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Effect of interlaminar imperfections on a behaviour of laminated plates

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Summary The effect of interlaminar bonding imperfections on behaviour of shells, investigated in [1], is elaborated here for a special case of laminated plates. The problem is stated for linear elastic materials, within the small displacement gradient theory. The obtained results are ilustrated by an example.

Einfluß der Zwischenschichtdefekte auf das Verhalten geschichteter Platten

Übersicht Der in [1] beschriebene Einfluß der Zwischenschichtdefekte auf das Verhalten geschichteter Schalen wird hier speziell für den Sonderfall geschichteter Platten studiert. Das Problem wird unter Berücksichtigung der linearen Elastizitätstheorie und der Theorie der kleinen Verschiebungsgradienten betrachtet. Die erhaltenen Ergebnisse werden anhand eines Beispiels illustriert.

1 Introduction

In the paper [1], a general approach to the formation of 2D-theories for laminated linear elastic shells with initial imperfections in the interlaminar bonding was proposed. These bonding imperfections are supposed to be sufficiently small (compared to the lamina thickness) and randomly distributed over lamina interfaces. The proposed approach takes also into account the effect of the interlaminar strain discontinuities on the shell behaviour. In this paper, following the line of approach given in [1], we elaborate a similar problem for thin laminated plates. To this end, we postulate a certain modified theory of laminated plates which makes it possible to evaluate the effect of interlaminar bonding imperfections on the behaviour of a plate. The considerations are carried out for linear elastic laminae and within a small displacement gradient theory. The obtained results are illustrated by an example.

Throughout the paper sub- and superscripts $\alpha, \beta, \ldots$ run over 1, 2, and A, B run over $1, \ldots, M$, unless otherwise stated; summation convention holds for $\alpha, \beta, \ldots$. Derivatives of an arbitrary function $f = f(\mathbf{x}, z, \tau)$ will be denoted by $f_{,\alpha} \equiv \partial f/\partial x_\alpha$, $f' \equiv \partial f/\partial z$, $\dot{f} \equiv \partial f/\partial \tau$, where $\mathbf{x} \equiv (x_1, x_2)$.

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Preliminary concepts and assumptions

The subject of considerations is a laminated (multilayered) plate, a fragment of which is shown in Fig. 1. A region occupied by an undeformed plate is parameterized by Cartesian coordinates $x_1, x_2, x_3 \equiv z$, where $\mathbf{x} \equiv (x_1, x_2) \in \Pi$, $z \in [-\delta/2, \delta/2]$, Π being plate midplane and δ the plate thickness. It is assumed that $z = 0$ is the symmetry plane of the plate. By τ we denote the time coordinate. The plate is made of $M+1$ laminae Λ_A, $A = 1, \ldots, M+1$. By Φ_A we denote the interface between the A-th and $A+1$-th laminae, which is situated on a coordinate plane $z = z_A$. The thickness of an arbitrary A-th lamina is constant and will be denoted by δ_A.
We assume that:

(i) the material of every lamina is macro-homogeneous linear elastic, and at every point has a plane of elastic symmetry $z = \text{const.}$;
(ii) on every interface Φ_A there are small bonding imperfections across which there is a unilateral contact without friction between adjacent laminae; the part of Φ_A occupied by bonding imperfections is denoted by Δ_A.

By $T_{\alpha\beta}$, $T_{\alpha 3}$, T_{33} we shall denote the components of the stress tensor at an arbitrary point of the plate (except of interlaminar planes). The components of a displacement vector will be denoted by u_α, u_3. The plate is loaded in the direction of z-axis by a constant body force b and by tractions $p(\mathbf{x}, \tau)$ on the upper boundary $z = -\delta/2$. The mass density is denoted by $\varrho = \varrho(z)$, being constant in every lamina. The principle of conservation of momentum and that of the moment of momentum for an arbitrary plate element bounded by coordinate planes:
$x_1 = \text{const.}$, $x_1 + dx_1 = \text{const.}$, $x_2 = \text{const.}$, $x_2 + dx_2 = \text{const.}$, $z = \pm\delta/2$ lead to the known equations:

$$Q_{\alpha,\alpha} + f = 0, \qquad M_{\alpha\beta,\beta} - Q_\alpha + m_\alpha = 0, \tag{1}$$

where we have denoted:

$$\begin{aligned}
Q_\alpha &\equiv \int_{-\delta/2}^{\delta/2} T_{\alpha 3}(\mathbf{x}, z, \tau)\, dz, \\
M_{\alpha\beta} &\equiv \int_{-\delta/2}^{\delta/2} z T_{\alpha\beta}(\mathbf{x}, z, \tau)\, dz, \\
f &\equiv p(\mathbf{x}, \tau) + \int_{-\delta/2}^{\delta/2} \varrho(z)\, [b - \ddot{u}_3(\mathbf{x}, z, \tau)]\, dz, \\
m_\alpha &\equiv - \int_{-\delta/2}^{\delta/2} \varrho(z)\, z \ddot{u}_\alpha(\mathbf{x}, z, \tau)\, dz.
\end{aligned} \tag{2}$$

In order to formulate the basic hypotheses of the proposed plate theory which describe both laminated structure and bonding imperfections, we have to introduce auxiliary functions: $h_A(z)$, $g_A(z)$, $\hat{g}_A(z)$, $z \in [-\delta/2, \delta/2]$ defined in [1], diagrams of which are shown in Fig. 2. In the sequel we denote $\nu_A \equiv \delta_A/\delta_{A+1}$. We shall use only derivatives of these functions, given by:

$$h_A'(z) = \begin{cases} 0 & \text{if} \quad z \in [-\delta/2, z_{A-1}) \cup (z_{A+1}, \delta/2], \\ 1 & \text{if} \quad z \in (z_{A-1}, z_A), \\ -\nu_A & \text{if} \quad z \in (z_A, z_{A+1}); \quad \nu_A \equiv \delta_A/\delta_{A+1}, \end{cases}$$

$$g_A'(z) = \begin{cases} 0 \quad \text{if} \quad z \in [-\delta/2, z_{A-1}] \cup (z_A, \delta/2], \\ (z - z_{A-1})/\delta_A \quad \text{if} \quad z \in (z_{A-1}, z_A), \end{cases}$$

$$\hat{g}_A'(z) = \begin{cases} 0 \quad \text{if} \quad z \in [-\delta/2, z_A) \cup [z_{A+1}, \delta/2], \\ (z_{A+1} - z)/\delta_{A+1} \quad \text{if} \quad z \in (z_A, z_{A+1}). \end{cases}$$

Here, we have denoted $z_0 \equiv -\delta/2$. We also introduce functions:

$$\alpha(z) \equiv 1 - 4z^2/\delta^2, \qquad z \in (-\delta/2, \delta/2),$$

$$\chi_A(\mathbf{x}) = \begin{cases} 1 \quad \text{if} \quad \mathbf{x} \in \Delta_A, \\ 0 \quad \text{if} \quad \mathbf{x} \in \Phi_A \setminus \Delta_A. \end{cases}$$

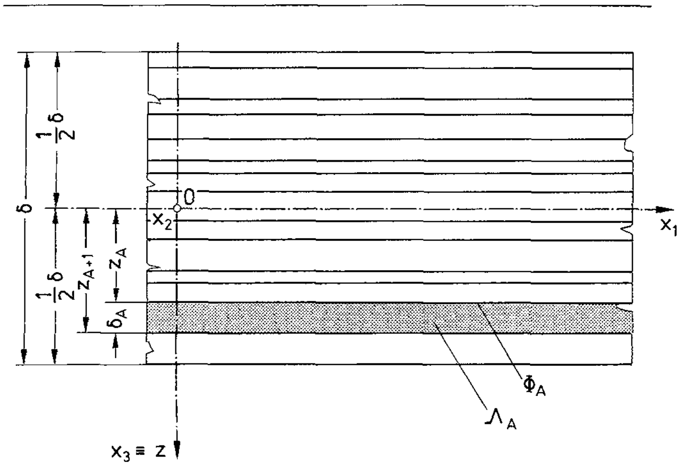

Fig. 1. Cross-section of a laminated plate

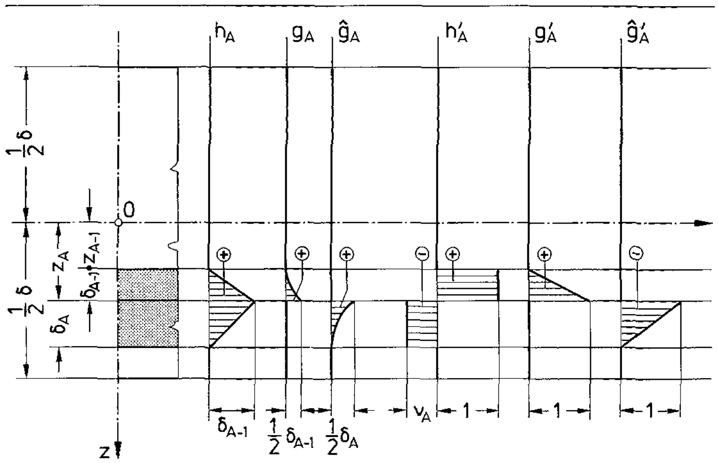

Fig. 2. Diagrams of shape functions and their derivatives

The formulation of the proposed plate theory will be based on three assumptions:

I *Strain state hypothesis* (SSH). The strain components $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\alpha 3}$ can be approximated in the form:

$$\varepsilon_{\alpha\beta}(\mathrm{x}, z, \tau) \cong \varphi_{(\alpha,\beta)}(\mathrm{x}, \tau)\, z\,,$$

$$\begin{aligned}\varepsilon_{\alpha 3}(\mathrm{x}, z, \tau) \cong{}& \alpha(z)\ [w_{,\alpha}(\mathrm{x}, \tau) + \varphi_\alpha(\mathrm{x}, \tau)]/2 \\ &+ \sum_{A=1}^{M} [h'_A(z)\ q^A_\alpha(\mathrm{x}, \tau) + g'_A(z)\ \chi_A(\mathrm{x})\ s^A_\alpha(\mathrm{x}, \tau) + \hat{g}'_A(z)\ \chi_A(\mathrm{x})\ \hat{s}^A_\alpha(\mathrm{x}, \tau)]\,,\end{aligned} \tag{3}$$

where $w(\mathrm{x}, \tau)$, $\varphi_\alpha(\mathrm{x}, \tau)$, $q^A_\alpha(\mathrm{x}, \tau)$, $s^A_\alpha(\mathrm{x}, \tau)$, $\hat{s}^A_\alpha(\mathrm{x}, \tau)$, are sufficiently regular functions, which constitute basic kinematic unknowns of the theory.

Bearing in mind introduced above form of shape functions h_A, g_A, $\hat{g}_A$, it can be seen that the unknown functions q^A_α are responsible for jumps of shear strains across interlamina planes Φ_A and unknowns s^A_α, $\hat{s}^A_\alpha$ describe possible jumps of displacements u_α, across imperfections Δ_A on every Φ_A. We shall assume that functions s^A_α, $\hat{s}^A_\alpha$ are linear dependent; the form of this dependents will be specified below.

II *Displacement state hypothesis* (DSH). The displacement components u_α, u_3 can be approximated by:

$$u_\alpha(\mathrm{x}, z, \tau) \cong \varphi_\alpha(\mathrm{x}, \tau)\, z\,, \qquad u_3(\mathrm{x}, z, \tau) \cong w(\mathrm{x}, \tau)\,. \tag{4}$$

The distribution of displacements u_α across the thickness of a laminated plate, which takes into account possible displacement jumps across imperfections, could be postulated in the form:

$$u_\alpha = \varphi_\alpha z + \sum_{A=1}^{M} [h_A(z)\ q^A_\alpha + g_A(z)\ s^A_\alpha + \hat{g}_A(z)\ \hat{s}^A_\alpha]\,.$$

In eqs. $(4)_1$ and $(3)_1$ we have neglected terms involving shape functions (but not their derivatives!), as small parameters of an order of a lamina thickness δ_A. Due to the introduced approximations, the strain compatibility conditions on a plate midplane $z=0$ are satisfied only with a certain approximation.

III *Stress state approximation* (SSA). In stress-strain relations of the linear elasticity, the stress component T_{33} will be neglected. Hence, stress-strain relations for $T_{\alpha\beta}$ are:

$$T_{\alpha\beta} = \bar{C}_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}\,, \qquad T_{\alpha 3} = 2C_{\alpha 3\beta 3}\varepsilon_{\beta 3}\,, \tag{5}$$

where $\bar{C}_{\alpha\beta\gamma\delta} \equiv C_{\alpha\beta\gamma\delta} - C_{\alpha\beta 33}C_{\gamma\delta 33}/C_{3333}$ and $C_{\alpha\beta\gamma\delta}$, C_{3333}, as well as $C_{\alpha 3\beta 3}$ are components of the elasticity tensor (the remaining components of this tensor are equal to zero).

Instead of SSA, in the form $T_{33}=0$, we could also introduce a less restrictive assumption postulated in Reissner plate theory, where T_{33} depends on the load p on the upper boundary plane of the plate.

We have assumed above that every lamina is made of a macro-homogeneous material; hence we shall denote

$$\bar{C}^A_{\alpha\beta\gamma\delta} \equiv \bar{C}_{\alpha\beta\gamma\delta}(z) \quad \text{for} \quad z \in (z_{A-1}, z_A)\,,$$

$$C^A_{\alpha 3\beta 3} \equiv C_{\alpha 3\beta 3}(z) \quad \text{for} \quad z \in (z_{A-1}, z_A)\,,$$

where $\bar{C}^A_{\alpha\beta\gamma\delta}$, $C^A_{\alpha 3\beta 3}$ are the known components of the elasticity tensor of A-th lamina. Here and in the sequel, $z_{A-1}=-\delta/2$ for $A=1$. The jump of shear stress $T_{\alpha 3}$ across any interface Φ_A, which is produced by shear strains done by bonding imperfections, has to be equal to zero; hence from (3)

$$C^A_{\alpha 3\beta 3}\chi_A(\mathrm{x})\; \overset{A}{s}_\beta(\mathrm{x},\tau) = C^{A+1}_{\alpha 3\beta 3}\chi_A(\mathrm{x})\; \overset{\wedge A}{s}_\beta(\mathrm{x},\tau)\,, \qquad \mathrm{x} \in \Pi\,. \tag{6}$$

Thus, functions w, φ_α, $\overset{A}{q}_\alpha$, $\overset{A}{s}_\alpha$, $\overset{\wedge A}{s}_\alpha$ can be taken as independent kinematic unknowns of the theory.

Now, for an arbitrary function $F=F(\mathrm{x},z,\tau)$ we define:

$$[\![F]\!]^A\;(\mathrm{x},\tau) = F^+(\mathrm{x},z_A,\tau) - F^-(\mathrm{x},z_A,\tau)\,,$$

where $F^+(\mathrm{x},z_A,\tau) \equiv \lim\limits_{z\searrow z_A} F(\mathrm{x},z,\tau)$, $F^-(\mathrm{x},z_A,\tau) \equiv \lim\limits_{z\nearrow z_A} F(\mathrm{x},z,\tau)$. We also introduce the following denotation:

$$T^A_{\alpha 3}(\mathrm{x},\tau) \equiv \lim_{z\nearrow z_A} T_{\alpha 3}(\mathrm{x},z,\tau)\,, \qquad \mathrm{x} \in \Pi\,,$$

for the shear stress acting on the interface Φ_A. Hence

$$\begin{aligned} T^A_{\alpha 3}(\mathrm{x},\tau) = 2C^A_{\alpha 3\beta 3}\{&2^{-1}\alpha_A(z)\;[w_{,\beta}(\mathrm{x},\tau) + \varphi_\beta(\mathrm{x},\tau)] \\ &- \nu_{A-1}\overset{A-1}{q}_\beta(\mathrm{x},\tau) + \overset{A}{q}_\beta(\mathrm{x},\tau) + \overset{A}{s}_\beta(\mathrm{x},\tau)\;\chi_A(\mathrm{x}) + \overset{\wedge A}{s}_\beta(\mathrm{x},\tau)\;\chi_A(\mathrm{x})\}\,. \end{aligned}$$

The interlamina conditions have now the form:

$$[\![T_{\alpha 3}]\!]^A\;(\mathrm{x},\tau) = 0 \quad \text{for} \quad \mathrm{x} \in \Pi\,, \tag{7}$$

$$T^A_{\alpha 3}(\mathrm{x},\tau)\;\chi_A(\mathrm{x}) = 0 \quad \text{for} \quad \mathrm{x} \in \Pi\,. \tag{8}$$

Equations (1)–(8) constitute the general formulation of the laminated plate theory with interlaminar bonding imperfections. It can be observed, that these relations lead to the system of governing equations for unknowns w, φ_α, $\overset{A}{q}_\alpha$, $\overset{A}{s}_\alpha$, $\overset{\wedge A}{s}_\alpha$. In the special case of homogeneous plates and neglecting interlaminar imperfections the obtained equations reduce to a certain version of Reissner plate theory, cf. [2], [3].

For sake of simplicity at the beginning of this section we have assumed that the material structure of the plate under consideration is symmetric with respect to the midplane $z=0$. However, the proposed theory could be also applied to laminated plates in which components $C_{\alpha\beta\gamma\delta}(z)$ of the elasticity tensor satisfy condition:

$$z_0 \int_{-\delta/2}^{\delta/2} C_{\alpha\beta\gamma\delta}(z)\; dz = \int_{-\delta/2}^{\delta/2} C_{\alpha\beta\gamma\delta}(z)\; z\, dz\,,$$

for a certain constant z_0. In this case, coordinate $z \in [-\delta/2, \delta/2]$ in all relations of the present paper has to be replaced by coordinate $\zeta \in [-\delta/2 - z_0, \delta/2 + z_0]$.

3
Averaged formulation of the theory

Relations of laminated plate theory proposed in Sect. 2 have purely theoretical meaning, and can not be applied to engineering problems since the exact distribution of imperfections is not known *a priori*. In the sequel, we shall assume that this distribution is random, and the mean density on every interface is known. Denote by $B_A(\mathrm{x})$ a sphere with a center at a point (x, z_A) and a radius δ_A. For an arbitrary function $f = f(\mathrm{x}, \tau)$, $\mathrm{x} \in \Pi$, we define an average value:

$$\langle f \rangle_A(\mathrm{x}) \equiv \frac{1}{\text{area}\,(B_A(\mathrm{x}) \cap \Phi_A)} \int\limits_{B_A(\mathrm{x}) \cap \Delta_A} f(y)\, dy.$$

Using the procedure similar to that introduced in [1], we shall convert to the averaged form of equations presented in Sect. 2. To this end, we assume that there exists a constant η_A such that $\eta_A \cong \langle \chi_A \rangle_A\,(\mathrm{x})$ for every $\mathrm{x} \in \Pi$. Hence η_A is a mean density of imperfections on every interface Φ_A. These imperfection densities $\eta_A \in [0, 1)$ are assumed to be known in every special problem under consideration.

Following [1] we shall replace interlamina conditions (7), (8) by

$$\langle [\![T_{\alpha 3}]\!]^A \rangle_A\,(\mathrm{x}, \tau) = 0 \quad \text{for} \quad \mathrm{x} \in \Pi, \tag{9}$$

$$\langle T^A_{\alpha 3} \chi_A \rangle_A\,(\mathrm{x}, \tau) = 0 \quad \text{for} \quad \mathrm{x} \in \Pi. \tag{10}$$

At the same time we introduce an averaged form of SSH, replacing eqs. (3) by

$$\begin{aligned} &\varepsilon_{\alpha\beta}(\mathrm{x}, z, \tau) \cong \varphi_{(\alpha,\beta)}(\mathrm{x}, \tau)\, z, \\ &\varepsilon_{\alpha 3}(\mathrm{x}, z, \tau) \cong \alpha(z)\,[w_{,\alpha}(\mathrm{x}, \tau) + \varphi_\alpha(\mathrm{x}, \tau)]/2 + \sum_{A=1}^{M} [h'_A(z)\, q^A_\alpha(\mathrm{x}, \tau) + g'_A(z)\, r^A_\alpha(\mathrm{x}) + \hat{g}'_A(z)\, \hat{r}^A_\alpha(\mathrm{x})], \end{aligned} \tag{11}$$

where $r^A_\alpha(\mathrm{x}, \tau) \equiv \langle \chi_A s^A_\alpha \rangle_A\,(\mathrm{x}, \tau)$, $\hat{r}^A_\alpha(\mathrm{x}, \tau) \equiv \langle \chi_A \hat{s}^A_\alpha \rangle_A\,(\mathrm{x}, \tau)$. It is easy to see, that the first of eqs. (3) does not change its form after averaging, since the averaging operation is applied exclusively to highly oscilating functions involving χ_A. Averaging (6) yields

$$C^A_{\alpha 3\beta 3} r^A_\beta(\mathrm{x}, \tau) = C^{A+1}_{\alpha 3\beta 3} \hat{r}^A_\beta(\mathrm{x}, \tau). \tag{12}$$

Similarly, formula for shear stress acting on the interface will be now assumed in the averaged form:

$$T^A_{\alpha 3}(\mathrm{x}, \tau) = 2C^A_{\alpha 3\beta 3}\{\alpha(z_A)\,[w_{,\beta}(\mathrm{x}, \tau) + \varphi_\beta(\mathrm{x}, \tau)]/2 - \nu_{A-1} q^{A-1}_\beta(\mathrm{x}, \tau) + q^A_\beta(\mathrm{x}, \tau) + r^A_\beta(\mathrm{x}, \tau)\}. \tag{13}$$

Eqs. (1), (2), (3), (4), (5), (9), (10) and (12) represent the formulation of laminated plate theory with interlaminar imperfections, which can be applied to the analysis of engineering problems provided that mean imperfection densities η_A are known. Aforementioned relations lead to the system of equations for unknown functions w, φ_α, q^A_α, r^A_α, $\hat{r}^A_\alpha$. The characteristic feature of the obtained relations is that unknowns q^A_α, describing strain discontinuities across interfaces, and r^A_α, $\hat{r}^A_\alpha$, due to the interlamina imperfections, cf. [1], can be eliminated. This will be done in the subsequent section.

4
Governing relations

For sake of simplicity we shall use denotations $C^A_{\alpha\beta} \equiv C^A_{\alpha 3\beta 3}$ and $\alpha_A \equiv \alpha(z_A) = 1 - 4z^2_A/\delta^2$. Interlaminar condition (9), after taking into account (5), (11) and (12) leads to the following system of linear equations for q^A_α

$$-\nu_{A-1} C^A_{\alpha\beta} q^{A-1}_\beta + (\nu_A C^{A+1}_{\alpha\beta} + C^A_{\alpha\beta})\, q^A_\beta - C^{A+1}_{\alpha\beta}\, q^{A+1}_\beta = \alpha_A [\![C_{\alpha\beta}]\!]^A\,(w_{,\alpha} + \varphi_\alpha)/2,$$

where $[\![C_{\alpha\beta}]\!]^A \equiv C^{A+1}_{\alpha\beta} - C^A_{\alpha\beta} = C^{A+1}_{\alpha 3\beta 3} - C^A_{\alpha 3\beta 3}$. After calculating a system of parameters $K^{AB}_{\alpha\beta}$ from equations

$$-\nu_{A-1} C^A_{\alpha\beta} K^{A-1,B}_{\beta\gamma} \delta^{A-1}_C + (\nu_A C^{A+1}_{\alpha\beta} + C^A_{\alpha\beta})\, K^{AB}_{\beta\gamma} \delta^A_C - C^{A+1}_{\alpha\beta} K^{A+1,B}_{\beta\gamma} \delta^{A+1}_C = \delta^B_C \delta_{\alpha\gamma}, \tag{14}$$

where δ^B_C, $\delta_{\alpha\gamma}$ are Kronecker symbols, we obtain

$$q^A_B = \sum_{B=1}^{M} K^{AB}_{\beta\gamma} [\![C_{\gamma\delta}]\!]^B\, \alpha_B(w_{,\delta} + \varphi_\delta)/2. \tag{15}$$

It is easy to see that for homogeneous plates $[\![C_{\alpha\beta}]\!]^A = 0$, and eqn. (15) yields $q_\alpha^A = 0$. This means that functions q_α^A describe jumps of elasticity tensors across interfaces of a laminated plate.

From condition (10), and using (5), (11), (12) we obtain

$$C_{\alpha\beta}^A r_\beta^A = C_{\alpha\beta}^{A+1} \hat{r}_\beta^A = \nu_{A-1} C_{\alpha\beta}^A \eta_A q_\beta^{A-1} - C_{\alpha\beta}^A \eta_A q_\beta^A - 2^{-1} C_{\alpha\beta}^A \alpha_A (w_{,\beta} + \varphi_\beta)\ \eta_A . \tag{16}$$

Keeping in mind definition of $T_{\alpha 3}^A$ in Sect. 3. and combining eqn. (13) with (16) we obtain

$$T_{\alpha 3}^A = 2(1-\eta_A)\ C_{\alpha 3\beta 3}^A [\alpha(z_A)\ (w_{,\beta} + \varphi_\beta)/2 - \nu_{A-1} q_\beta^{A-1} + q_\beta^A] .$$

Substituting the right hand side of eqn. (15) into above formula, and denoting

$$H_{\alpha 3\beta 3}^A \equiv -C_{\alpha 3\beta 3}^A \sum_{B=1}^{M} (K_{\gamma\delta}^{AB} - \nu_{A-1} K_{\gamma\delta}^{A-1,B})\ [\![C_{\delta\beta}]\!]^B , \tag{17}$$

we arrive at the final form of the shear stresses acting on the interfaces:

$$T_{\alpha 3}^A(\mathbf{x}, \tau) = \alpha(z_A)\ (1-\eta_A)\ (C_{\alpha 3\beta 3}^A - H_{\alpha 3\beta 3}^A)\ [w_{,\beta}(\mathbf{x}, \tau) + \varphi_\beta(\mathbf{x}, \tau)] . \tag{18}$$

Due to the fact that every lamina is thin we shall apply an approximation

$$T_{\alpha 3}(\mathbf{x}, z, \tau) \cong T_{\alpha 3}^A(\mathbf{x}, \tau) \quad \text{for} \quad z \in (z_{A-1}, z_A) .$$

Using formulae (2), $(5)_1$, $(11)_1$ and (18), after denoting

$$\begin{aligned} B_{\alpha\beta\gamma\delta} &\equiv \int_{-\delta/2}^{\delta/2} C_{\alpha\beta\gamma\delta}(z)\ z^2\, dz , \\ A_{\alpha\beta}^A &\equiv \alpha(z_A)\ (C_{\alpha 3\beta 3}^A - H_{\alpha 3\beta 3}^A)\ \delta_A , \end{aligned} \tag{19}$$

we obtain

$$\begin{aligned} M_{\alpha\beta}(\mathbf{x}, \tau) &= B_{\alpha\beta\gamma\delta} \varphi_{\gamma,\delta}(\mathbf{x}, \tau) , \\ Q_\alpha(\mathbf{x}, \tau) &= \sum_{A=1}^{M} (1-\eta_A)\ A_{\alpha\beta}^A [w_{,\beta}(\mathbf{x}, \tau) + \varphi_\beta(\mathbf{x}, \tau)] , \qquad \mathbf{x} \in \Pi . \end{aligned} \tag{20}$$

Moreover, using (4) and denoting

$$\mu \equiv \int_{-\delta/2}^{\delta/2} \varrho(z)\ dz , \qquad I \equiv \int_{-\delta/2}^{\delta/2} \varrho(z)\ z^2\, dz , \tag{21}$$

we shall rewrite Eqs. (1) to the form

$$\begin{aligned} Q_{\alpha,\alpha}(\mathbf{x}, \tau) + p(\mathbf{x}, \tau) + \mu b &= \mu \ddot{w}(\mathbf{x}, \tau) , \\ M_{\alpha\beta,\beta}(\mathbf{x}, \tau) - Q_\alpha(\mathbf{x}, \tau) &= I \ddot{\varphi}_\alpha(\mathbf{x}, \tau) , \qquad \mathbf{x} \in \Pi . \end{aligned} \tag{22}$$

Eqs. (20), (22), together with denotations (17), (19), (21), where $K_{\gamma\delta}^{AB}$ have to be calculated from (14), constitute the system of governing equations of the proposed theory of linear elastic laminated plates with initial interlaminar imperfections. The effect of interlaminar imperfections on the plate behaviour is described by the form of moduli $A_{\alpha\beta}^A$. The obtained equations have to be considered together with three boundary conditions, and initial conditions for functions w and φ_α.

If all mean densities of imperfections η_A are equal, $\eta_A = \eta$, then denoting

$$A_{\alpha\beta} \equiv \sum_{A=1}^{M} \alpha(z_A)\ (C_{\alpha 3\beta 3}^A - H_{\alpha 3\beta 3}^A)\ \delta_A , \tag{23}$$

we shall rewrite constitutive equations (20) to the form

$$\begin{aligned} Q_\alpha(\mathbf{x}, \tau) &= (1-\eta)\ A_{\alpha\beta} [w_{,\beta}(\mathbf{x}, \tau) + \varphi_\beta(\mathbf{x}, \tau)] , \\ M_{\alpha\beta}(\mathbf{x}, \tau) &= B_{\alpha\beta\gamma\delta} \varphi_{(\gamma,\delta)}(\mathbf{x}, \tau) , \qquad \mathbf{x} \in \Pi . \end{aligned} \tag{24}$$

Substituting right hand side of (24) into eqs. (22), we obtain the system of equations of the proposed laminated plate theory

$$(1-\eta)\ A_{\alpha\beta}(w_{,\alpha\beta}+\varphi_{\beta,\alpha})+p+\mu b=\mu\ddot{w},$$
$$B_{\alpha\beta\gamma\delta}\varphi_{\gamma,\delta\beta}-(1-\eta)\ A_{\alpha\beta}(w_{,\beta}+\varphi_{\beta})=I\ddot{\varphi}_{\alpha}, \tag{25}$$

which describe the effect of interlaminar imperfections on the laminated plate behaviour in the explicit form.

5 Example of application

Now, we shall apply the general results obtained in the previous sections given by eqs. (22), (24) and (25). We shall investigate the effect of interlaminar imperfections on the free vibrations of a laminated rectangular plate. The plate is simply supported on oposite edges $x_1=0$, $x_1=L$. In this case, the problem depends only on arguments $x_1\in[0,L]$, $\tau\in[-\infty,+\infty]$. By means of $\varphi_1=0$, eqs. (25) reduce to the form

$$(1-\eta)\ A[w_{,11}(x,\tau)+\varphi_{,1}(x,\tau)]=\mu\ddot{w}(x,\tau),$$
$$B\varphi_{,11}(x,\tau)-(1-\eta)\ A[w_{,1}(x,\tau)+\varphi(x,\tau)]=I\ddot{\varphi}(x,\tau), \tag{26}$$

where we have denoted $A\equiv A_{11}$, $\varphi\equiv\varphi_1$, $B\equiv B_{1111}$, $x\equiv x_1$, and where we have neglected loads $p(x,\tau)$ and b. Solutions of eqs. (26) can be assumed in the form

$$w(x,\tau)=C\ \sin(n\pi x/L)\ \exp(i\omega_n\tau),$$
$$\varphi(x,\tau)=D\ \cos(n\pi x/L)\ \exp(i\omega_n\tau), \tag{27}$$

where $n=1,2,\ldots$, and C, D are arbitrary constants. Substituting (27) into (26) we obtain a system of linear equations for C, D:

$$[-(1-\eta)\ A(n\pi/L)^2+\omega_n^2\mu]\ C-(1-\eta)\ A(n\pi/L)\ D=0,$$
$$-(1-\eta)\ A(n\pi/L)\ C-[B(n\pi/L)^2+(1-\eta)\ A-\omega_n^2 I]\ D=0. \tag{28}$$

Nontrivial solutions exist only if the determinant of this system is equal to zero. This condition leads to the following equation for vibration frequencies ω_n:

$$\mu I\omega_n^4-\{\mu B(n\pi/L)^2+(1-\eta)\ A[\mu+I(n\pi/L)^2]\}\ \omega_n^2+(1-\eta)\ AB(n\pi/L)^4=0. \tag{29}$$

Since $\mu I/L^4$ can be treated as a small parameter of an order $(\delta/2)^4$, then the solutions of eqn. (29) can be obtained in the approximate form:

$$I\omega_n^2=(B/2)\ (n\pi/L)^2+(1-\eta)\ (A/2\mu)\ [\mu+I(n\pi/L)^2]\pm\{(B/2)\ (n\pi/L)^2+(1-\eta)\ A[\mu+I(n\pi/L)^2]$$
$$-(1-\eta)\ IAB(n\pi/L)^4\ \{\mu B(n\pi/L)^2+(1-\eta)\ A[\mu+I(n\pi/L)^2]\}^{-1}\}. \tag{30}$$

As it is known for thin plates, the effect of rotational inertia term $I\ddot{\varphi}$ in eqs. (26) on frequency ω is rather small; hence, setting $I\rightarrow 0$ and assuming that ω_n^2 is bounded, we see that eqn. (30) holds only if one takes into account the branch with sign "$-$". In this case, we obtain the final formula:

$$\omega_n^2=(1-\eta)\ AB(n\pi/L)^4\ \{\mu B(n\pi/L)^2+(1-\eta)\ A[\mu+I(n\pi/L)^2]\}^{-1}. \tag{31}$$

This result illustrates explicitly the effect of interlaminar imperfections (represented by a mean imperfection density η) on the free vibration frequency ω_n for the laminated plate under consideration. On passing to a delaminated plate, i.e. setting $\eta\rightarrow 1$, we obtain from (30) that $\omega_n\rightarrow 0$.

Taking into account sign "+" in Eq. (30) we would arrive at the high frequency vibrations:

$$\omega_n^2=(B/I)\ (n\pi/L)^2+(1-\eta)\ (A/I)\ [1+(I/\mu)\ (n\pi/L)^2]$$
$$-(1-\eta)\ IAB(n\pi/L)^4\ \{\mu B(n\pi/L)^2+(1-\eta)\ A[\mu+I(n\pi/L)^2]\}^{-1}.$$

The last term in the above formula is relatively small, compared to the first and second one, and can be neglected. Passing to a total delamination $\eta\rightarrow 1$ we would obtain $\omega_n^2\rightarrow(B/I)\ (n\pi/L)^2$. This result follows immediately from eqs. (28) under assumption that $C=0$, $D\neq 0$. In this case (which has purely theoretical meaning) we would deal exclusively with rotational vibrations.

6
Conclusions

It has been shown, in the framework of the Reissner-type plate theory in which one deals with three basic unknown functions, how the laminated structure of a plate can be described. To this end, we have to calculate parameters $K_{\alpha\beta}^{AB}$ from systems of linear algebraic equations (14). After that we have to determine coefficients $H_{\alpha 3\beta 3}^{A}$ from eqs. (17) and $A_{\alpha\beta}^{A}$, $B_{\alpha\beta\gamma\delta}$ from eqs. (19). The additional unknowns q_{α}^{A}, which describe jumps of material properties across lamina interfaces, have been eliminated from governing equations. Let us observe, that this situation does not hold if we apply the broken line hypothesis [4]–[7]. Here, the number of basic unknowns can be, for example, equal to $3(M+2)$, where $(M+1)$ is the number of lamina [8]. At the same time, we have taken into account possible bonding imperfections on the interfaces between adjacent laminae. It needs to be emphasized that the resulting relations (20)–(22) have a relatively simply form, and can be applied to the analysis of engineering problems illustrated by the example given above.

References

1. **Woźniak, Cz.; Woźniak, M.**: Composite shells with interlaminar imperfections. Arch. Appl. Mech. 63 (1993) 543–550
2. **Timoshenko, S.; Woinowsky-Krieger, S.**: Theory of plates and shells. New York: McGraw-Hill 1959
3. **Reissner, E.**: On the theory of bending of elastic plates. J. Math. Phys. 23 (1944) 184–191
4. **Reissner, E.**: On bending of elastic plates. Quart. Appl. Math. 5 (1947) 55–68
5. **Chepiga, B. E.**: K utochnenoi teorii sloistykh obolochek. Prikl. Mekh. 12 (1976) 45–49
6. **Epstein, M.; Glockner, P. G.**: Nonlinear analysis of multilayered shells. Int. J. Solids Struct. 13 (1977) 1081–1089
7. **Ogibalov, P. M.; Suvorova, Yu. V.**: Mekhanika armirovannykh plastikov. Moskau: Izd. MGU 1965
8. **Baczyński, Z. F.**: On modelling thermoelastic processes in layered composites. Acta Mechanica 88 (1991) 245–251