}}$ are defined in $(2.2)_3$. It is worth pointing out that in the H-W space, an *infinite number of non-coaxial tensors* having the same principal values but different orientation of principal axes reduce to a *single point representation*[4)], what actually means that a kind of strong filter operates in the H-W space.

Tensorial decomposition of tensor $\boldsymbol{\sigma}$ into spherical (isotropic) and deviatoric (anisotropic) parts $(3.5)_1$ should be carefully distinguished from the *vectorial decomposition* of *octahedral traction vector* ($\mathbf{t}_{oct}$) into *octahedral normal stress vector* ($\boldsymbol{\sigma}_{oct}$) and *octahedral shear stress vector* ($\boldsymbol{\tau}_{oct}$), which values can be expressed in terms of principal values as follows:

$$\mathbf{t}_{oct} = \boldsymbol{\sigma}\mathbf{n}_{oct} = \frac{1}{\sqrt{3}}(\sigma_{\mathrm{I}}\mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\mathbf{n}_{\mathrm{III}}),$$

$$\mathbf{t}_{oct} = t_{oct}\mathbf{n}_t = \sigma_{oct}\mathbf{n}_{oct} + \tau_{oct}\mathbf{n}_\tau, \quad t_{oct} = ||\mathbf{t}_{oct}|| = \sqrt{\sigma_{oct}{}^2 + \tau_{oct}{}^2} = \frac{1}{3}||\boldsymbol{\sigma}||,$$

$$\text{(3.16)} \qquad \begin{aligned} \sigma_{oct} &= \mathbf{t}_{oct}\cdot\mathbf{n}_{oct} = \mathbf{n}_{oct}\cdot\boldsymbol{\sigma}\mathbf{n}_{oct} \\ &= \frac{1}{3}(\sigma_{\mathrm{I}}\mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\mathbf{n}_{\mathrm{III}})\cdot(\mathbf{n}_{\mathrm{I}} + \mathbf{n}_{\mathrm{II}} + \mathbf{n}_{\mathrm{III}}) = \sigma_m, \end{aligned}$$

$$\tau_{oct} = \sqrt{||\mathbf{t}_{oct}||^2 - \sigma_{oct}^2} = \sqrt{\frac{2J_2}{3}} = \frac{r}{\sqrt{3}}.$$

An alternative isomorphic coordinates for meridional cross-sections in H-W space is pair of variables ($\sigma_{oct} = \sigma_m$, $\tau_{oct} = r/\sqrt{3}$). These coordinates were already used in 1929 by Burzyński – with denotation ($\omega_1 \leftrightarrow \sigma_{oct}$, $\omega_2 \leftrightarrow \tau_{oct}$),

[4)] *Note*: In mathematical terms, H-W space can also be interpreted (understood) as a set of *numerical markers of stress tensors orbits* (*with respect to proper rotations group – special orthogonal group SO_3*); every point of H-W space represents one orbit, an interested reader can find more information on this aspect in Rychlewski's book [27].

in his work devoted to the formulation of extended plastic yield strength criterion for linearly elastic, isotropic solids taking into account the influence of pressure (first invariant), cf. formula (12) in [3]. Burzyński, in this work, correctly disregarded the Lode angle influence on such criterion, as it cancels out from the expression for elastic energy in the case of isotropic, linearly elastic solids, which is clearly expounded here below.

The stress tensor invariants σ_m, r and σ_{ef} present in (3.14) have well-known clear physical interpretations of pressure (with negative sign), shear magnitude (norm) and *total shear effort* (intensity) of the material. The Lode angle θ_L as yet *does not have a clear physical interpretation.* From a mathematical standpoint, the Lode angle describes the angle between the projection of stress tensor (vector) and projection of principal axis I versor (corresponding to the greatest principal value of stress tensor) on the octahedral (deviatoric) plane, cf. Fig. 1.

Upon $(3.8)_6$, $(3.15)_4$ and $(3.15)_8$ it is easy to show by direct calculation that the following formulas are valid, cf. also $(3.8)_{1-3}$:

$$\sqrt{\frac{3}{2}}s_{\mathrm{I}} = \mathbf{N}^{\sigma_I}_{oct} \cdot \mathbf{s} = r\cos(\theta_L) \Rightarrow \cos(\theta_L) = \sqrt{\frac{3}{2}}\frac{s_I}{r},$$

(3.17)

$$\cos(\theta_L - 120^\circ) = \sqrt{\frac{3}{2}}\frac{s_{\mathrm{II}}}{r}, \qquad \cos(\theta_L + 120^\circ) = \sqrt{\frac{3}{2}}\frac{s_{\mathrm{III}}}{r}.$$

The above expressions show that the ordering of principal values actually does not influence the effective definition and/or interpretation of the Lode angle.

Formulas for the Lode angle in terms of principal values of deviator are inconvenient because principal values of deviator have to be computed first – a costly operation, before the Lode angle value can be determined. Formulas for the Lode angle expressed in terms of stress invariants – cf. $(3.8)_7$, are much more convenient numerically. The present author, in his historical survey, found Novozhilov's paper from 1951 to be the earliest publication in which third invariant of deviator is explicitly expressed in terms of a trigonometric function, cf. formula (1.13) in [19]. Actually, Novozhilov's angle ζ is defined not with cosine but with sine function ($\sin(\zeta) \equiv -\overline{J}_3$). The ζ is just the negative of the skewness angle introduced here below with $(4.2)_1$ ($\zeta = -\theta_{sk}$).

Summarizing the present state-of-the-art review regarding eigenproperties of second order symmetric tensors, it can be stated that the tensor treated conceptually as oriented geometrical object exhibits fixed characteristics described by the tensor invariants. A separate feature is the orientation of the tensor in a space – laboratory frame, described with its Euler angles. Depending on the internal symmetries of the tensor, its specific orientation can lead to the appearance or not of certain (directional) effects and can influence their magnitude. Such directional sensitivity of a tensor becomes fully exposed only upon its interaction with other tensorial objects, which takes place through Euler angles characterizing uniquely mutual relative orientation of two tensors. A similar situation exists in the case of colliding objects when it is important whether one object collides with its front or with its back with the other object and under what angle. It is worth to indicate that isotropic tensors are insensitive to their directional orientation in space, if it can be said they have one. Examination of very interesting aspects of the influence of symmetry of causes on the symmetry of effects can be found in RYCHLEWSKI's book [27].

4. NEW STRUCTURAL PARAMETRIZATION OF SECOND ORDER TENSOR EIGENPROPERTIES – THE CONCEPT OF ISOTROPY ANGLE (θ_{iso}) AND SKEWNESS ANGLE (θ_{sk})

A great variety of tensor eigenproperties parameterizations based on invariants is possible, and the usefulness and applicability of a specific parameterization set depend on a particular area of interest. Let us introduce a new set of invariant parameters characterizing second order symmetric tensors. The set seems to be especially convenient because it leads to the simplification of formulas expressing tensor properties and thus makes a more lucid characterization of real physical phenomena described by it. The new generic structural parameterization transparently and clearly corresponds to the internal structure of the tensor object and can also be conveniently used for constructing derivative sets matched to specific applications.

Let us first introduce the concept of *isotropy angle*, defined as follows:

$$
\begin{aligned}
&\sin(\theta_{iso}) \equiv \operatorname{sign}(\sigma_m)\frac{||\boldsymbol{\sigma}_{sph}||}{||\boldsymbol{\sigma}||} = \frac{\sqrt{3}\sigma_m}{||\boldsymbol{\sigma}||} = \frac{z}{||\boldsymbol{\sigma}||} \in \langle -1, 1\rangle,\\
&\cos(\theta_{iso}) \equiv \frac{||\mathbf{s}||}{||\boldsymbol{\sigma}||} = \frac{\sqrt{2J_2}}{||\boldsymbol{\sigma}||} \in \langle 0, 1\rangle, \quad \theta_{iso} \in \langle -90^\circ, 90^\circ\rangle, \quad ||\boldsymbol{\sigma}|| = \sqrt{3\sigma_m^2 + 2J_2}.
\end{aligned} \tag{4.1}
$$

A graphical representation of the isotropy angle is shown in Fig. 2a. The isotropy angle enables extraction of the spherical (isotropic) part and deviatoric (anisotropic) part of the tensor in a very straightforward and convenient manner. The sine and cosine functions of isotropy angle can also be treated as convenient normalized factors (indexes) describing the magnitude of spherical and/or deviatoric parts relative to the overall magnitude of the tensor (its modulus).

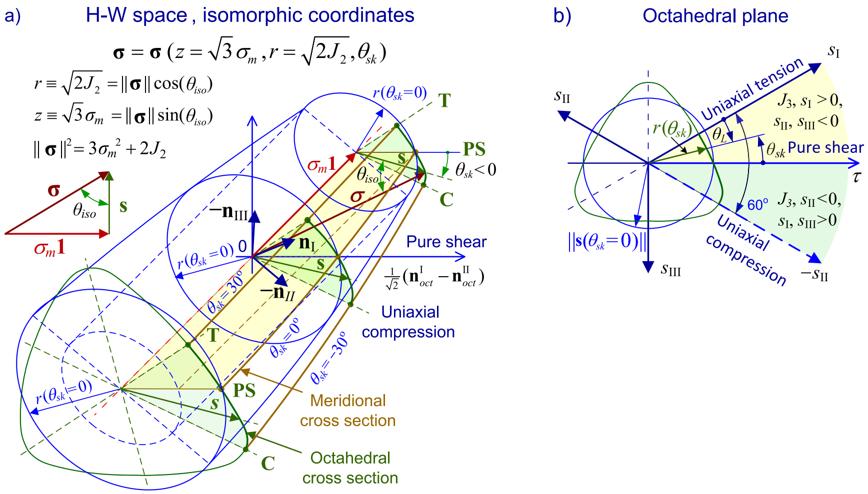

FIG. 2. a) Illustrative drawing of direct sum decomposition, in the H-W principal values space, of the second order symmetric tensor into spherical (isotropic) and deviatoric (anisotropic) parts ($\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \mathbf{s}$, $\sigma_m \mathbf{1} \perp \mathbf{s}$). Segments drawn with green color in octahedral planes mark respective projections of hypothetical critical surface for some *isotropic material* (three projections of principal axes on the octahedral plane make symmetry axes of the isotropic material critical surface), e.g., plastic flow yield surface $\sigma_{0.2}\ (z, r, \theta_{sk})$. b) Isomorphic cylindrical coordinates of tensor deviator drawn in the octahedral plane; θ_{sk} is the skewness angle and θ_L denotes the Lode angle.

Let us further introduce the concept of *skewness angle* defined as follows:

$$
\begin{aligned}
\sin(3\theta_{sk}) &\equiv \frac{1}{2} \cdot \frac{3\sqrt{3}J_3}{J_2^{\frac{3}{2}}} = \sqrt{2} \cdot \frac{3\sqrt{3}J_3}{r^3} \\
&= \frac{27}{2} \cdot \frac{J_3}{\sigma_{ef}^3} = \overline{J}_3 \in \langle -1, 1 \rangle, \quad \theta_{sk} \in \langle -30^\circ, 30^\circ \rangle, \\
\mu_L &\equiv \frac{2\sigma_{\mathrm{II}} - \sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}}{\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}} = \frac{3s_{\mathrm{II}}}{s_{\mathrm{I}} - s_{\mathrm{III}}} \\
&= -\sqrt{3}\tan\theta_{sk}, \quad \cos(3\theta_L) = \sin(3\theta_{sk}), \quad \theta_L = 30^\circ - \theta_{sk},
\end{aligned} \tag{4.2}
$$

where μ_L denotes the so-called Lode parameter introduced by Lode in 1925, cf., e.g., [11].

The skewness angle in mathematical terms describes the departure of the actual tensor deviator from the corresponding reference pure shear, i.e., deviator with the same modulus but with a third invariant equal to zero $J_3 = 0$, cf. Fig. 2b. The reasons why pure shear is considered appropriate comparison reference, what physical interpretation can be assigned to skewness angle, and why "skewness" is an appropriate name for the mode angle θ_{sk} are elucidated in the next section. The skewness angle has a very simple (linear) connection with the Lode angle, at present, universally used in multiaxial stress studies, cf. $(4.2)_5$. The isotropy angle and skewness angle can be easily normalized to the range $\langle -1, 1\rangle$, dividing them by $90°$ and $30°$, respectively.

Let us introduce *generic structural parameterization* of the second order symmetric tensor in the following form:

$$(4.3)\qquad \begin{gathered} (||\boldsymbol{\sigma}||, \theta_{iso}, \theta_{sk}), \quad ||\boldsymbol{\sigma}|| \equiv \sqrt{\sigma_{ij}\sigma_{ij}} \in \langle 0, \infty), \\ \theta_{iso} \in \langle -90°, 90°\rangle, \quad \theta_{sk} \in \langle -30°, 30°\rangle. \end{gathered}$$

This set of parameters delivers convenient – a kind of military structural characterization of the tensor object. The $||\boldsymbol{\sigma}||$ immediately tells about the overall strength of the entity, θ_{iso} allows to quickly evaluate what is the magnitude of general purpose (non-oriented) forces $||\boldsymbol{\sigma}_{sph}|| = ||\boldsymbol{\sigma}|| \sin(|\theta_{iso}|) = \sqrt{3}|\sigma_m| = |z|$ and the magnitude of special (oriented) resources $||\mathbf{s}|| = ||\boldsymbol{\sigma}|| \cos(\theta_{iso}) = r$, while θ_{sk} informs about a kind of versatility of the oriented resources of the tensor entity. The newly proposed generic structural parameterization can be conveniently adapted for the purposes of specific areas of application to form derivative parameterizations. In particular, already introduced various sets of invariants can be expressed in terms of newly proposed set, e.g., Murzewski's isomorphic coordinates $(3.14)_1$ or principal values of stress tensor $(3.15)_1$, upon uncomplicated operations can be expressed as follows:

$$(z = ||\boldsymbol{\sigma}|| \sin(\theta_{iso}), \;\; r = ||\boldsymbol{\sigma}|| \cos(\theta_{iso}), \;\; \theta_L = 30° - \theta_{sk})$$

– Murzewski's isomorphic coordinates,

$$(4.4)\qquad \begin{gathered} \sigma_{\mathrm{I}} = \frac{1}{\sqrt{3}}z + \sqrt{\frac{2}{3}}r\sin(60° + \theta_{sk}), \quad \sigma_{\mathrm{II}} = \frac{1}{\sqrt{3}}z + \sqrt{\frac{2}{3}}r\sin(-\theta_{sk}), \\ \sigma_{\mathrm{III}} = \frac{1}{\sqrt{3}}z - \sqrt{\frac{2}{3}}r\sin(60° - \theta_{sk}). \end{gathered}$$

Substituting $\cos(\theta_{iso})$ for $||\mathbf{s}||/||\boldsymbol{\sigma}||$ upon $(4.1)_2$, and $\theta_L = 30° - \theta_{sk}$ upon $(4.2)_4$ into $(3.11)_1$, simple manipulations lead to an extremely simple and elucidating formula for the anisotropy factor:

$$(4.5)\qquad \eta_{ani} = \cos(\theta_{iso}) \cdot \cos(\theta_{sk}), \quad \cos(\theta_{iso}) \in \langle 0, 1\rangle, \quad \cos(\theta_{sk}) \in \left\langle \frac{\sqrt{3}}{2}, 1 \right\rangle.$$

The first term in formula (4.5) clearly shows that the anisotropy degree of second order symmetric tensor grows with a growing fraction of its deviatoric part, reaching maximum for tensors being pure deviators ($\cos(\theta_{iso} = 0) = 1$). The second term shows that *the most anisotropic deviators* are *pure shears*

($\cos(\theta_{sk} = 0) = 1$). The anisotropy factor decreases with the deviatoric part departing from the respective comparison pure shear mode. In the case of pure deviators, it drops from 1 to a minimum value of $\eta_{ani} = \sqrt{3}/2 = 0.866$ reached at uniaxial tension or uniaxial compression ($\theta_{sk} = \pm 30°$).

Proposition on how to explain the reasons for this rather puzzling behavior of anisotropy factor diminishing with departure from pure shears, cf. Fig. 3a, is discussed in Sec. 5.

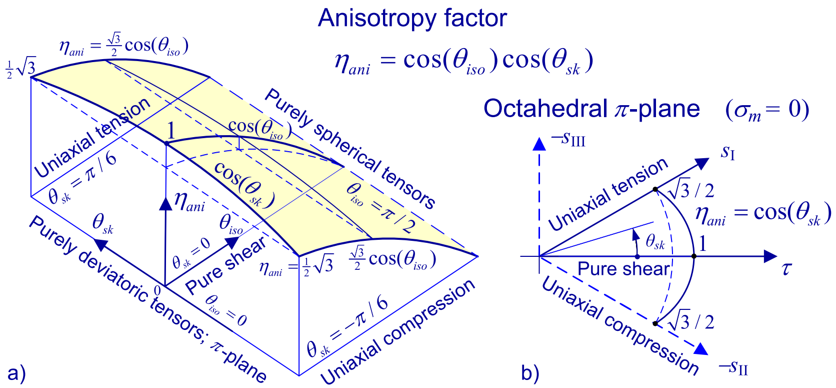

FIG. 3. a) Graphical illustration of the variation of anisotropy factor in dependence on isotropy angle θ_{iso} and skewness angle θ_{sk}, b) graphical illustration of the variation of anisotropy factor in octahedral π plane ($\sigma_m = 0$).

5. The special character of pure shear mode – elementary unit of the microstructure of deviators

In order to recognize the physical interpretation of the skewness angle defined through the normalized third invariant of deviator $\overline{J}_3$ – cf. (4.2), let us give some thought to the problem of what is the most elementary (atom) nontrivial form of second order tensor. It immediately comes to mind that it is the tensor, which has a single nonzero entry on diagonal in its matrix representation $\mathrm{diag}(a, 0, 0)$, and the option of the single nonzero off-diagonal component is excluded due to symmetry requirement. Such representation has, for example, the uniaxial tension (extension) and/or uniaxial compression tensors. However, upon further reflection, it can be realized that the uniaxial tension tensor is not as simple as it seems, and in fact several elemental (atom) components can be distilled from it along the lines of deviatoric decomposition $(3.5)_1$. The most elementary component of uniaxial tensor that can actually be identified as irreducible to more simple modes, is the spherical tensor, *spherical elementary mode*, having three identical in value diagonal components $\mathrm{diag}\left(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a\right)$. It can be physically interpreted as describing the simplest three-dimensional layout of action of forces in physical space, i.e., forces acting uniformly in all three physical directions, or alternatively, three-dimensional kinematics of displacements taking place uniformly in physical space, describing volume change. The remaining deviator of uniaxial tensor, on the other hand, proves to be always decomposable, in general in *infinitely many ways*, into the two most elementary irreducible deviator modes involving so-called *pure shears*, for example, $\operatorname{diag}\left(\frac{1}{3}a, -\frac{1}{3}a, 0\right) + \operatorname{diag}\left(\frac{1}{3}a, 0, -\frac{1}{3}a\right) = \operatorname{diag}\left(\frac{2}{3}a, -\frac{1}{3}a, -\frac{1}{3}a\right)$. The pure shear, *deviatoric elementary mode*, can be physically interpreted as the most simple two-dimensional (plane) layout of action of forces in physical space, i.e., forces operating uniformly in all parallel planes with fixed common normal axis (alternatively, two-dimensional kinematics of displacements taking place uniformly in planes with common normal axis, and proportional to the distance from some fixed plane, similarly as it is in the case of sliding tile of cards).

Let us recall, after Blinowski and Rychlewski [2], the precise mathematical definition of pure shear. A second order tensor $\boldsymbol{\tau}$ is called a *pure shear* when the following conditions are fulfilled; some other equivalent definitions can be found in the original publication [2]:

$$I_1 = \operatorname{tr}(\boldsymbol{\tau}) = 0, \quad I_3 = \det(\boldsymbol{\tau}) = 0 \;\Rightarrow\; J_3 = \operatorname{tr}(\boldsymbol{\tau}^3)/3 = 0. \tag{5.1}$$

Depending on the selection of the basis, the following two very characteristic, easily recognizable tensor representations of pure shear can be specified as follows:

$$\begin{gathered} \boldsymbol{\tau} = t\,(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\,, \quad \boldsymbol{\tau} = t\,(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2), \\ t\mathbf{e}_1 = \boldsymbol{\tau}\mathbf{e}_2, \quad t\mathbf{e}_2 = \boldsymbol{\tau}\mathbf{e}_1, \quad t\mathbf{n}_1 = \boldsymbol{\tau}\mathbf{n}_1, \quad -t\mathbf{n}_2 = \boldsymbol{\tau}\mathbf{n}_2, \\ \mathbf{n}_1 = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}, \quad \mathbf{n}_2 = (\mathbf{e}_2 - \mathbf{e}_1)/\sqrt{2}, \quad \mathbf{n}_3 = \mathbf{e}_3, \end{gathered} \tag{5.2}$$

where versors $\mathbf{e}_1$, $\mathbf{e}_2$ are called *shear directions*, and the plane determined by the pairs $(\mathbf{e}_1, \mathbf{e}_2)$ or $(\mathbf{n}_1, \mathbf{n}_2)$ is called a *shear plane*. The line along versor $\mathbf{e}_3$ or $\mathbf{n}_3$ is called the *shear axis*. It is clear that pure shears are planar tensors, cf. Fig. 4.

In accordance with the above nomenclature, two very useful classes of pure shears can be distinguished: shears with the *common shear direction* and shears with the *common shear axis*. A two-parameter family of pure shears with common shear direction $\boldsymbol{\tau}^{(n)}$ and a family of pure shears with common shear axis $\boldsymbol{\tau}^{(k)}$ can be expressed in the following mathematical form:

$$\begin{gathered} \boldsymbol{\tau}^{(n)} = \mathbf{n} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{n}, \qquad \boldsymbol{\tau}^{(k)} = \mathbf{y} \otimes \mathbf{z} + \mathbf{z} \otimes \mathbf{y}, \\ (\mathbf{x} \cdot \mathbf{n} = 0)\,, \qquad (\mathbf{y} \cdot \mathbf{k} = 0,\ \mathbf{z} \cdot \mathbf{k} = 0,\ \mathbf{y} \cdot \mathbf{z} = 0)\,, \\ \boldsymbol{\tau}^{(n)} \sim \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix}, \quad \mathbf{n} = \mathbf{e}_3, \quad \boldsymbol{\tau}^{(k)} \sim \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{k} = \mathbf{e}_3. \end{gathered} \tag{5.3}$$

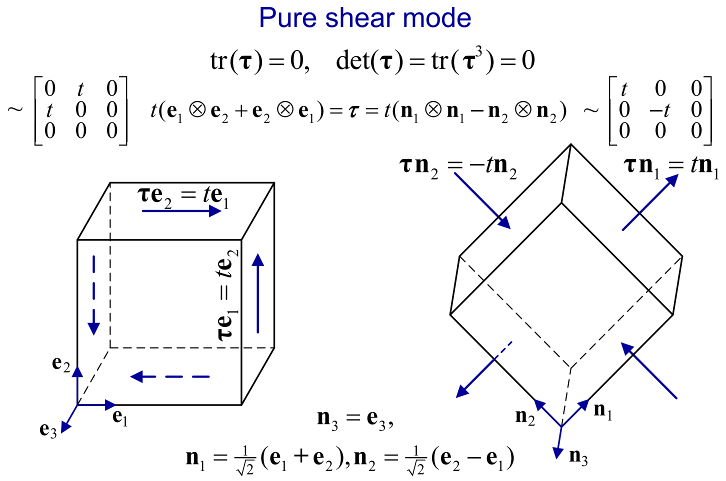

FIG. 4. Graphical illustration of the pure shear tensor $\boldsymbol{\tau}$ shown in two bases (coordinate frames) rotated by 45°, which results in two very characteristic pure shear tensorial representations.

Thus, all possible pure shears having common shear direction $\mathbf{n}$ parallel to axis $\mathbf{e}_3$ can be generated with freely selected vector $\mathbf{x}$ orthogonal to direction $\mathbf{n}$, cf. $(5.3)_1$. All pure shears with common shear axis $\mathbf{k}$ parallel to axis $\mathbf{e}_3$ can be generated with arbitrarily selected vector $\mathbf{y}$ orthogonal to shear axis $\mathbf{k}$ and vector $\mathbf{z}$ mutually orthogonal to vectors $\mathbf{k}$ and $\mathbf{y}$, cf. $(5.3)_2$.

The pure shears prove to be excellent modeling idealizations of many commonly encountered, actual physical situations. For example, *uniform plastic slip* deformation can be understood in modeling terms as a group of pure shears with a common axis, while the formation of a *compound martensitic twin* can be understood as a pair of two pure shears with a common shear direction. Experimental setups leading to pure shear stress or strain are very frequently used in experimental mechanics to determine material properties. This issue is discussed here in more details in Sec. 8.

BLINOWSKI and RYCHLEWSKI demonstrated in [2] that the population of all pure shears generates a complete subspace of all deviators. They also proved that *any deviator*, *in infinitely many ways*, can be decomposed into a sum of *two orthogonal pure* shears, cf. formula (3.9) in [2] and accompanying text. Hence, pure shears can be regarded as elementary building blocks of deviators subspace. However, pure shears do not create a linear subspace because the sum of two pure shears is not always a pure shear. It is worth noting that the sum of any number of pure shears will never result in a spherical tensor. All pure shears have the same "shape" in the sense that any and all pure shears can be obtained from arbitrary, preselected unit pure shear $\boldsymbol{\tau}_0$ by its proportional scaling t and rotating ($\boldsymbol{\tau} = t\mathbf{Q}\boldsymbol{\tau}_0\mathbf{Q}^{\mathrm{T}}$; $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{1}$). It is interesting to note that strictly non-orthogonal bases – any two basis tensors are not orthogonal, and/or strictly orthogonal bases of the five-dimensional subspace of deviators can be created composed of pure shears only, cf., e.g., pages 488 and 498 in [2]. Blinowski and Rychlewski, in their pivotal publication [2], rather open up than terminate research devoted to pure shears, and reveal many more attractive properties of pure shears.

The above discussion indicates the special role played by pure shears, which can be regarded as two-dimensional (plane), irreducible basic modes (atoms) of shearing making component parts of any deviator. This feature justifies their use as comparison reference elements for any other deviatoric mode of second order tensor and introduction of skewness angle notion as an appropriate quantitative parameter characterizing departure of specific given deviator from its comparison's pure shear. The following section presents an attempt to explain in what physical sense such difference is and its other consequences.

6. Statistical interpretation of second order symmetric tensor invariants, skewness angle as a measure of entropic part of stress tensor anisotropy

It is interesting to note that there exist very simple and straightforward connections between principal invariants of deviator J_α and quantities known as statistical central moments μ_i, namely,

$$\begin{aligned}
&\mu_1(\mathbf{s}) = \overline{\mu} = \frac{1}{3}(s_{\mathrm{I}} + s_{\mathrm{II}} + s_{\mathrm{III}}) = \frac{1}{3}J_1 = 0,\\
&\mu_2(\mathbf{s}) \equiv \frac{1}{3}\left(s_{\mathrm{I}}^2 + s_{\mathrm{II}}^2 + s_{\mathrm{III}}^2\right) = \frac{1}{3}\mathrm{tr}\left(\mathbf{s}^2\right) = \frac{1}{3}(2J_2),\\
&\mu_3(\mathbf{s}) \equiv \frac{1}{3}\left(s_{\mathrm{I}}^3 + s_{\mathrm{II}}^3 + s_{\mathrm{III}}^3\right) = \frac{1}{3}\mathrm{tr}\left(\mathbf{s}^3\right) = J_3,\\
&\mu_i \equiv \sum_{k=1,n} \frac{1}{n}(x_k - \overline{x})^i.
\end{aligned} \tag{6.1}$$

Please note that also, in the statistical sense, there exists an "orthogonal" decomposition of the tensor $\boldsymbol{\sigma}$ into spherical and deviatoric parts in the sense that for the spherical part, only the first central moment is different from zero and all the remaining central moments are equal to zero ($\mu_1(\sigma_m\mathbf{1}) = \sigma_m,\ \mu_i(\sigma_m\mathbf{1}) = 0,$ $i = 2, ...$), while for the deviator part it is ($\mu_1(\mathbf{s}) = 0,\ \mu_i(\mathbf{s}) \neq 0,\ i = 2, ...$).

Substituting the relations $(6.1)_{1,2}$ into the formula for the *Fisher-Pearson skewness coefficient* – cf., e.g., formula (20.2.2.9) in Polyanin and Manzhirov [22], and comparing it with expression $(4.2)_1$ reveals the existence of the following connection:

$$g_1 \equiv \frac{\mu_3}{\sqrt{\mu_2}^3} = 3\sqrt{3}\frac{J_3}{\sqrt{(2J_2)}^3} \tag{6.2}$$
$$= 3\sqrt{3}\frac{s_{\mathrm{I}}}{r}\frac{s_{\mathrm{II}}}{r}\frac{s_{\mathrm{III}}}{r} = \frac{1}{\sqrt{2}}\overline{J}_3 \quad \Rightarrow \quad \sin(3\theta_{sk}) = \sqrt{2}g_1 \in \langle -1, 1 \rangle.$$

The connection $(6.2)_2$ delivers grounds for assigning the name "skewness angle" to the angle defined with formula $(4.2)_1$. At the same time, this relation supplies lead to revealing one more unexpected physical interpretation of the skewness angle of statistical character, which allows for an explanation of the mysterious, at first sight, reduction of anisotropy degree of the tensor $\boldsymbol{\sigma}$ with the increasing departure of its deviator from pure shear – cf. Fig. 3 and accompanying text.

In statistical literature, there exist very well-known interpretations of central moments. The first is understood and linked with the mean value of the population of objects. The square root of the second central moment is called standard deviation and is understood as describing the magnitude of scatter, or magnitude of non-uniformity, or disorder of the population around its mean. The third central moment normalized with standard deviation is understood as describing the non-symmetry or "skewness" of the population toward the left or right wing of its statistical distribution.

It seems that these classical understandings of central moments must be modified to take into account the specific situation of applying them to the deviator of a tensor, i.e., a quantity in which the first central moment is always equal to zero. It seems that the classical interpretations should be shifted by one in view of zeroing of the first central moment of the deviator, i.e., the second central moment of the deviator should be interpreted as the mean value, and the third central moment should be interpreted as scatter or disorder of the population about the mean. This proposal finds support in NOVOZHILOV's work [18], in which he demonstrated that the second principal invariant, which is linearly proportional to the second central moment of deviator ($\mu_2 = (2/3)J_2$), is proportional to the average shear stress of tensor $\boldsymbol{\sigma}$ calculated over all directions on the unit sphere:

$$\tau_{av} = \sqrt{\frac{3}{5}}\sqrt{\mu_2} \Longleftarrow \tau_{av} = \frac{1}{\sqrt{5}}\sqrt{2J_2}, \quad \sqrt{\mu_2} = \frac{1}{\sqrt{3}}\sqrt{2J_2},$$
$$\tau_{av} \equiv \sqrt{\frac{1}{\Omega}\int \tau^2 \,\mathrm{d}\Omega}, \tag{6.3}$$
$$\tau^2 = \sigma_{\mathrm{I}}^2 l_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 l_{\mathrm{II}}^2 + \sigma_{\mathrm{III}}^2 l_{\mathrm{III}}^2 - (\sigma_{\mathrm{I}} l_{\mathrm{I}}^2 + \sigma_{\mathrm{II}} l_{\mathrm{II}}^2 + \sigma_{\mathrm{III}} l_{\mathrm{III}}^2)^2,$$

where τ_{av} is the average shear stress over all possible directions on the unit sphere, also called by Novozhilov *shear stress intensity*, τ is shear stress operating on elementary surface $\mathrm{d}\Omega$ of unit sphere, σ_{I}, σ_{II}, σ_{III} are principal stresses, l_{I}, l_{II}, l_{III} are direction cosines determining the orientation of normal to surface $\mathrm{d}\Omega$ in relation to principal directions of tensor $\boldsymbol{\sigma}$.

Pursuing the above line of approach, it is shown that the normalized third invariant of tensor deviator ($\overline{J}_3$) can be assigned an extra interpretation of the *standard deviation of "directional dipoles"*, i.e., a parameter describing orientational (directional) spread or disorder of population of elementary shears around their average value τ_{av}. For that purpose, let us first indicate that whenever some directional entities have an influence on some total (macroscopic) orientational property, an analogy with magnetic and/or electric dipoles immediately comes to mind. Then, the more ordered directional units, the bigger the overall orientational effect. As we have already indicated in the previous section, the tensor deviator can be treated as a macroscopic parameter describing the overall (average) action of the population of pure shears ("directional dipoles"). A good measure of such overall directional effect of the action of the population of pure shears is maximum shear stress $\tau_{\max}$. Novozhilov has shown that the following relations are valid, cf. formulas (2.1)–(2.6) in [18]:

$$\tau_{\max} = \frac{1}{2}\left(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}\right) = \frac{1}{\sqrt{2}} r \cos\theta_{sk} = \sqrt{\frac{5}{2}}\tau_{av}\cos\theta_{sk}, \quad \frac{\sqrt{3}}{2} \leq \cos\theta_{sk} \leq 1,$$

$$\Rightarrow 1.39 = \frac{\sqrt{3}}{2}\cdot\sqrt{\frac{5}{2}} = \frac{\tau_{\max}\left(\theta_{sk} = \pm 30^\circ\right)}{\tau_{av}} \tag{6.4}$$

$$\leq \frac{\tau_{\max}\left(\theta_{sk}\right)}{\tau_{av}} = \sqrt{\frac{5}{2}}\cos\theta_{sk} \leq \frac{\tau_{\max}\left(\theta_{sk} = 0^\circ\right)}{\tau_{av}} = \sqrt{\frac{5}{2}} = 1.58.$$

Inequalities $(6.4)_3$ show that the ratio of maximum shear stress to average shear stress $\tau_{\max}/\tau_{av}$ ($\tau_{av} = r/\sqrt{5} = 0.447r$) attains maximum value – attains maximum directional effect for pure shear mode and minimum value for uniaxial tension or uniaxial compression.

In view of the presented above argumentation, this effect can be explained when one accepts that the population of micro-shears generating macro pure shear mode is the most ordered directionally, and the population of micro-shears generating macro uniaxial tension/compression mode is the most scattered directionally. It is known from thermodynamics that entropy is a good measure of the degree of internal ordering of any system.

Corollary 1. Decrease of the value of anisotropy factor of the second order tensor with the departure of its deviator from the pure shear mode – growth of absolute magnitude of the skewness angle, can be attributed to increase of *internal entropy of the tensor*. This latter understood as the growth of orientational scatter in the population of micro pure shears generating a specific mode of the tensor deviator.

The above gives grounds to call the term $\cos(\theta_{sk})$ *entropic part of tensor anisotropy*, while $\cos(\theta_{iso})$ can be called *deviator modulus part of tensor anisotropy*.

The classical interpretation of the third central moment also does not lose its validity. This is so because bias (skewness) of micro pure shears statistical distribution generating specific macro stress state can be identified to be the reason for shifting direction of its deviator from the direction of respective reference pure shear mode $0.5(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}})$ toward either direction of the projection of the first (σ_{I}) or the third (σ_{III}) principal stress on the octahedral plane. Speaking otherwise, although different pure shears are activated when generating uniaxial tension mode and different ones when generating uniaxial compression mode, their directional scatter about mean value is the same.

The present discussion can be concluded with the statement that the value of the skewness angle delivers twofold information. Firstly, it informs about the magnitude of *internal entropy* of the tensor reflected in the value of the anisotropy factor – the greater the internal order, the bigger the value of the anisotropy factor. Secondly, it informs about the skewness of the population of micro pure shears generating the examined macro stress mode, which is reflected in the departure of the macro stress deviator direction from the direction of the corresponding pure shear mode on the octahedral plane.

7. Stress tensor in interaction with external system (environment)

7.1. Independence of linear elastic isotropic material elastic energy from the skewness angle

In the previous sections, stress tensor was considered an autonomous object. However, it is an examination of its interaction with other tensorial objects, which allows for effective modeling of real physical phenomena. One of the classical problems of mechanics, still open and attracting the interest of many researchers, is the formulation of criteria of material strength. Such criteria are commonly proposed employing the concept of *elastic energy* stored in the material macroelement under consideration. In order to obtain an expression for elastic energy of macroelement, the interaction between the stress tensor and fourth order Hooke's tensor – describing elastic properties of the material under consideration, has to be considered. The most simple case of Hooke's tensor, actually the most commonly encountered in continuum mechanics literature and being a starting point for more advanced studies, is the tensor describing properties of *linear elastic, isotropic material.* This tensor leads to the following linear constitutive relations between stress and strain tensors:

$$
\begin{aligned}
&\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} = \lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon},\\
&\qquad\qquad \boldsymbol{\varepsilon} = \mathbf{C}\cdot\boldsymbol{\sigma} = \frac{1}{9K}(\mathrm{tr}\boldsymbol{\sigma})\,\mathbf{1} + \frac{1}{2\mu}\mathbf{s}, \quad \mathbf{S}\cdot\mathbf{C} = \mathbf{I}^{(4s)},\\
&\sigma_m = K\varepsilon_v, \quad \mathbf{s} = 2\mu\boldsymbol{\varepsilon}^d, \quad \varepsilon_v \equiv \mathrm{tr}\,(\boldsymbol{\varepsilon}),\\
&\qquad\qquad \boldsymbol{\varepsilon}^d \equiv \boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_v\mathbf{1}, \quad \mathbf{1}\,(\delta_{ij}), \quad \mathbf{1}\otimes\mathbf{1}\,(\delta_{ij}\delta_{kl}),\\
&3K = 3\lambda + 2\mu = \frac{E}{1-2\nu}, \quad 2\mu = \frac{E}{1+\nu},\\
&\qquad\qquad E = \frac{\mu\,(3\lambda+2\mu)}{\lambda+\mu}, \quad 2\nu = \frac{\lambda}{\lambda+\mu},\\
&\mathbf{S} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}^{(4s)} \quad \left(S_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right),\\
&\qquad\qquad \mathbf{I}^{(4s)}\left(I^{(4s)}_{ijkl} = \frac{1}{2}\;\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right),
\end{aligned}
\tag{7.1}
$$

where $\mathbf{S}$, $\mathbf{C}$ denote isotropic stiffness and compliance tensors of elasticity, respectively, λ, μ denote Lame constants, $\mu = G$ is called the shear modulus, E, K are Young's and bulk modulus, respectively, ν denotes Poisson's ratio, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^d$ are strain and strain deviator, respectively, and $\mathbf{I}^{(4s)}$ is a fourth order symmetric unit tensor.

Relations $(7.1)_{4,5}$ correspond to relation $(7.1)_1$ upon its decomposition into spherical and deviatoric parts. Taking the dot product (full contraction) of stress $\boldsymbol{\sigma} - (3.5)_3$, and strain $\boldsymbol{\varepsilon} - (7.1)_2$ leads to the following expression for elastic energy stored in a unit volume of the linear elastic isotropic material ($\sigma_m\mathbf{1}\cdot\mathbf{s} = 0$):

$$
\begin{aligned}
&\Phi(\boldsymbol{\sigma}) \equiv \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \frac{1}{2K}\sigma_m^2 + \frac{1}{4\mu}\mathbf{s}\cdot\mathbf{s} = \frac{1}{18K}I_1^2 + \frac{1}{2\mu}J_2,\\
&\Phi(\boldsymbol{\sigma}) = \Phi(\sigma_m\mathbf{1}) + \Phi(\mathbf{s}).
\end{aligned}
\tag{7.2}
$$

There can be formulated two important remarks ensuing from the formula (7.2).

Remark 1. Elastic energy of linear elastic isotropic materials decouples into two parts, first depending on pressure and second depending on the shearing part of the stress tensor, only – or speaking alternatively, first depending on the first invariant of stress I_1 and second depending on the second invariant of stress deviator J_2, only. The mixed term is not present. When the energy is expressed in terms of small strains tensor, then the parts are connected with volumetric and distortional parts of the strain tensor, respectively.

Remark 2. Elastic energy of linear elastic, isotropic material does not depend on the skewness angle θ_{sk} (Lode angle θ_L) – it does not depend on the third principal invariant of stress tensor deviator J_3.

It is worth to examine in detail how it occurs that the Lode angle θ_L (skewness angle θ_{sk}) cancels out from the expression for elastic energy. For that purpose, formulas $(3.8)_{1-3}$ are instrumental:

$$
\begin{aligned}
\mathbf{s}\cdot\mathbf{s} &= \left(\frac{2}{3}\sigma_{ef}\right)^2 \left[\cos(\theta_L)\mathbf{N}_{\mathrm{I}} + \cos\left(\theta_L - \frac{2}{3}\pi\right)\mathbf{N}_{\mathrm{II}} + \cos\left(\theta_L + \frac{2}{3}\pi\right)\mathbf{N}_{\mathrm{III}}\right] \\
&\quad \cdot \left[\cos(\theta_L)\mathbf{N}_{\mathrm{I}} + \cos\left(\theta_L - \frac{2}{3}\pi\right)\mathbf{N}_{\mathrm{II}} + \cos\left(\theta_L + \frac{2}{3}\pi\right)\mathbf{N}_{\mathrm{III}}\right] \\
&= \left(\frac{2}{3}\sigma_{ef}\right)^2 \left[\cos^2(\theta_L) + \cos^2\left(\theta_L - \frac{2}{3}\pi\right) + \cos^2\left(\theta_L + \frac{2}{3}\pi\right)\right] \\
&= \left(\frac{2}{3}\sigma_{ef}\right)^2 \frac{3}{2} = \frac{2}{3}\sigma_{ef}^2 = 2J_2.
\end{aligned}
\tag{7.3}
$$

While formula (7.2) is very common knowledge, the present author has not encountered in the literature an explicitly formulated statement similar to that in Remark 2. Probably because it is so obvious, it very often escapes attention or is somehow forgotten.

In numerous works devoted to more advanced materials research, linear elastic, isotropic constitutive relation is assumed for the investigated material behavior to search in subsequent steps for "elastic energy criterion" of material effort, which contains the Lode angle as an argument. In view of Remark 2, such an approach leads to methodological inconsistencies, at best. Their removal requires clearly formulated and well-justified additional assumptions in each specific case, usually missing at present. What factors then can be identified to be responsible for very often encountered in experimental works dependence of, e.g., critical stress of plastic yielding on skewness (Lode) angle, while at the same time material exhibits, with acceptable approximation, linear elastic and isotropic behavior. There can be identified at least three such causes:

- (i) material is actually *not linear elastic*,
- (ii) material is *not isotropic*,
- (iii) so-called *internal constraints* operate in the material – of force or kinematic character, and of known or unknown physical origins.

The first factor (i) can be identified to be the primary reason, it is a standard that elastic energy functions proposed for rubberlike and/or polymeric materials as potentials for deriving their constitutive relations are proposed to be functions of all three principal invariants of the strain tensor. Thus, it is rightly assumed that their elastic energy depends on the skewness (Lode) angle of the strain tensor. Actually, the second factor also plays a role in polymers. Elasticity in polymeric materials is physically generated by a change of internal entropy of these materials and not internal energy, cf., e.g., Müller [17] pp. 111–112. Due to that, polymeric material, even when isotropic at zero loading, changes its internal symmetry usually into a transversely isotropic one when subjected to moderate strains. It returns to its original symmetry (isotropy) upon removal of loading. The second factor (ii) does not require additional comments. A typical situation when the third factor (iii) becomes important is in the case of, for example, composite materials in which some kind of reinforcement elements are present.

7.2. Mechanism of formation of six invariants of the stress tensor

Let us return to the eigenstates problem and Rychlewski's energy orthogonal decomposition of stress tensor [24], cf. formula (2.5) and accompanying text. In the discussed above case of linear elastic material, isotropy is a special case of interaction of stress tensor with Hooke's tensor, which resulted in fully decoupled decomposition of stress tensor into two parts spherical–deviatoric (pressure––shear). In the most general case of linear elastic material, the anisotropic material stress tensor can be decomposed into six parts, an energy orthogonal in the sense of $(2.5)_4$. This sextuple is independent of any vector basis – reference frame $\{\mathbf{e}_i\}$. Hence, its components actually make *six linearly independent stress tensor invariants*. We may notice that in this manner, the very basic feature underlying the tensor notion – i.e., invariance, is presented in its full light. The statement about six invariants may seem to contradict an earlier statement that only three linearly independent tensor invariants can be generated, but this is not the case, as can be realized from the commentary below.

A very interesting loop has come full circle. Ricci-Curbastro motivated by the idea of quadratic forms invariance, devised objects – and the whole mathematical apparatus, which predict that in the case of second order symmetric tensor its representation's six components transform in a specific manner with the change of reference frame. Next, it was identified that from these six components there can always be formed a set of *three linearly independent invariants* independent from the coordinates frame, and a set of another three parameters changing with coordinates' frame change. An analogy of free vector comes to mind when a tensor is considered an autonomous object. When the tensor is considered in some environment, in interaction with other tensors, then *six linearly independent invariants* can be formed out of its representation components – and the analogy of anchoring the vector to a fixed reference point (frame) comes to mind.

Let us consider the following situation in order to better understand in what sense anchoring of the tensor takes place. Take two autonomous tensors, e.g., stress tensor $\boldsymbol{\sigma}$ and not coaxial with it strain tensor $\boldsymbol{\varepsilon}$ – a typical situation for non-isotropic materials. Each of these tensors is fully described by three invariants and three Euler angles. The Euler angles characterize the orientation of each tensor with respect to any conceivable reference frame. While these angles change with the change of reference frame, the *relative orientation* of specific stress tensor with respect to specific non-coaxial strain tensor does not change. Upon their interaction – e.g., taking their scalar product, only their relative orientation is important, which manifests itself in the possibility of generating six invariants, as indicated by formula $(2.5)_1$. So, anchoring of the tensor means that the orientation of the principal axes of the first or the second tensor takes over the role of the reference frame – and any other reference frame is not needed, as it does not play any role.

Rychlewski's energy orthogonal decomposition (2.5) resulting from the eigenvalues problem (2.4) delivers yet another very inspiring and prolific hint for research works. It indicates that when with the initiation of some physical phenomenon, e.g., plastic yield flow, there can be associated some fourth order tensor such as, e.g., Hooke's tensor in the case of elastic energy plastic flow yield criterion, then loadings inducing the phenomenon can be divided, in the most general case, into six *classes of loadings*, depending on specific symmetry of Hooke's tensor. This suggests, for example, that when the safety of a structure is to be assured to preclude plastic yield flow, then for each identified class of loadings different value of safety coefficient can be expected to be appropriate. Issues discussed in this section can only be signaled due to space limitations. More detailed treatment of interaction of second order tensor with environment – other tensors and the consequences of it – deserve and require separate research efforts and works.

8. Some remarks on conditions and applicability of biaxial (planar) loadings for experimental examination of the influence of skewness angle on materials behavior

8.1. Simple shear versus planar shear experimental testing layouts

A considerable attention has been devoted in the present work to the theoretical issues connected with pure shear mode. Let us discuss at present two major experimental layouts leading to the actual physical realization of the pure shear mode as defined by (5.1), i.e., so-called "simple shear" and "planar shear" experimental testing layouts. In experimental mechanics, they are usually considered in terms of strain rather than stress tensor. When the tested material is isotropic, then this distinction bears no conceptual difference because, in such a case, a simple equivalence of states exists between stress and strain tensors due to their coaxiality. Many misunderstandings exist in the literature regarding the difference between simple shear versus planar shear testing layouts. Both these testing layouts belong to the class of biaxial tests and make the practical realization of a pure shear state. In order to clarify the issues, it is pointed out that no difference between planar shear and simple shear exists in terms of definition (5.1), as in both cases trace and determinant of strain tensor during experimental testing are with very good accuracy equal to zero ($\mathrm{tr}(\boldsymbol{\varepsilon}) = 0, \det(\boldsymbol{\varepsilon}) = 0$). In the case of testing materials for which one or both of these conditions are not fulfilled, variation of specimen thickness is also measured to introduce relevant corrections. The difference exists between the kinematics of two testing layouts, i.e., the motion of material points. While different kinematics results in different deformation gradients, the principal values of stretch tensor $\mathbf{U}$ ($\mathbf{F} = \mathbf{RU}$) are exactly the same in both layouts, though differently situated in the laboratory frame, cf. Fig. 5. Here, only the most important characteristics of simple and planar shear are succinctly and explicitly formulated to possibly facilitate the decision on selecting one or the other experimental layout for attaining specific experimental research tasks. An interested reader can find a more detailed discussion of planar shear and simple shear, for example, in Ogden [20] and Ziółkowski [33].

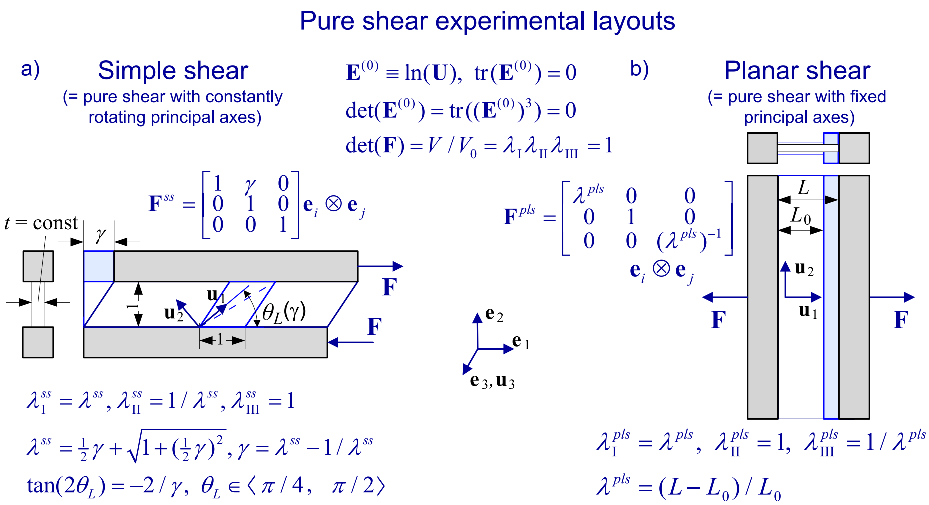

Fig. 5. Difference between experimental testing layouts of *simple shear* (a) and *planar shear* (b) is shown graphically. The $\mathbf{u}_i$ denote Lagrangian principal axes, $\boldsymbol{\theta}_L$ denotes the orientation angle of Lagrangian principal axes with respect to fixed laboratory frame, $\mathbf{E}^{(\mathbf{0})}$ denotes logarithmic Lagrangian strain measure, and $\boldsymbol{\lambda}_J$ are principal stretches.

The *major matching feature* of simple shear and planar shear testing layouts is the *identical strain pattern* shared by both layouts, finding reflection in identical values of principal stretches. In the case of simple shear, they are $\lambda_{\mathrm{I}}^{ss} = \lambda^{ss}$, $\lambda_{\mathrm{II}}^{ss} = 1/\lambda^{ss}$, $\lambda_{\mathrm{III}}^{ss} = 1$, where $\lambda^{ss} = \gamma/2 + \sqrt{1 + (\gamma/2)^2}$ and γ denotes the so-called *shear parameter*. In the case of planar shear, principal stretches take the form: $\lambda_{\mathrm{I}}^{pls} = \lambda^{pls}$, $\lambda_{\mathrm{II}}^{pls} = 1$, $\lambda_{\mathrm{III}}^{pls} = 1/\lambda^{pls}$, where $\lambda^{pls} = \Delta L/L_0$. When λ^{ss} is equated to λ^{pls} one to one correspondence can be immediately found between γ and ΔL. In both testing schemes, volume is preserved $\det(\mathbf{F}) = \lambda_{\mathrm{I}}^{ss}\lambda_{\mathrm{II}}^{ss}\lambda_{\mathrm{III}}^{ss} = \lambda_{\mathrm{I}}^{pls}\lambda_{\mathrm{II}}^{pls}\lambda_{\mathrm{III}}^{pls} = \mathrm{d}v/\,\mathrm{d}V = 1$, where $\mathrm{d}v$ denotes the elementary volume in the actual configuration and, $\mathrm{d}V$ is the elementary volume in the initial configuration.

The *major distinctive feature* differing simple shear from planar shear is that *principal axes constantly rotate* with the advancement of shear loading in simple shear layout, but, in planar shear layout, *principal axes remain constant (fixed)* relative to the laboratory frame at all times.

The simple shear testing layout is very popular (standard) in experimental testing of behavior and/or properties of metallic materials. The planar shear testing layout is very often used (standard) in examining polymeric materials. Besides strain pattern, many additional factors may influence choosing one layout or the other. For example, the stiffness of metallic samples prevents early warping of the sample during simple shear testing. On the other hand, testing metallic sheets in planar shear scheme might require considerably larger forces in comparison to the simple shear scheme of testing. It is also worth indicating that loadings used in testing of metallic samples as a standard *do not induce a change of symmetry of the material.*

In the case of polymeric materials, loadings used in their testing as a standard *do induce a change of their symmetry* – due to the entropic origin of polymeric elasticity, e.g., initially isotropic polymeric material changes its symmetry to transversely isotropic under testing load. From the above discussion, it can be concluded that execution of simple shear and planar shear tests on the same material allows to evaluate the influence of principal axes rotation on the behavior of the material.

8.1.1. Efficient test for determining whether linear elastic material is isotropic. A very interesting experimental application of pure shear modes is that the results of five tests in which linear elastic material is submitted to a set of five linearly independent pure shear loadings enable to uniquely determine experimentally whether the material is elastically isotropic. The above suggestion is a direct consequence resulting from Theorem 4.1 in Blinowski, cf. [2]. At the same time, it gives information what is the minimum number of tests necessary for finding out whether the material is isotropic. Indeed, when the linear elastic material is submitted to five tests with pure shear loadings, for example, the ones listed by Blinowski in proof of Theorem 4.1 with the following representations in the fixed laboratory frame:

$$\overbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_1 =}, \quad \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_2 =}, \quad \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_3 =}, \quad \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}}^{\boldsymbol{\tau}_4 =}, \quad \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{\boldsymbol{\tau}_5 =}, \tag{8.1}$$

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \neq 0, \quad \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \neq 0, \quad \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_4 \neq 0, \quad \boldsymbol{\tau}_4 \cdot \boldsymbol{\tau}_5 \neq 0,$$

and in response to these loadings, the shear moduli determined from elaborated experimental data in charts $\boldsymbol{\varepsilon}_i = (1/(2\mu_i))\boldsymbol{\tau}_i$ will show to have the same value $\mu_i = \mu$, $i = 1, ..., 5$. Then, this will prove that the tested material is isotropic linear elastic. The technical realization of such a testing cannot be further discussed here due to limited space.

8.2. Deficiency of biaxial (planar) tests for finding experimentally material behavior sensitivity to skewness (Lode) angle

In 1959 Davies and Connelly introduced the so-called *triaxiality factor*, defined as the quotient of stress first principal invariant divided by effective stress $\left(\eta_{DC} \equiv I_1/\sqrt{3J_2} = 3\sigma_m/\sigma_{ef};\ \ \sigma_{ef} \neq 0\right)$, cf. formula (35) in [5]. They were motivated in this proposal by supposition, correct in view of their own and later research, that spherical tension ($\sigma_m > 0$) called by them rather exotically *triaxial tension*, has a strong influence on the loss of ductility of metals, and the need to have some parameter to describe this effect. The name triaxial tension for spherical tension is rather unfortunate because it gives a false impression that no (negative) pressure but general three-dimensional multiaxial stress states are subject of description with this parameter. The triaxiality factor gained considerable attention and use when Wierzbicki and his collaborators pointed out that not only spherical tension (negative pressure) but also Lode angle can considerably influence ductility and other properties of metals. The important issue in this was that in 2005 Wierzbicki and Xue found that in the class of biaxial tests ($\sigma_{\mathrm{III}} = 0$) a unique relation exists between the Lode angle – normalized principal third invariant of deviator, and the triaxiality factor in the form $\overline{J}_3 = \cos(3\theta_L) = -(27/2)\eta(\eta^2 - 1/3)$ – cf. formula (8) in Bai and Wierzbicki [1]. Wierzbicki and collaborators adopted a slightly modified definition of triaxiality factor than the original one ($\eta \equiv \sigma_m/\sigma_{ef} = \eta_{DC}/3$). It seems to be worth introducing the concept of *isomorphic triaxiality factor*, naturally corresponding to *isomorphic cylindrical coordinates*, defined as follows – cf. formulas (4.1):

$$\eta_i \equiv \frac{z}{r}, \quad \eta_i = \tan(\theta_{iso}) = \frac{3}{\sqrt{2}}\eta. \tag{8.2}$$

The class of *biaxial tests* is defined by the condition that always one of the principal values of the stress tensor is equal to zero. According to the ordering convention of principal values, this could be the smallest, middle or the largest principal value ($\sigma_{\mathrm{III}} \leq \sigma_{\mathrm{II}} \leq \sigma_{\mathrm{I}}$), but usually, it is written conventionally that the third principal value is zero, regardless of the standard ordering convention. During any kind of biaxial tests, in view of $\sigma_{\mathrm{III}} = 0$, two control parameters, e.g., two principal values of stress $(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$, uniquely determine any set of *three principal stress invariants* fully characterizing properties of stress tensor treated as a sovereign object, e.g., $\{\sigma_m, J_2, J_3\}$. Some other convenient pair of control parameters can be selected, for example $(\sigma_m, \Delta\sigma \equiv (\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}))$. Simple transformations lead to the following relations valid for biaxial stress states, cf. (3.12):

$$
\begin{aligned}
&\sigma_{\mathrm{III}} = 0 \ \Rightarrow\ \sigma_m = \frac{\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}}{3}, \quad \Delta\sigma = (\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}),\\
&s_{\mathrm{I}} = \sigma_{\mathrm{I}} - \sigma_m, \quad s_{\mathrm{II}} = \sigma_{\mathrm{II}} - \sigma_m, \quad s_{\mathrm{III}} = -\sigma_m,\\
&J_2 = s_{\mathrm{III}}^2 - (s_{\mathrm{I}} s_{\mathrm{II}}) = \frac{1}{3}\left[\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 - \sigma_{\mathrm{I}}\sigma_{\mathrm{II}}\right] = \frac{1}{4}\left[3\sigma_m^2 + \Delta\sigma^2\right],\\
&J_3 = s_{\mathrm{III}}\,(s_{\mathrm{I}} s_{\mathrm{II}}) = \frac{1}{27}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}})(\sigma_{\mathrm{I}} - 2\sigma_{\mathrm{II}})(2\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})\\
&\quad = \frac{1}{4}\sigma_m[\Delta\sigma^2 - \sigma_m^2] = \sigma_m[J_2 - \sigma_m^2].
\end{aligned}
\tag{8.3}
$$

Taking advantage of relation $(8.3)_6$, it is easy to show that the following inequalities are always valid for any planar (two-dimensional) stress state – during any biaxial test:

$$
\begin{aligned}
&\sigma_{ef} = \sqrt{3J_2} = \sqrt{\frac{9}{4}\left[\sigma_m^2 + \frac{1}{3}\Delta\sigma^2\right]} \geq \frac{3}{2}\,|\sigma_m| \geq 0,\\
&||\mathbf{s}|| = \sqrt{2J_2} \geq \sqrt{\frac{3}{2}}|\sigma_m| = 1.225|\sigma_m| \geq 0.
\end{aligned}
\tag{8.4}
$$

The observation can be formulated in the form of the following property:

Property 1. The modulus of the deviatoric (shearing) part of any non-zero (non-trivial) planar stress state is always greater than the modulus of its spherical part.

The direct conclusion from Property 1 is that *no purely spherical (isotropic) planar tensor exists* or equivalently *the only purely spherical planar tensor is a zero tensor*. Thus, it can be stated that in the case of any non-trivial planar stress tensor – biaxial tests domain, its shearing part dominates over its spherical part.

Wierzbicki and Xue's constraint relation valid for biaxial tests can be expressed in the equivalent form of classical third-power polynomial equation:

$$\eta^3 - \frac{1}{3}\eta + \frac{2}{27}\sin(3\theta_{sk}) = 0, \quad \eta \equiv \frac{\sigma_m}{\sigma_{ef}},$$
(8.5)
$$\overline{J}_3 = \sin(3\theta_{sk}) = \frac{27}{2}\left[\frac{1}{3}\eta - \eta^3\right],$$

where the Lode angle is replaced by the skewness angle ($\overline{J}_3 \equiv \cos(3\theta_L)$).

This equation can be solved with the same method as the one used for finding stress principal values from the characteristic equation, cf. (3.8). The solution can be written in the following form:

(8.6)
$$\sin(60^\circ - \theta_{sk}) = \frac{3}{2}\eta, \qquad \eta \in \left\langle \frac{2}{3}, \frac{1}{3} \right\rangle, \qquad \theta_{sk} \in \langle -30^\circ, 30^\circ \rangle,$$
$$\text{when} \quad \sigma_{\mathrm{III}} = 0 \le \sigma_{\mathrm{II}} \le \sigma_{\mathrm{I}},$$
$$\sin(\theta_{sk}) = \frac{3}{2}\eta, \qquad \eta \in \left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle, \qquad \theta_{sk} \in \langle -30^\circ, 30^\circ \rangle,$$
$$\text{when} \quad \sigma_{\mathrm{III}} \le \sigma_{\mathrm{II}} = 0 \le \sigma_{\mathrm{I}},$$
$$\sin(60^\circ + \theta_{sk}) = -\frac{3}{2}\eta, \qquad \eta \in \left\langle -\frac{1}{3}, \frac{2}{3} \right\rangle, \qquad \theta_{sk} \in \langle -30^\circ, 30^\circ \rangle,$$
$$\text{when} \quad \sigma_{\mathrm{III}} \le \sigma_{\mathrm{II}} \le \sigma_{\mathrm{I}} = 0;$$
$$\mathrm{sign}(\theta_{sk}) = \mathrm{sign}(\overline{J}_3), \qquad \mathrm{sign}\,(\eta) = \mathrm{sign}\,(\sigma_m)\,.$$

In the above, the standard denotation convention of principal stresses was employed ($\sigma_{\mathrm{III}} \le \sigma_{\mathrm{II}} \le \sigma_{\mathrm{I}}$) and the following identities: $4\sin^3(\theta_{sk}) - 3\sin(\theta_{sk}) + \sin(3\theta_{sk}) = 0$, $\sin(\theta - 120^\circ) = -\sin(60^\circ + \theta)$, $\sin(\theta + 120^\circ) = \sin(60^\circ - \theta)$.

Explicit relations (8.6) linking the triaxiality factor and skewness angle ($\eta \leftrightarrow \theta_{sk}$), cf. Fig. 6, are *three bijections* (one to one relations) in three sharing edges

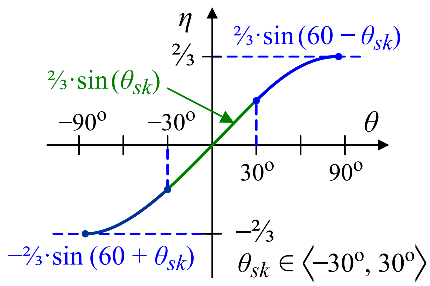

FIG. 6. Triaxiality factor as a function of the skewness angle $\eta = \eta\,(\theta_{sk})$.

separate subdomains, which altogether make the entire *two-parameter* domain (half-plane) of biaxial tests stress states, see also Fig. 7.

From relation (8.5) it can be easily concluded that the constant value of triaxiality factor η corresponds to the constant value of skewness (Lode) angle θ_{sk}. However, it is not obvious, what are the curves of constancy paths of these parameters in the biaxial tests domain (plane). The following Theorem I is helpful in this issue.

Theorem 1. The radial lines (rays) coming out from the origin ($\sigma_{\rm I} = 0$, $\sigma_{\rm II} = 0$) of the coordinates frame of the biaxial tests domain, i.e., half-plane ($\sigma_{\rm II} \leq \sigma_{\rm I}$), are lines of constant values of triaxiality factor $\eta = \text{const}$, and, at the same time, lines of constant values of skewness (Lode) angle $\theta_{sk} = \text{const}$.

Proof. The radial lines running from the origin can be described as follows:

$$\sigma_{\rm II} = a\sigma_{\rm I}\,(a = \text{const}) \Rightarrow \sigma_m = \left(\frac{1}{3}\right)(\sigma_{\rm I} + \sigma_{\rm II}) = \left(\frac{1}{3}\right)(1+a)\,\sigma_{\rm I},$$

$$\begin{aligned} \sigma_{ef} &= \sqrt{\sigma_{\rm I}^2 + \sigma_{\rm II}^2 - \sigma_{\rm I}\sigma_{\rm II}} = \sqrt{1-a+a^2}|\sigma_{\rm I}| \\ &\Rightarrow \eta = \frac{\sigma_m}{\sigma_{ef}} = \frac{1}{3}\frac{(1+a)}{\sqrt{1-a+a^2}}\text{sign}(\sigma_{\rm I}), \end{aligned} \tag{8.7}$$

$$a = \text{const} \Leftrightarrow \eta = \text{const} \Leftrightarrow \theta_{sk} = \text{const}.$$

In the case $\sigma_{\rm I} = 0$, $\sigma_{\rm II}$ can take any value, and it is $\eta = -\frac{1}{3} = \text{const}$, $\theta_{sk} = -30^0 = \text{const}$. Q.E.D.

Theorem 2. The relations $\sigma_m(\sigma_{ef}, \theta_{sk})$, $\sigma_{ef}(\sigma_m, \theta_{sk})$, $\theta_{sk}(\sigma_m, \sigma_{ef})$ resulting from relations (8.6), valid for plane stress states, are bijections (one to one relations) in three sharing edges but otherwise separate subdomains of the whole domain of biaxial tests stress states, except on the line $\sigma_m = \frac{1}{3}(\sigma_{\rm I} + \sigma_{\rm II}) = 0$, on which $\eta = \theta_{sk} = 0$ for any value of $\sigma_{ef} = \sqrt{3}|\sigma_{\rm I}|$.

Proof. It is straightforward to find that the *skewness angle* θ_{sk} maintains constant value on *radial lines* running from the origin ($\sigma_{\rm I} = 0, \sigma_{\rm II} = 0$) of biaxial tests domain coordinates frame, *the mean value of stress* σ_m maintains constant value on *45° slanted lines*, and the *effective stress* σ_{ef} maintains constant value on *ellipsoids with centers in the origin* ($\sigma_{\rm I} = 0, \sigma_{\rm II} = 0$) of biaxial tests domain, cf. also Fig. 7.

In view of the above, at any specific point of three complementary subdomains of the biaxial tests domain, the value of any variable chosen from the triple set $\{\sigma_m, \sigma_{ef}, \theta_{sk}\}$ can be uniquely determined by the values of the two remaining ones. On line $\sigma_m = 0$ it is $\sigma_m = \sigma_{\rm I} + \sigma_{\rm II} = 0 \Rightarrow J_3 = 0$, $\sigma_{ef} = \sqrt{3}\sigma_{\rm I} \Rightarrow \eta = \theta_{sk} = 0$. Q.E.D.

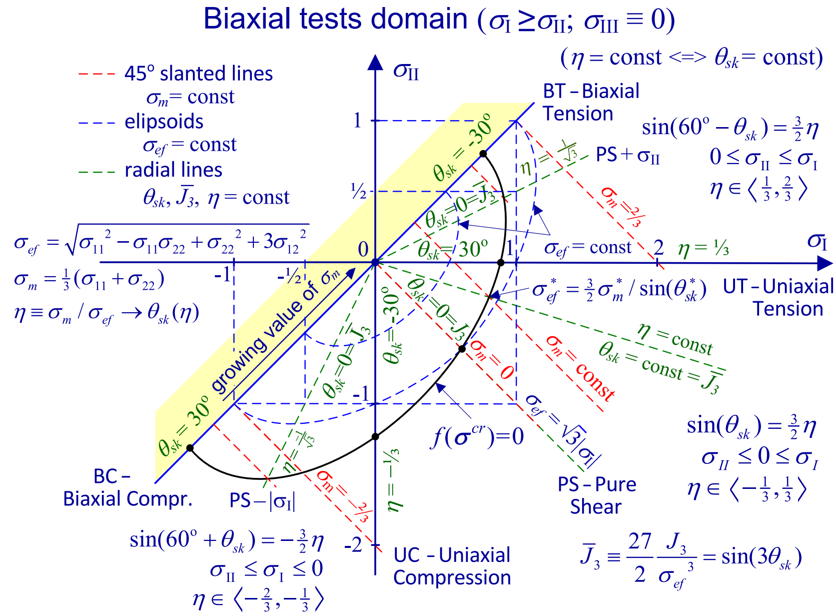

FIG. 7. Graphical illustration of the biaxial tests domain parameterization in terms of parameters $(\sigma_m, \sigma_{ef}, \theta_{sk})$. The $f(\sigma^{cr}) = 0$ marks some hypothetical convex, critical surface (e.g., plastic yield), for which critical values of effective stress depend on skewness angle. The second quarter of the chart $(0, \sigma_{\mathrm{I}} \geq \sigma_{\mathrm{II}} \leq 0)$ makes the domain of states attainable in standard tests on the tubular specimen (with no internal pressure).

An important open problem of *experimental mechanics of materials* comprises determining *critical stress states surfaces* conditioning initiation of some physical processes in materials, for example, plastic yield flow, damage, cracking or the start of phase transition. The following observation can be formulated by taking advantage of Theorems 1 and 2.

Corollary 2. In the case of the convex critical surface, with the aid of any type of biaxial test, for any fixed value of the mean stress (pressure) $\sigma_m = \sigma_m^*$, critical effective stress σ_{ef}^* can be determined for only a single value of the skewness (Lode) angle θ_{sk}^*.

In the case of the *convex critical surface*, with the aid of any type of biaxial test, for any fixed value of the skewness (Lode) angle $\theta_{sk} = \theta_{sk}^*$, critical effective stresses σ_{ef}^* can be determined for only three values of the mean stress (pressure) $\sigma_m = \sigma_m^*$.

The direct conclusion from Corollary 2 is that planar (biaxial) tests, among them very common tension (compression)-torsion tests on tubular samples – also the ones with internal pressure, are *not sufficient for executing methodologically correct experimental examination of the influence of skewness (Lode) angle on materials behavior*. This is so because upon executing only biaxial tests, no adequate experimental data results can be collected to reliably separate the influence of mean stress and/or skewness angle on the possible variations of critical effective stresses. One value for any fixed pressure and/or three values for any fixed skewness angle are indeed insufficient for such a purpose. This observation delivers a clear incentive for the development and use of experimental techniques in which all three parameters characterizing stress state can be independently controlled to induce in the specimen not only planar stress state (two-dimensional) but fully three-dimensional stress state loadings. They should make possible determining critical effective stress, or other critical parameters, for example, effective fracture strains in the whole range of skewness angle values at freely prescribed, fixed mean stress.

Relations (8.6) valid for *biaxial (plane) tests* show that in such a case, the values of the triaxiality factor must always remain in the range $\eta \in \langle -\frac{2}{3}, \frac{2}{3} \rangle$, while in the general case of *three-dimensional multiaxial tests*, the triaxiality factor can take any value from the range $\eta \in \langle -\infty, +\infty \rangle$. In many *experimental mechanics* publications, in which results from *biaxial tests* are presented, values of triaxiality factor exceeding the two-third value $\frac{2}{3} < \eta$ can be observed, which may seem to be *incorrect*. However, experimental observation of the *triaxiality factor greater than* $\frac{2}{3}$ rather indicates that the *biaxiality condition of test was lost*, and in the sample general (three-dimensional) stress state started to exist. This delivers a hint to develop experimental methodologies in which the *triaxiality factor* is used as an effective *indicator of passing from a plane state of stress to a three-dimensional state of stress.*

Relations (8.6) delivering explicit connection between triaxiality factor and skewness angle $\theta_{sk}(\eta)$ are *very convenient for numerical computations*, because they enable the determination of the value of the skewness (Lode) angle θ_{sk} from the value of the triaxiality factor η *much more efficiently numerically* than, e.g., from Wierzbicki and Xue's formula, not to speak about computing it from definition formula what necessitates computation of the third invariant of deviator (J_3). Selection of the correct subformula (8.6) is very easy because it can be decided upon *the value of* η falling into a specific range of values. For example, when $\eta^* = 0.51$, then it belongs to the range $\eta^* \in \langle \frac{1}{3}, \frac{2}{3} \rangle$; hence, $\theta^*_{sk} = 60° - \sin^{-1}\left(\frac{3}{2}\eta^*\right) = 10.1°$. The reverse of formulas (8.6) is not so convenient because to select the correct reverse subformula (8.6) to compute $\eta\left(\theta_{sk}\right)$, a combination of signs of two non-zero principal values of stress tensor $(\operatorname{sign}(\sigma_{\mathrm{I}}),\ \operatorname{sign}(\sigma_{\mathrm{II}}))$ must be known.

Since the popularization of *triaxiality factor* η by Wierzbicki [1], it started to be very frequently used in charts as an argument (governing parameter) – often together with Lode angle in a three-dimensional charts, to present experimental results obtained in *biaxial tests* in order to present the influence of Lode angle on various properties of metals and other materials. However, the present study shows that the *triaxiality factor*, in general, *is not a convenient operand* to be used *for the presentation of experimental biaxial tests' results.* This is so because when taken at its face value, it contains tangled together information on two in principle linearly independent parameters characterizing stress tensor (loading), i.e., σ_m and σ_{ef}. Such an entanglement projected to the presented results makes them somehow blurred. In the case of *biaxial tests*, one to one relation exists between the triaxiality factor and skewness (Lode) angle, i.e., the *constant value of the triaxiality factor corresponds to the constant value of skewness (Lode) angle.* Due to that, it is advisable to directly use *skewness angle as governing parameter in charts* presenting experimental results from biaxial tests. Possibly, with information indicating the mode of loading: tensioning ($0 < \sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}$), mixed ($\sigma_{\mathrm{I}} < 0 < \sigma_{\mathrm{II}}$) or compressive ($\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}} < 0$). In this manner, specific information presented in the chart from the biaxial test will be delivered in a transparent, methodologically unambiguous manner.

Results obtained in the present section deliver sound methodological grounds for rational and effective designing of experimental programs aiming at determining mechanical properties of complex materials, as they enable precise evaluation of which material characteristics can be acquired in biaxial test and which one can only be obtained in truly triaxial tests.

9. Concluding remarks

In the paper, a concise historical survey on the tensor notion was presented. The survey gives grounds for the view that the key features, which decided that tensors nowadays became the language of all advanced technological sciences, are linearity and invariance. In view of the universal use of tensors to model real phenomena in all kinds of applied sciences, a profound understanding of what the tensors actually are, what their specific features are, and how they mutually interact is of great importance. The present work addressed these issues on the example of the stress tensor, a generic instance of a second order symmetric tensor. It was pointed out that tensors can be viewed and/or understood from several perspectives as either: algebraic objects, linear transformations or oriented geometrical objects. This versatility might be another feature that determined the attractiveness of tensors. Attention was focused here on identifying and finding possibly the best manner of description of eigenproperties of second order symmetric tensors. The results and conclusions obtained here specifically for stress tensor are of general character, and *mutatis mutandis* translate to second order symmetric tensors, which may have other interpretations in modeling real physical phenomena and/or objects.

The executed analysis showed that it is convenient to introduce new parameterization of second order tensor eigenproperties by introducing the concept of isotropy angle and skewness angle. The first parameter allows for very quick and transparent, at first sight, evaluation/separation of isotropic and anisotropic parts of the tensor, which actually coincides with the decomposition of the tensor into spherical and deviatoric parts. The special role played by pure shear modes, which explains and justifies using them as comparison reference states in the definition of the skewness angle, is elucidated. At the same time, a strong rationale for replacing Lode angle with skewness angle in the characterization of second order tensors, in view of a much clearer and more comprehensive physical interpretation of the skewness angle, is given. A new, very simple formula was presented for the anisotropy factor of the stress tensor, based on the tensor orbit notion, expressed in terms of isotropy angle and skewness angle. An original statistical interpretation of principal invariants of tensor deviator was developed, which allowed explaining why the anisotropy factor of the second order tensor diminishes with the departure of the tensor from pure shear mode. The reason for that can be attributed to the growth of internal entropy of the tensor understood as an increasing orientational disorder of elementary pure shears population generating given macro stress state. It was shortly outlined that interaction of second order tensor with other tensorial objects, e.g., fourth order Hooke's tensor representing elastic properties of a material, results in the possibility of constructing not only three but six parameters of second order symmetric tensor remaining invariant upon change of coordinates frame (basis). It was indicated that this gives premises for introducing and developing the *weighted effective stress* notion, which takes into account interaction of stress tensor with other tensorial objects characterizing material to improve the classical effective stress notion.

Several observations were presented concerning multiaxial tests targeted at examining the influence of skewness angle on material behavior. In particular, it was clearly demonstrated that the only difference between the two, very popular in experimental mechanics, testing layouts (simple shear and planar shear tests), is that, in the first case, principal axes rotate constantly with increasing loading while in the second case, the orientation of principal axes remains all the time constant with respect to the laboratory reference frame. The kinematics, deformation gradient $\mathbf{F}$, in the two tests is different, but the principal stretches (strains) in both cases are exactly the same. New original formulae have been derived, which deliver explicit relation between the so-called triaxiality factor and skewness angle (normalized third invariant of deviator) valid for biaxial tests. It was shown that executing only biaxial tests does not allow for the methodologically correct determination of critical surfaces, e.g., plastic yielding, phase transition start or initiation of fracture (effective fracture strain). This is so because, in the case of biaxial tests for any fixed value of mean stress (pressure), only a single critical value can be determined of some indicator (e.g., effective stress, thermodynamic force of phase transition, effective fracture strain, etc.) corresponding to only one value of skewness (Lode) angle of critical stress. This situation calls for the development of new experimental testing layouts enabling independent control of all three invariant parameters characterizing stress loading of the test specimen. Such testing layouts should enable determining critical values of stress (strain) loadings for the entire range of skewness angles $\theta_{sk} \in \langle -30^\circ, 30^\circ \rangle$ at a fixed value of pressure.

In view of rapid developments in computer technology, there can be noticed a very strong need and demand for developing efficient methods of visualization of not only vector but also second order (and higher) tensor fields. The classical principal axes ellipsoid can be noted as the first attempt of this kind, which completely fails when some principal values are negative. The present study shows that due to the very rich structure of second order tensors, the task of reasonably visualizing second order tensor fields grasping simultaneously all the characteristics and flavors of these objects is rather desperate. At the same time, it shows that the way out of this dilemma is proper structuralization of the second order tensor to single out, construct parameters proper for describing the specific problem of interest, and only later visualization of the fields of such parameters. Otherwise, the visualization results may prove to be obscure, incomprehensible and intricate. The proposed new parametrization of second order tensor eigenproperties can be very helpful in such tasks.

Supplement

In order to make the work as self-contained as possible essential definitions connected with the *external symmetry* of tensors are recalled here, cf., e.g., [21].

Definition S1. A set of second order tensors $\mathbf{Q}$ with properties:

$$\mathcal{O} = \left\{ \mathbf{Q} \in \mathcal{T}^2; \quad \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{1}, \quad \det \mathbf{Q} = \pm 1 \right\}, \tag{S.1}$$

is a group and is called the *group of orthogonal tensors.*

Definition S2. A subset of orthogonal tensors for which $\det(\mathbf{Q}) = 1$:

$$\mathscr{R} = \{ \mathbf{Q} \in \mathcal{T}^2; \quad \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{1}, \quad \det(\mathbf{Q}) = 1 \}, \;\; \mathscr{R} \subset \mathcal{O}, \tag{S.2}$$

is a group and is called the *proper orthogonal group or rotation group* (SO_n).

Definition S3. A *group of external symmetry* of tensor $\mathbf{T} \in \mathcal{T}^p$ (p denotes the order of the tensor) we call a subset of all orthogonal tensors $\mathbf{Q}$, which satisfy the following condition:

$$\mathcal{O}_T = \{ \mathbf{Q} \in \mathcal{O}; \quad \mathbf{Q} * \mathbf{T} = \mathbf{T} \}, \;\; \mathcal{O}_T \subseteq \mathcal{O}. \tag{S.3}$$

Tensors $\mathbf{T}$ that satisfy condition (S.3) are called *symmetric* with respect to orthogonal transformations $\mathbf{Q} \in \mathcal{O}_T$.

Definition S4. Tensor is *isotropic* when the group of its external symmetry is the whole set of orthogonal tensors $\mathcal{O}_T = \mathcal{O}$, cf. (S.1).

Definition S5. Tensor is *hemitropic* (also called *proper-isotropic*) when the group of its external symmetry is the whole set of proper orthogonal tensors $\mathcal{O}_T = \mathscr{R}$, cf. (S.2).

Note: The above definitions plainly show that symmetry property is a characteristic of a tensor as an integrated entity of basis and representation (components in the basis) and not the tensor components matrix (representation) only.

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