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The dynamic modelling of elastic wavy plates

B. Michalak, Cz. Woźniak, M. Woźniak

Summary The aim of the contribution is to formulate an engineering theory describing the dynamic behaviour of periodically waved shell-like elements, called wavy plates. On the basis of the proposed theory, the effect of coupling between free macro- and micro-vibrations of a wavy plate is investigated. It is also shown that the homogenized model of wavy plates (obtained by scaling down the wavelength parameters) cannot be applied in the analysis of dynamic problems.

Key words shell, modelling, dynamics, periodic structure, homogenization

1
Introduction

The subject of the paper is a thin periodic shell-like structure made of a linear-elastic homogeneous material, which is referred to as a wavy plate. An example of this structure is shown in Fig. 1. It is assumed that the wavelengths l_1, l_2 are small enough compared to the minimum characteristic length dimension L of the projection of the structure on the plane Ox_1x_2. Hence, the parameter $l := \sqrt{(l_1)^2 + (l_2)^2}$ in the description of the wavy plate midsurface geometry can be treated as a certain microstructure length parameter. At the same time the thickness δ of the shell under consideration is supposed to be constant and small compared both to the microstructure length parameter l and to the midsurface minimum curvature radius R, $\delta \ll l$, $\delta \ll R$. If follows that on a microscale (taking into account terms of an order l) the wavy plate behaves as a thin shell, while on a macroscale (neglecting terms of an order l), we deal with certain special plate behaviour.

From a formal point of view, the structure under consideration can be described in the framework of the well-known theories for thin elastic shells. However, due to the micro-periodic shape of the wavy plate midsurface, this direct description of the wavy plates leads to shell equations with periodic highly oscillating coefficients. These are too complicated to be used in the analysis of engineering problems and numerical calculations. That is why the aim of this research is to formulate a simplified mathematical model of the wavy plates, which can be applied as a tool for investigation of special problems.

The main feature of the proposed mathematical model of wavy plates is that it describes the effect of the microstructure length dimensions l_1, l_2 on the macrobehaviour of the structure. It will be shown that this effect plays a crucial role in the analysis of dynamic problems. Hence, the proposed model

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Bohdan Michalak
Technical University of Łódź, Department of Structural Mechanics, Al. Politechniki 6, Pl-93950 Łódź, Poland

Czesław Woźniak
Institute of Fundamental Technological Research, Polish Academy of Sciences, Center of Mechanics, Świętokrzyska 21, PL-00049 Warszawa, Poland

Margaret Woźniak
Technical University of Łódź, Department of Geotechnical and Structure Engineering, Al. Politechniki 6, PL-93950 Łódź, Poland

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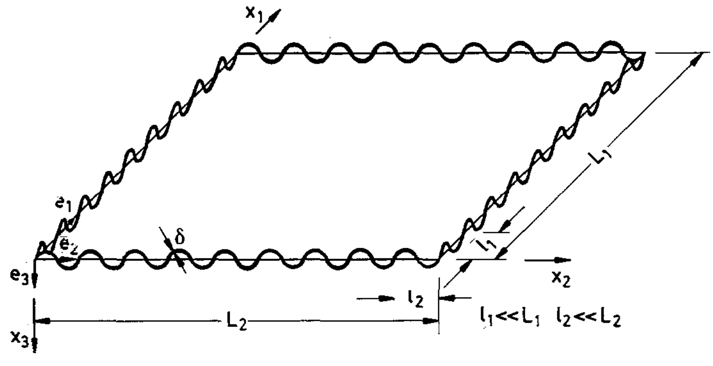

Fig. 1. A scheme of the wavy plate

will be referred to as the microstructural theory of wavy plates. The asymptotic approximation of the obtained equations, where the microstructure is scaled down, $l \searrow 0$, will be considered as a homogenized theory of wavy plates.

The general line of the approach is similar to that used in [1, 2] for the formulation of what is called the refined macrodynamics of periodic composite materials. To make this paper self-consistent, the concepts used in the formulation of the refined macrodynamics will be recalled.

The starting point of this contribution is the direct description of wavy plates in the framework of the theory of thin elastic shells in Sec. 2. In Sec. 3 we formulate the macromodelling hypotheses and the modelling procedure leading to the microstructural theory of wavy plates. The governing equations of this theory are summarized in Sec. 4. In Sec. 5 we derive the homogenized theory of the wavy plates. The examples of applications for both models and the comparison of results are given in Sec. 6. The conclusions in Sec. 7 close the paper. The considerations are carried out in the framework of the linear theory; a more general approach is reserved for a separate study.

Throughout the paper, indices $i, j, k\ldots$ run over 1, 2, 3, being related to the orthogonal carthesian coordinates x_1, x_2, x_3 with vector basis $\mathbf{e}_i$ shown in Fig. 1. Indices α, β, $\gamma, \ldots$ run over 1, 2 and are related to the midsurface shell parameters θ^1, θ^2. We also introduce non-tensorial indices $a, b, c, \ldots$ which run over the sequence $1, \ldots, n$. The summation convention holds for all aforementioned sub- and superscripts. The time coordinate is denoted by t, and the overdot stands for a time differentiation.

2
Direct description of wavy plates

Let the midsurface of the undeformed wavy plate under consideration be given by $x^i = R^i(\theta^1, \theta^2)$, $(\theta^1, \theta^2) \in \Pi$, where Π is a regular plane region. The explicit form of the above equations be described by $x^1 = \theta^1$, $x^2 = \theta^2$, $x^3 = z(\theta^1, \theta^2)$, where $z(\cdot)$ is a piecewise smooth function satisfying conditions $z(\theta^1, \theta^2) = z(\theta^1 + l_1, \theta^2) = z(\theta^1, \theta^2 + l_2)$ in the whole domain of its definition. In the sequel $\mathbf{x} = (x^1, x^2) = (\theta^1, \theta^2)$ stands for an arbitrary point belonging to $\bar{\Pi}$. Under denotation $\Delta \equiv (0, l_1) \times (0, l_2)$, function $z(\cdot)$ will be referred to as the Δ-periodic function, Δ being the representative plate element of a wavy plate. Setting $l := \sqrt{(l_1)^2 + (l_2)^2}$, it is assumed that $l/L \ll 1$, L being the smallest characteristic length dimension of Π. Hence, l will be called the microstructure length parameter of the microperiodic wavy plate under consideration. Using the known denotations: $G^i_\alpha \equiv R^i_{,\alpha}$, $\mathbf{g}_\alpha \equiv G^i_\alpha \mathbf{e}_i$, $\mathbf{n} \equiv \mathbf{g}_1 \times \mathbf{g}_2 / |\mathbf{g}_1 \times \mathbf{g}_2|$ we obtain the metric tensors of the undeformed midsurface $a_{\alpha\beta} = G^i_\alpha G_{\beta i}$, $b_{\alpha\beta} = n_i G^i_{\alpha|\beta}$ and a Ricci tensor $\varepsilon_{\alpha\beta}$ as Δ-periodic functions and define $a = \det a_{\alpha\beta}$. Here and in the sequel, a vertical line before the subscripts stands for the covariant derivative in the metric $a_{\alpha\beta}$, and $\left\{ {\mu \atop \alpha\beta} \right\}$ denote the pertinent Christoffel symbol related to the midsurface of a wavy plate. Introducing the displacement field of the wavy plate midsurface: $\mathbf{u} = u^i(\mathbf{x}, t)\,\mathbf{e}_i$, denoting by $\mathbf{p} = p^i(\mathbf{x}, t)\,\mathbf{e}_i$ the external forces, and by ρ the mass density related to this midsurface, in the framework of the linear approximated theory for thin elastic shells [3], we obtain the following system of equations:

(i) strain-displacement equations

$$\varepsilon_{\alpha\beta} = G^i_{(\alpha} u_{i,\beta)}, \quad \kappa_{\alpha\beta} = n^i \left(u_{i,\alpha\beta} - \left\{ {\gamma \atop \alpha\beta} \right\} u_{i,\gamma} \right), \tag{1}$$

(ii) stress-strain equations

$$n^{\alpha\beta} = DH^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}, \quad m^{\alpha\beta} = BH^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}, \tag{2}$$

where

$$H^{\alpha\beta\gamma\delta} = a^{\alpha\gamma}a^{\beta\delta} - \nu\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}, \quad D \equiv E\frac{\delta}{1-\nu^2}, \quad B \equiv E\frac{\delta^3}{12(1-\nu^2)},$$

E, ν being the Young modulus and Poisson's ratio, respectively,

(iii) equations of motion in the weak form

$$\int_{\Pi} (n^{\alpha\beta}\delta\varepsilon_{\alpha\beta} + m^{\alpha\beta}\delta\kappa_{\alpha\beta})\sqrt{a}\,dx^1dx^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi} \rho\dot{u}^i\delta u_i\sqrt{a}\,dx^1dx^2 = \int_{\Pi} p^i\delta u_i\sqrt{a}\,dx^1dx^2, \tag{3}$$

where

$$\delta\varepsilon_{\alpha\beta} = G^i_{(\alpha}\delta u_{i,\beta)}, \quad \delta\kappa_{\alpha\beta} = n^i\left(\delta u_{i,\alpha\beta} - \begin{Bmatrix}\mu\\ \alpha\beta\end{Bmatrix}\delta u_{i,\mu}\right),$$

which have to hold for any virtual displacement field δu_i, such that $\delta u_i(\mathbf{x}) = 0$ for every $\mathbf{x} = (x^1, x^2)\in\partial\Pi$.

In order to simplify the considerations, boundary conditions are not specified here, and the approximated version of the shell theory is used.

It can be seen that the coefficients in Eqs. (1)–(3) are Δ-periodic functions. Due to the highly oscillating character of these functions, the direct description of wavy plates does not constitute a proper mathematical tool for investigations of engineering problems, and it will be used only as the starting point of the modelling procedure.

3
Modelling procedure

Following [1, 2] we introduce two auxiliary concepts. To this end, let λ_F be a small parameter characterizing the accuracy of calculations of a certain function F in the proposed macrodescription of the microperiodic shell-like structures under consideration. A function $F(\mathbf{x}, t)$, $\mathbf{x} = (x^1, x^2)\in\Pi$ will be called l-macrofunction (related to λ_F) if for every $\mathbf{x}, \mathbf{y}\in\Pi$, such that $\|\mathbf{x}-\mathbf{y}\| < l$, condition $|F(\mathbf{x}, t) - F(\mathbf{y}, t)| < \lambda_F$ holds. If a function F is continuous in Π together with its derivatives, and if similar conditions with pertinent calculation accuracy parameters hold also for all derivatives of F (including time derivatives), then F will be called *the regular l-macrofunction.*

Let $f(\mathbf{x}, \mathbf{y})$ be an integrable function, such that $f(\cdot, \mathbf{y})$ is a regular l-macrofunction defined on Π and $f(\mathbf{x}, \cdot)$ is a Δ-periodic function. In the sequel, we shall use the denotation

$$\langle f\rangle(\mathbf{x}) = \frac{1}{l_1 l_2}\int_{\Delta} f(\mathbf{x}, \mathbf{y})\,dy_1dy_2,$$

If f is independent of $\mathbf{x}$ then $\langle f\rangle$ is constant.

Let $h^a(\cdot)$, $a = 1, \ldots, n$, be a system of n linear independent continuous Δ-periodic functions defined on $\mathbb{R}^2$, having continuous first and second derivatives and such that $\langle h^a\rangle = 0$. Moreover, let for every $\mathbf{x}\in\Pi$ functions $h^a(\cdot)$ satisfy conditions $h^a(\mathbf{x})\in O(l^2)$, $h^a_{,\alpha}(\mathbf{x})\in O(l)$, $h^a_{|\alpha\beta}(\mathbf{x})\in O(1)$ (i.e. the values of $h^a_{|\alpha\beta}$ are independent of the microstructure length parameter l). Let us also assume that the displacement field u_i, restricted to an arbitrary but fixed element $(x^1, x^1 + l_1)\times(x^2, x^2 + l_2)$ of Π, can be approximated by a linear combination of functions $h^a(\cdot)$, $a = 1, \ldots, n$, superimposed on a certain uniform deformation. Under the aforementioned conditions, functions h^a will be referred to as *the microshape functions*. For more detailed discussion of this concept the reader is referred to [1].

Macro-Kinematic Hypothesis (MKH). The displacement field $u_i(\mathbf{x}, t), \mathbf{x}\in\Pi$ of the wavy plate can be assumed in the form

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}, t) + h^a(\mathbf{x})V^a_i(\mathbf{x}, t), \quad \mathbf{x} = (x^1, x^2)\in\Pi, \quad t\geqq 0, \tag{4}$$

where $U_i(\cdot, t)$, $V^a_i(\cdot, t)$ are regular l-macrofunctions and $h^a(\cdot)$ are the microshape functions, postulated in every problem under consideration. Similarly, the virtual displacements δu_i in Eq. (3) will be given by

$$\delta u_i(\mathbf{x}) = \delta U_i(\mathbf{x}) + h^a(\mathbf{x})\delta V^a_i(\mathbf{x}), \quad \mathbf{x} = (x^1, x^2)\in\Pi, \tag{5}$$

where δU_i, δV^a_i are arbitrary linearly independent regular l-macrofunctions, such that $\delta u_i(\mathbf{x}) = 0$ for every $\mathbf{x}\in\partial\Pi$.

Fields $U_i(\cdot)$, $V_i^a(\cdot)$ are the new kinematic basic unknowns being called macrodisplacements and correctors, respectively.

Macro-Modelling Assumption (MMA). In the course of a modelling procedure, terms $O(\lambda_F)$ can be neglected as compared to values of F, where F runs over all macrofunctions U_i, V_i^a, δU_i, δV_i^a and their derivatives.

In order to explain the meaning of MMH, let us observe that for any integrable Δ-periodic function f and continuous l-macrofunction F defined on Π, the following formula holds:

$$\int_\Pi f(\mathbf{x})F(\mathbf{x})\,dx_1dx_2 = \langle f\rangle \int_\Pi F(\mathbf{x})\,dx_1dx_2 + O(\lambda_F).$$

Moreover, if h is a microshape function then

$$(hF)_{,\alpha} = h_{,\alpha}F + O(\lambda_F), \quad (hF)_{|\alpha\beta} = h_{|\alpha\beta}F + O(\lambda_F) + O(\lambda_{\nabla F}),$$

where ∇F stands for a gradient of F. Using MMH, terms $O(\lambda_F)$, $O(\lambda_{\nabla F})$ in the above approximation formulae can be neglected.

Since $h^a V^a_{i,\alpha} \in O(\lambda_V)$, $h^a V^a_{i|\alpha\beta} \in O(\lambda_{\nabla V})$, where λ_V, $\lambda_{\nabla V}$ stand for calculation accuracies related to V_i^a, $V^a_{i,\alpha}$, respectively, Eqs. (1), (2) yield, by means of MKH and MMA, the relations

$$n^{\alpha\beta} = DH^{\alpha\beta\gamma\delta}G^i_\gamma U_{i,\delta} + DH^{\alpha\beta\gamma\delta}G^i_\gamma h^a_{,\delta} V^a_i, \tag{6}$$

$$m^{\alpha\beta} = BH^{\alpha\beta\gamma\delta}n^i U_{i,\gamma\delta} - BH^{\alpha\beta\gamma\delta}n^i \begin{Bmatrix}\mu\\ \gamma\delta\end{Bmatrix} U_{i,\mu} + BH^{\alpha\beta\gamma\delta}n^i h^a_{|\gamma\delta} V^a_i.$$

The aforementioned macromodelling hypotheses lead from Eqs. (3)-(6) to a certain averaged model of a wavy plate.

Let us define the following 3×3-matrices:

$$\mathbf{D}^{\alpha\beta} \equiv D\langle H^{\delta\alpha\gamma\beta}\mathbf{g}_\delta \otimes \mathbf{g}_\gamma \sqrt{a}\rangle, \qquad \mathbf{D}^{a\alpha} \equiv D\langle H^{\beta\alpha\gamma\delta}\mathbf{g}_\beta \otimes \mathbf{g}_\gamma h^a_{,\delta}\sqrt{a}\rangle,$$

$$\mathbf{D}^{\beta} \equiv D\left\langle H^{\alpha\delta\gamma\beta}\begin{Bmatrix}\lambda\\ \alpha\delta\end{Bmatrix}\mathbf{g}_\lambda \otimes \mathbf{g}_\gamma \sqrt{a}\right\rangle, \qquad \mathbf{D}^{a} \equiv D\left\langle H^{\alpha\beta\gamma\delta}\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix}\mathbf{g}_\lambda \otimes \mathbf{g}_\gamma h^a_{,\delta}\sqrt{a}\right\rangle,$$

$$\mathbf{D}^{ab} \equiv D\langle H^{\alpha\beta\gamma\delta}\mathbf{g}_\alpha \otimes \mathbf{g}_\gamma h^a_{,\beta} h^b_{,\delta}\sqrt{a}\rangle, \qquad \mathbf{B}^{\alpha\beta\gamma\delta} = B\langle H^{\alpha\beta\gamma\delta}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\rangle, \tag{7}$$

$$\mathbf{B}^{\alpha\beta\gamma} \equiv B\left\langle H^{\alpha\beta\mu\delta}\begin{Bmatrix}\gamma\\ \mu\delta\end{Bmatrix}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\right\rangle, \qquad \mathbf{B}^{a\alpha\beta} \equiv B\langle H^{\alpha\beta\gamma\delta}h^a_{|\gamma\delta}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\rangle,$$

$$\mathbf{B}^{\alpha\gamma} \equiv B\left\langle H^{\beta\pi\mu\nu}\begin{Bmatrix}\alpha\\ \beta\pi\end{Bmatrix}\begin{Bmatrix}\gamma\\ \mu\nu\end{Bmatrix}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\right\rangle, \quad \mathbf{B}^{a\alpha} \equiv B\left\langle H^{\beta\gamma\mu\nu}\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}h^a_{|\mu\nu}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\right\rangle,$$

$$\mathbf{B}^{ab} \equiv B\langle H^{\alpha\beta\gamma\delta}h^a_{|\alpha\beta}h^b_{|\gamma\delta}\mathbf{n}\otimes\mathbf{n}\sqrt{a}\rangle,$$

and introduce the system of vector macrofunctions given by

$$\mathbf{N}^\alpha = \mathbf{D}^{\alpha\beta}\mathbf{U}_{,\beta} + \mathbf{D}^{a\alpha}\mathbf{V}^a,$$

$$\mathbf{N} = \mathbf{D}^{\beta}\mathbf{U}_{,\beta} + \mathbf{D}^{a}\mathbf{V}^a,$$

$$\mathbf{N}^a = \mathbf{D}^{ab}\mathbf{V}^b + \mathbf{D}^{a}\mathbf{U}_{,\beta}, \tag{8}$$

$$\mathbf{M}^{\alpha\beta} = \mathbf{B}^{\alpha\beta\gamma\delta}\mathbf{U}_{,\gamma\delta} + \mathbf{B}^{\alpha\beta\gamma}\mathbf{U}_{,\gamma} + \mathbf{B}^{a\alpha\beta}\mathbf{V}^a,$$

$$\mathbf{M}^{\alpha} = \mathbf{B}^{\alpha\gamma\delta}\mathbf{U}_{,\gamma\delta} + \mathbf{B}^{\alpha\gamma}\mathbf{U}_{,\gamma} + \mathbf{B}^{a\alpha}\mathbf{V}^a,$$

$$\mathbf{M}^{a} = \mathbf{B}^{ab}\mathbf{V}^b + \mathbf{B}^{a\alpha\beta}\mathbf{U}_{,\alpha\beta} + \mathbf{B}^{a\alpha}\mathbf{U}_{,\alpha},$$

which will be called the macrostress resultants. Moreover, define $\tilde{\rho} \equiv \rho\sqrt{a}$ and assume that p^i are l-macrofunctions. Substituting the right-hand sides of Eqs. (4), (5), (6) into Eq. (3), and applying MMA to the aforementioned approximation formulae, we obtain the condition

$$\int_{\Pi} [\mathbf{M}^{\alpha\beta}\cdot\delta\mathbf{U}_{,\alpha\beta} + (\mathbf{M}^{\alpha} + \mathbf{N}^{\alpha})\cdot\delta\mathbf{U}_{,\alpha} + \mathbf{N}\cdot\delta\mathbf{U} + (\mathbf{N}^{a} + \mathbf{M}^{a})\cdot\delta\mathbf{V}^{a}]\, dx^1 dx^2$$
$$+\frac{d}{dt}\int_{\Pi} (\langle\tilde{\rho}\rangle\dot{\mathbf{U}}\cdot\delta\mathbf{U} + \langle\tilde{\rho}h^a h^b\rangle\dot{\mathbf{V}}^{b}\cdot\delta\mathbf{V}^{a})\, dx^1 dx^2 = \int_{\Pi} \mathbf{p}\cdot\delta\mathbf{U}\, dx^1 dx^2, \tag{9}$$

which has to hold for every $\delta\mathbf{U}$, $\delta\mathbf{V}^{a}$, such that virtual macrodisplacements $\delta\mathbf{U}_i$ together with their derivatives are equal to zero on $\partial\Pi$. After applying the divergence theorem and du Bois-Reymond lemma, we arrive at the system of equations in macrodisplacements U_i and correctors V_i^a. These equations represent the proposed simplified mathematical model of the wavy plates and will be discussed in the subsequent section. 181

4
Micro structural theory (MST)

Governing equations of the averaged theory of wavy plates derived from (9) have the form

$$\mathbf{M}^{\alpha\beta}_{,\alpha\beta} - (\mathbf{M}^{\alpha} + \mathbf{N}^{\alpha})_{,\alpha} + \mathbf{N} + \langle\tilde{\rho}\rangle\ddot{\mathbf{U}} = \mathbf{p}, \tag{10}$$

$$\mathbf{N}^{a} + \mathbf{M}^{a} + \langle\tilde{\rho}h^a h^b\rangle\ddot{\mathbf{V}}^{b} = \mathbf{0}.$$

Define $\Pi^{\circ} := \{\mathbf{x} = (x_1, x_2) \in \Pi : \mathbf{x} + \Delta \subset \Pi\}$. It can be shown that for every $\mathbf{x} \in \Pi^{\circ}$ and every instant t, the introduced macrostress resultants have the following physical interpretation:

$$\mathbf{N}^{\alpha}(\mathbf{x}, t) = \langle n^{\alpha\beta}\mathbf{g}_{\beta}\sqrt{a}\rangle(\mathbf{x}, t), \qquad \mathbf{N}(\mathbf{x}, t) = \left\langle n^{\alpha\beta}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}\mathbf{g}_{\gamma}\sqrt{a}\right\rangle(\mathbf{x}, t),$$

$$\mathbf{N}^{a}(\mathbf{x}, t) = \langle n^{\alpha\beta}\mathbf{g}_{\alpha}h^{a}_{,\beta}\sqrt{a}\rangle(\mathbf{x}, t), \qquad \mathbf{M}^{\alpha\beta}(\mathbf{x}, t) = \langle m^{\alpha\beta}\mathbf{n}\sqrt{a}\rangle(\mathbf{x}, t), \tag{11}$$

$$\mathbf{M}^{\alpha}(\mathbf{x}, t) = \left\langle m^{\beta\gamma}\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\mathbf{n}\sqrt{a}\right\rangle(\mathbf{x}, t), \quad \mathbf{M}^{a}(\mathbf{x}, t) = \langle m^{\alpha\beta}h^{a}_{|\alpha\beta}\mathbf{n}\sqrt{a}\rangle(\mathbf{x}, t), \quad \mathbf{x} \in \Pi^{\circ}.$$

Equation $(10)_1$ describes, in the averaged form, the macrodynamic behaviour of wavyplates, due to the well known form $-\langle\tilde{\rho}\rangle\ddot{U}^i$ of inertial forces. On the other hand, in Eq. $(10)_2$ we deal with a new kind of inertia terms given by $\langle\tilde{\rho}h^a h^b\rangle\ddot{V}^b_i$, characterizing what can be called microinertial properties of the structure under consideration. The mechanical interpretation of Eq. (10) is strictly related to the postulated form of Eq. (4), where $h^a V^a_i$ represent certain disturbances of displacements caused by the periodic changes in the shell geometry.

Equations (10) and (8) will be called equations of motion and constitutive equations of the averaged theory of wavyplates, respectively. Since $h^a \in O(l^2)$, then the microinertial modulae $\langle\tilde{\rho}h^a h^b\rangle$ satisfy condition $\langle\tilde{\rho}h^a h^b\rangle \in O(l^4)$, and hence the aforementioned equations describe the microstructure length-scale effect on the behaviour of the structure under consideration. That is why the proposed theory will be referred to as *the microstructural theory of wavy plates* (MST). Substituting the right-hand sides of Eq. (8) into Eq. (10), we obtain $3 + 3n$ equations for 3 macrodisplacements U_i and $3n$ correctors V^a_i. It has to be emphasized that the resulting differential equations have constant coefficients, defined by Eq. (8), and hence represent a good computational model of the wavy plates, which can be easily applied to the analysis of particular engineering problems. In particular, components N^{α}_3, N_3, N^a_3 and $M^{\alpha\beta}_{\gamma}$, M^{α}_{γ}, M^{a}_{α} describe the coupling between the plate and plane problems. The characteristic feature of the microstructural theory of wavy plates is that the correctors V^a_i are governed by the ordinary differential equations involving second-order time derivatives. Hence, the correctors play the role of internal dynamic variables, being independent of the boundary conditions. Equations (10), (8) have to be considered together with the boundary and initial conditions for macrodisplacements and with the initial conditions for correctors. The exact form of the boundary conditions will be discussed separately. It can be seen that for trivial initial conditions for correctors, and under assumption that $z(\theta^1, \theta^2) = 0$, the correctors are equal to zero. From Eqs. (7)–(10), after simple manipulations, we obtain the well-known equations of the Kirchhoff theory for homogeneous plates.

If the obtained microstructural theory of wavy plates is based on the approximate theory of shallow shells, then $\{^{\alpha}_{\beta\gamma}\} \cong 0$, $b_{\alpha\beta} \cong z_{,\alpha\beta}$, and terms $\mathbf{N}$ and $\mathbf{M}^{\beta}$ drop out from the above equations. In this case, we deal with what will be called the microstructural theory of slightly wrinkled plates, which is to be investigated separately.

5
Homogenized theory (HT)

The homogenization methods for periodic composite materials, leading to the effective modulus theories, cf. [4, 5], are based on the asymptotic approximation in which the microstructure of the body is scaled down. Applying the asymptotic approximation $l \searrow 0$ to the equations of MST, we shall neglect microinertial terms $\langle \tilde{\rho} h^{a} h^{b} \rangle$ in Eq. $(10)_2$, as well as terms $\mathbf{D}^{a\alpha}$, $\mathbf{D}^{a}$, $\mathbf{D}^{ab}$ in Eq. (8). In the framework of HT, we have to replace Eq. (10), by $\mathbf{M}^{a} = \mathbf{0}$. In this case, using the constitutive equations for $\mathbf{M}^{a}$, and bearing in mind that the linear transformation $\mathbb{R}^{3n} \to \mathbb{R}^{3n}$ given by the $3n \times 3n$ matrix of elements $\mathbf{B}^{ab}$ is invertible (this statement can be proved by rather lengthy calculations, which are not exposed here), we can eliminate correctors V^{a}_{i} from the governing equations by means of the formula

$$\mathbf{V}^{a} = -\mathbf{K}^{ab}[\mathbf{B}^{b\alpha\beta}\mathbf{U}_{,\alpha\beta} + \mathbf{B}^{b\alpha}\mathbf{U}_{,\alpha}], \tag{12}$$

where $\mathbf{K}^{ab}$ determine the linear transformation, inverse to that given by $\mathbf{B}^{ab}$

$$\mathbf{K}^{ab}\mathbf{B}^{bc} = \boldsymbol{\delta}^{ac},$$

where $\boldsymbol{\delta}^{ac}$ determine the identity transformation $\mathbb{R}^{3n} \to \mathbb{R}^{3n}$. Denoting

$$\tilde{\mathbf{B}}^{\alpha\beta\gamma\delta} \equiv \mathbf{B}^{\alpha\beta\gamma\delta} - \mathbf{B}^{a\alpha\beta}\mathbf{K}^{ab}\mathbf{B}^{b\gamma\delta}, \tag{13}$$

$$\tilde{\mathbf{B}}^{\alpha\beta\gamma} \equiv \mathbf{B}^{\alpha\beta\gamma} - \mathbf{B}^{a\alpha}\mathbf{K}^{ab}\mathbf{B}^{b\beta\gamma}, \quad \tilde{\mathbf{B}}^{\alpha\beta} \equiv \mathbf{B}^{\alpha\beta} - \mathbf{B}^{a\alpha}\mathbf{K}^{ab}\mathbf{B}^{b\beta},$$

we obtain the constitutive equations of HT in the form

$$\begin{aligned}
\mathbf{N}^{\alpha} &= \mathbf{D}^{\alpha\beta}\mathbf{U}_{,\beta},\\
\mathbf{N} &= \mathbf{D}^{\beta}\mathbf{U}_{,\beta},\\
\mathbf{M}^{\alpha\beta} &= \tilde{\mathbf{B}}^{\alpha\beta\gamma\delta}\mathbf{U}_{,\gamma\beta} + \tilde{\mathbf{B}}^{\gamma\alpha\beta}\mathbf{U}_{,\gamma},\\
\mathbf{M}^{\alpha} &= \tilde{\mathbf{B}}^{\alpha\beta\gamma}\mathbf{U}_{,\beta\gamma} + \tilde{\mathbf{B}}^{\alpha\beta}\mathbf{U}_{,\beta}.
\end{aligned} \tag{14}$$

The equations of motion reduce to Eq. $(10)_1$

$$\mathbf{M}^{\alpha\beta}_{,\alpha\beta} - (\mathbf{M}^{\alpha} + \mathbf{N}^{\alpha})_{,\alpha} + \mathbf{N} + \langle \tilde{\rho} \rangle \ddot{\mathbf{U}} = \mathbf{p}. \tag{15}$$

Equations (14), (15) represent the system of govering equations for the homogenized theory of wavy plates. The material properties of wavy plates in the framework of HT are determined by the constant coefficients (13) which can be referred to as the effective modulae. That is why the homogenized theory of wavy plates can be also called the effective modulus theory. The basic unknowns are three macrodisplacements U_i, satisfying the system of three partial differential equations obtained by substituting (14) into (15). Equations (14), (15) have to be considered together with the pertinent boundary and initial conditions for macrodisplacements.

Comparing both the microstructural and homogenized models of wavy plates with the known theories of Reissner or Kirchhoff plates, it can be seen that the wavy plates under consideration cannot be described using the plate theories. This statement is implied by the fact that for wavy plates the decoupling between in-plane and out-plane problems is not possible.

6
Example

In order to compare MST and HT, we shall investigate the simple problem of the cylindrical bending of a wavy plate, in which the basic unknowns depend exclusively on variables $x_2 = \theta^2$ and t, being independent of the θ^1-coordinate. In the framework of MST, we obtain the system of equations for $U_i = U_i(x_2, t)$ and $V^{a}_{i} = V^{a}_{i}(x_2, t)$ by substituting the right-hand sides of Eq. (8) into Eq. (10). After some manipulations, bearing in mind formulae (7), and neglecting external loadings, this system will take the form

$$
\begin{aligned}
&M^{122}_{,22} - M^{12}_{,2} - N^{12}_{,2} + N^{1} + \langle\tilde{\rho}\rangle \ddot{U}^{1} = 0,\\
&M^{222}_{,22} - M^{22}_{,2} - N^{22}_{,2} + N^{2} + \langle\tilde{\rho}\rangle \ddot{U}^{2} = 0,\\
&M^{322}_{,22} - M^{32}_{,2} - N^{32}_{,2} + N^{3} + \langle\tilde{\rho}\rangle \ddot{U}^{3} = 0,\\
&N^{a/1} + M^{a/1} + \langle\tilde{\rho} h^{a} h^{b}\rangle \ddot{V}^{b1} = 0,\\
&N^{a/2} + M^{a/2} + \langle\tilde{\rho} h^{a} h^{b}\rangle \ddot{V}^{b2} = 0,\\
&N^{a/3} + M^{a/3} + \langle\tilde{\rho} h^{a} h^{b}\rangle \ddot{V}^{b3} = 0,
\end{aligned}
\tag{16}
$$

where the right-hand sides of formulae (8) with notations (7) have to be substituted into (16). Let us restrict considerations to the analysis of free vibrations for the unbounded wavy plate. In this case, we shall look for solutions of Eq. (16) in the form

$$
\begin{aligned}
&U_1 = 0, \quad U_2 = A_2 \sin(kx_2)\cos(\omega_2 t), \quad U_3 = A_3 \sin(kx_2)\cos(\omega_3 t),\\
&V^{a}_{1} = 0, \quad V^{a}_{2} = C^{a}_{2} \cos(kx_2)\cos(\omega_3 t), \quad V^{a}_{3} = C^{a}_{3}\cos(kx_2)\cos(\omega_2 t),
\end{aligned}
\tag{17}
$$

where $k := 2\pi/L$ is the wave number, L being the vibration wavelength, $L \gg l$ and A_2, A_3, C^{a}_{2}, C^{a}_{3} are constants.

Substituting the right-hand sides of Eq. (17) into Eq. (16) combined with Eq. (8), we obtain non-trivial solutions only if

$$
\begin{vmatrix}
(\omega_2)^2\langle\tilde{\rho}\rangle - C_{22} & 0 & 0 & C_{26}\\
0 & (\omega_3)^2\langle\tilde{\rho}\rangle - C_{33} & C_{35} & 0\\
0 & C_{53} & (\omega_3)^2\langle\tilde{\rho} hh\rangle - C_{55} & 0\\
C_{62} & 0 & 0 & (\omega_2)^2\langle\tilde{\rho} hh\rangle - C_{66}
\end{vmatrix} = 0,
\tag{18}
$$

where we have denoted

$$
\begin{aligned}
&C_{22} = C^{4}_{22} + C^{2}_{22},\\
&C^{4}_{22} \equiv B\langle H^{2222}(n^2)^2\sqrt{a}\rangle k^4,\\
&C^{2}_{22} \equiv [B\langle H^{2222}(\{{}^{2}_{22}\} n^2)^2\sqrt{a}\rangle + D\langle H^{2222}\sqrt{a}\rangle]k^2,\\
&C_{66} \equiv B\langle H^{2222}(n^3 h_{,22})^2\sqrt{a}\rangle + D\langle H^{2222}(G^{3}_{2} h_{,2})^2\sqrt{a}\rangle,\\
&C_{26} = C_{62} \equiv [-D\langle H^{2222} G^{3}_{2} h_{,2}\sqrt{a}\rangle - B\langle H^{2222}\{{}^{2}_{22}\} n^2 n^3 h_{,22}\sqrt{a}\rangle]k,\\
&C_{33} = C^{4}_{33} + C^{2}_{33},\\
&C^{4}_{33} \equiv B\langle H^{2222}(n^3)^2\sqrt{a}\rangle k^4,\\
&C^{2}_{33} \equiv [B\langle H^{2222}(\{{}^{2}_{22}\} n^3)^2\sqrt{a}\rangle + D\langle H^{2222}(G^{3}_{2})^2\sqrt{a}\rangle]k^2,\\
&C_{55} \equiv B\langle H^{2222}(n^2 h_{,22})^2\sqrt{a}\rangle + D\langle H^{2222}(h_{,2})^2\sqrt{a}\rangle,\\
&C_{35} = C_{53} \equiv [-D\langle H^{2222} G^{3}_{2} h_{,2}\sqrt{a}\rangle - B\langle H^{2222}\{{}^{2}_{22}\} n^3 n^2 h_{,22}\sqrt{a}\rangle]k.
\end{aligned}
\tag{19}
$$

Equation (18) implies the decoupling between ω_2 and ω_3. Under additional notations

$$(\mu_2)^2 = \frac{D\langle H^{2222} h_{,2} h_{,2} \sqrt{a}\rangle + B\langle H^{2222} N^2 N^2 h_{,22} h_{,22} \sqrt{a}\rangle}{\langle \tilde{\rho} hh\rangle},$$

$$(\mu_3)^2 = \frac{D\langle H^{2222} G_2^3 G_2^3 h_{,2} h_{,2} \sqrt{a}\rangle + B\langle H^{2222} N^3 N^3 h_{,22} h_{,22} \sqrt{a}\rangle}{\langle \tilde{\rho} hh\rangle}, \tag{20}$$

we obtain from (18) the resulting relation for the free vibration frequencies in the form

$$(\omega_2)^2 = \frac{C_{22} - (C_{26})^2/C_{66}}{\langle \tilde{\rho}\rangle} + \left[(\omega_2)^2 - \frac{C_{22}}{\langle \tilde{\rho}\rangle}\right]\left(\frac{\omega_2}{\mu_3}\right)^2,$$

$$(\omega_3)^2 = \frac{C_{33} - (C_{35})^2/C_{55}}{\langle \tilde{\rho}\rangle} + \left[(\omega_3)^2 - \frac{C_{33}}{\langle \tilde{\rho}\rangle}\right]\left(\frac{\omega_3}{\mu_2}\right)^2. \tag{21}$$

Assuming that $O(l) = O(\delta)$, let us introduce a small parameter ε setting $O(\varepsilon) = O(l)$. After that, solutions ω_2, ω_3 to Eq. (21) are

$$(\omega_2')^2 = \frac{C_{22}^4}{\langle \tilde{\rho}\rangle} + \frac{C_{22}^2 - [C_{26}]^2/C_{66}}{\langle \tilde{\rho}\rangle}\left[1 - \frac{\langle \tilde{\rho} hh\rangle}{\langle \tilde{\rho}\rangle}\left(\frac{C_{26}}{C_{66}}\right)^2\right] + o(\varepsilon^4),$$

$$(\omega_2'')^2 = \frac{C_{66}}{\langle \tilde{\rho} hh\rangle} + \frac{[C_{26}]^2/C_{66}}{\langle \tilde{\rho}\rangle}\left\{1 - \frac{\langle \tilde{\rho} hh\rangle}{\langle \tilde{\rho}\rangle}\left[\left(\frac{C_{26}}{C_{66}}\right)^2 - \frac{C_{22}^2}{C_{66}}\right]\right\} + o(\varepsilon^4),$$

$$(\omega_3')^2 = \frac{C_{33}^4}{\langle \tilde{\rho}\rangle} + \frac{C_{33}^2 - [C_{35}]^2/C_{55}}{\langle \tilde{\rho}\rangle}\left[1 - \frac{\langle \tilde{\rho} hh\rangle}{\langle \tilde{\rho}\rangle}\left(\frac{C_{35}}{C_{55}}\right)^2\right] + o(\varepsilon^4), \tag{22}$$

$$(\omega_3'')^2 = \frac{C_{55}}{\langle \tilde{\rho} hh\rangle} + \frac{[C_{35}]^2/C_{55}}{\langle \tilde{\rho}\rangle}\left\{1 - \frac{\langle \tilde{\rho} hh\rangle}{\langle \tilde{\rho}\rangle}\left[\left(\frac{C_{35}}{C_{55}}\right)^2 - \frac{C_{33}^2}{C_{55}}\right]\right\} + o(\varepsilon^4).$$

Let the shell midsurface be given by $z = l\sin(2\pi\theta^2/l)$. In this case, formulae (19), for a wavy type plate with a constant thickness δ and a constant mass density $\tilde{\rho}$, after notation $\lambda := \delta/l$, $q := kl$, yield

$$C_{22}^4 = \frac{E}{\delta(1-\nu^2)} 0.001716\,\lambda^4 q^4,$$

$$C_{22}^2 = \frac{E}{\delta(1-\nu^2)}(0.4736\,\lambda^2 + 0.080575)\,\lambda^2 q^2,$$

$$C_{26} = \frac{E\delta}{1-\nu^2}(-0.076803 + 1.2727\,\lambda^2)\,q,$$

$$C_{66} = \frac{E\delta^3}{1-\nu^2}\left(7.7067 + \frac{0.7708}{l^2}\right),$$

$$C_{33}^4 = \frac{E}{\delta(1-\nu^2)} 0.004998\,\lambda^4 q^4, \tag{23}$$

$$C_{33}^2 = \frac{E}{\delta(1-\nu^2)}(0.7991\,\lambda^2 + 0.076803)\,\lambda^2 q^2,$$

$$C_{35} = \frac{E\delta}{1-\nu^2}(-0.076803 + 1.2727\,\lambda^2)\,q,$$

$$C_{55} = \frac{E\delta^3}{1-\nu^3}\left(2.4847 + \frac{0.076803}{l^2}\right).$$

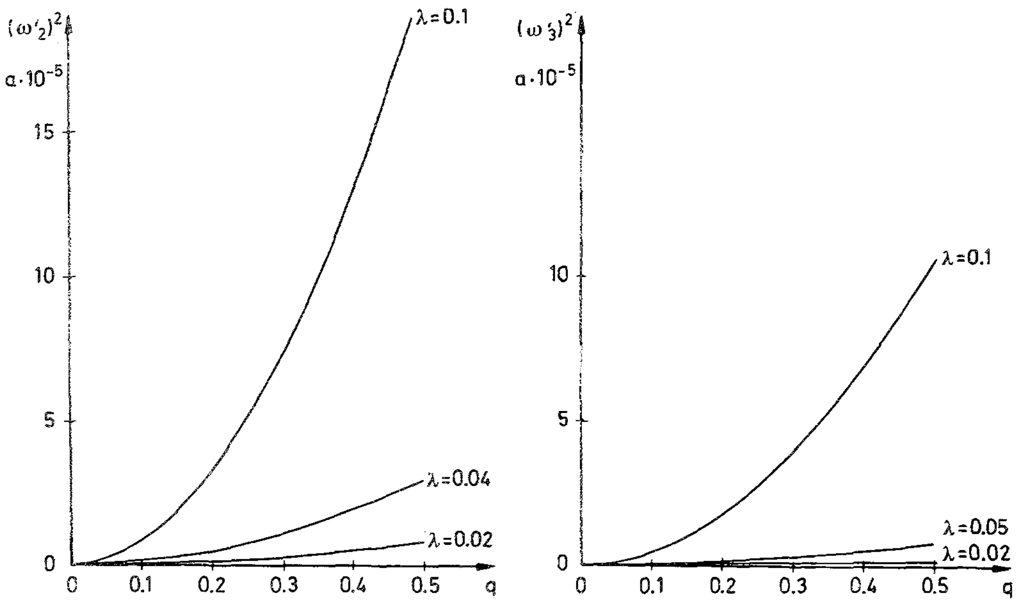

Fig. 2. Free lower vibration frequencies ω'_2, ω'_3 versus the non-dimensional wave number $q = kl$; ratio $\lambda = \delta/l$ is used as a parameter

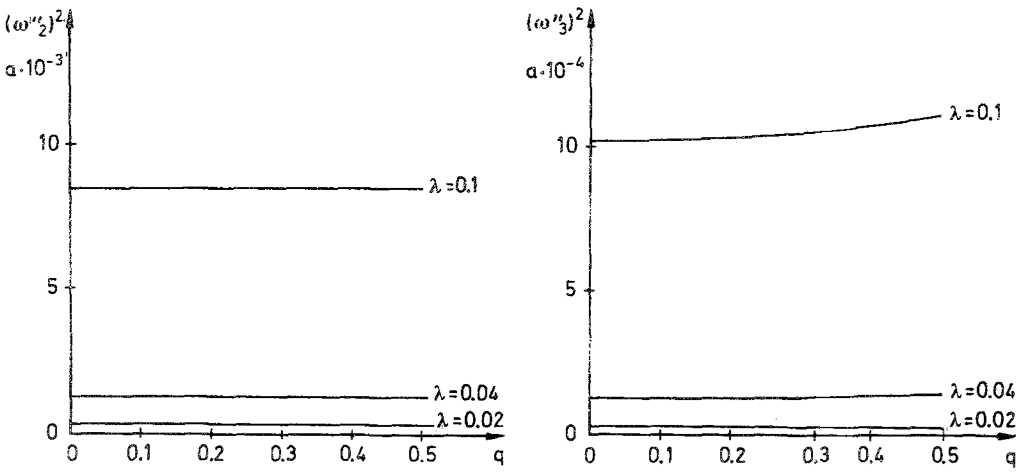

Fig. 3. Free higher vibration frequencies ω''_2, ω''_3 versus the non-dimensional wave number $q = kl$; ratio $\lambda = \delta/l$ is used as a parameter

In Figs. 2, 3 the diagrams of the free vibration frequency ω versus the non-dimensional wave number q are shown, where the ratio $\lambda = \delta/l$ is used as a parameter.

Equation (18) holds in the framework of the proposed microstructural theory of the wavy type plates. For the homogenized asymptotic approximation (obtained by scaling down wavy microstructure) we obtain

$$(\breve{\omega}_2)^2 = \frac{C_{22} - \dfrac{(C_{26})^2}{C_{66}}}{\langle\tilde{\rho}\rangle}, \quad (\breve{\omega}_2)^2 = (\omega_2)^2 + o\left(\frac{l}{L}\right)^2,$$

$$(\breve{\omega}_3)^2 = \frac{C_{33} - \dfrac{(C_{35})^2}{C_{55}}}{\langle\tilde{\rho}\rangle}, \quad (\breve{\omega}_3)^2 = (\omega_3)^2 + o\left(\frac{l}{L}\right)^2. \qquad (24)$$

Comparing the obtained results, given by Eqs. (22) and (24), it can be seen that lower vibration frequencies obtained from HT can be treated as certain approximations of the pertinent frequencies resulted from MST. However, the homogenized theory is not able to describe higher vibration frequencies, since the effect of the microstructure length dimensions on the dynamic behaviour of the wavy plate in the framework of HT is neglected.

7
Conclusions

From the above example it follows that the proposed microstructural theory of wavy plates can be successfully applied to the analysis of special dynamic problems. In order to compare results related to MST and HT, we have restricted ourselves to a simple illustrative example. Nevertheless, the example leads to the conclusion that the homogenized theory (effective modulus theory) of wavy plates cannot be used in the analysis of dynamic problems, and the effect of the microstructure length dimension on the time dependent processes plays an important role. On the other hand, for quasi-stationary processes, the homogenized theory of wavy plates can constitute a convenient tool for engineering investigations. Generally speaking, problems of wavy plates, very difficult when analysed within the framework of the shell theory, can be described by the relatively simple differential equations with constant coefficients both within the framework of the microstructural and homogenized theories.

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