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TRAVELING WAVES FOR REACTION-DIFFUSION PDE COUPLED TO DIFFERENCE EQUATION WITH NONLOCAL DISPERSAL TERM AND TIME DELAY

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Abstract. We consider a class of biological models represented by a system composed of reaction-diffusion PDE coupled with difference equations (renewal equations) in n-dimensional space, with nonlocal dispersal terms and implicit time delays. The difference equation generally arises, by means of the method of characteristics, from an age-structured partial differential system. Using upper and lower solutions, we study the existence of monotonic planar traveling wave fronts connecting the extinction state to the uniform positive state. The corresponding minimum wave speed is also obtained. In addition, we investigate the effect of the parameters on this minimum wave speed and we give a detailed analysis of its asymptotic behavior.

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1. Introduction

The existence of traveling front solutions to nonlinear parabolic equations was first shown in connection with the celebrated Fisher-KPP equation. It was proposed in 1937 by Fisher [13] and in a generalized form by Kolmogorov, Petrovskii and Piskunov [21]. In [13], Fisher showed, using logistic growth combined with diffusion that, after an advantageous gene allele was established in a population, there would be a wave of advance with the velocity proportional to the square root of diffusion and the selective advantage of the allele.

Traveling wave fronts have been widely studied for reaction-diffusion equations modeling a variety of biological phenomena (see, *e.g.*, [5, 21, 32] and the references therein), and for time-delayed reaction-diffusion PDEs (see, *e.g.*, [2, 3, 11, 12, 14, 22–24, 28, 29, 31, 33, 34, 37, 38]). Yet, in many problems of structured population dynamics, we have to take into account their spatial diffusion. A large number of investigators simply added a diffusion term to the corresponding reduced delay differential equation. However, in recent years it has become recognized that there are modeling difficulties with this approach (see [7, 30]). The problem is that individuals have not been at the same point in space at previous times. To address this difficulty, we use a general approach

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by which certain differential equations with delay can be obtained from age-structured population models or renewal equations (see [7, 30, 33]). In this way, we obtain nonlocal dispersal reaction-diffusion PDEs with time delays.

In this paper we will analyse the following system of delayed reaction-diffusion PDE coupled to difference equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &=& D\Delta u(t,x) - f(u(t,x)) + h\left(\kappa \displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) w(t-r,y)\mathrm{d}y\right), \\ w(t,x) &=& g(u(t,x)) + (1-\kappa)\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r,x-y)w(t-r,y)\mathrm{d}y, \end{cases} \tag{1.1}$$

with $t>0$, $x\in\mathbb{R}^n$ and the initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad w(\theta,x) = w_0(\theta,x), \quad \text{for } \theta \in [-r,0].$$

$\kappa\in[0,1]$, Δ is the Laplacian operator, $D>0$ is the diffusion coefficient, $r>0$ is the time-delay, f, g and h are positive functions, and Γ_d is the heat kernel given, for $d>0$, by

$$\Gamma_d(t,x) = \frac{1}{(4d\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4\mathrm{d}t}\right), \quad t>0,\ x\in\mathbb{R}^n.$$

It satisfies

$$\int_{\mathbb{R}^n} \Gamma_d(t,x)\mathrm{d}x = 1, \quad \text{for all } \ t>0.$$

As it will be shown in Lemma 1.1, the system (1.1) can be derived from a system of a reaction-diffusion equation and an age-structured population equation, modeling a typical interaction of two subpopulations of individuals, represented schematically in Figure 1.

A particular case of the system (1.1) was first introduced in [2] to study hematopoietic stem cell population dynamics. This model has the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = D\dfrac{\partial^2 u(t,x)}{\partial x^2} - (\delta(u(t,x)) + \beta(u(t,x)))u(t,x) \\ \qquad\qquad\qquad +2(1-K)e^{-\gamma r}\displaystyle\int_{\mathbb{R}} \Gamma_d(r,x-y)w(t-r,y)\mathrm{d}y, \\ w(t,x) = \beta(u(t,x))u(t,x) + 2Ke^{-\gamma r}\displaystyle\int_{\mathbb{R}} \Gamma_d(r,x-y)w(t-r,y)\mathrm{d}y, \end{cases} \tag{1.2}$$

where u represents the density of quiescent cells and w the density of new proliferating cells (see [4, 25]). The special case $K=0$, was treated in [24]. The coefficients D and d represent the diffusion rates in the quiescent and proliferating phases, respectively. The delay r represents the duration of the proliferating phase and γ the apoptosis rate (programmed cell death rate). The coefficient $\kappa := 1-2Ke^{-\gamma r}$, where $2Ke^{-\gamma r}$ describes the part of divided cells (with the coefficient 2 representing the division) that return to the proliferating phase to divide again. The functions f, g and h are given by $f(u) := (\delta(u)+\beta(u))u$, $g(u) := \beta(u)u$ and $h(w) := \dfrac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}w$, where δ is a natural death rate in the resting phase (taking also into account the differentiation), and β is the re-introduction rate from the resting phase to the proliferating one, and $2(1-K)e^{-\gamma r}$ is the part of dividing cells that enter directly the resting phase.

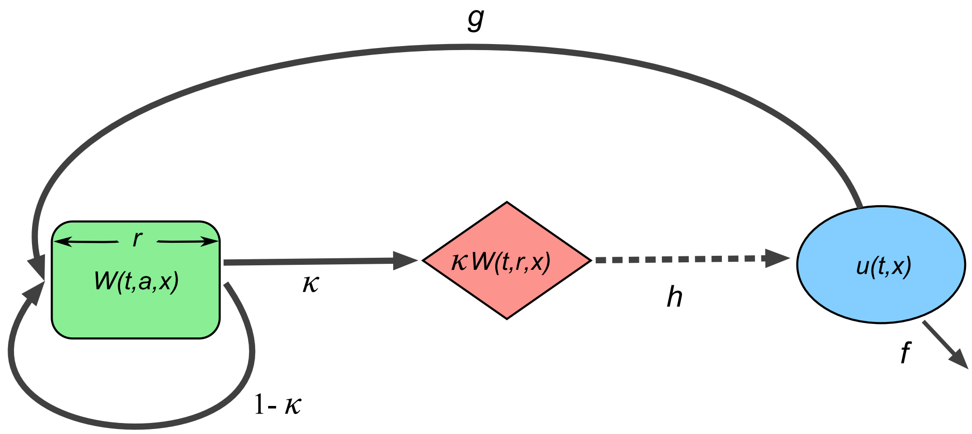

FIGURE 1. Transfer diagram of the interactions between the compartments for model (1.3). The continuous lines represent transition between compartments, and entrance and exit of individuals by production and losses in the population. The dashed line represents the transmission (or transformation) of κW into u.

Inspired by the model (1.2), we consider in this paper a more general model (1.1), with two phases, u-phase and w-phase. In the u-phase, $-f(u)$ represents locally the difference between the production and losses in the population. It contains, in particular, mortality and transition to the w-phase or to any other phase (not represented here). We assume that the u-population cannot survive without the presence of the w-population, *i.e.*, $-f \leq 0$. In the w-phase, the function g represents the new w individuals coming from the u-phase. We assume that every individual in the w-phase (coming from the u-phase) has a limited duration r. A great number of population dynamics phenomena can be modeled using two phases' approach. In general, we consider an active phase (corresponding to u-phase) and an inactive one (w-phase). The inactive phase can have several interpretations. For a prey population, it can represent a refuge where the population is protected from predators. For predators, it may be a resting phase where the predator is not hunting. In epidemiology, it may represent a period of temporary immunity due to vaccination or by taking certain drugs [1].

The above characterised interaction between the u- and w-phase can be modeled by the following system of a reaction–diffusion and an age-structured population model equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) - f(u(t,x)) + h(\kappa W(t,r,x)), \\ \dfrac{\partial W}{\partial t}(t,a,x) + \dfrac{\partial W}{\partial a}(t,a,x) = d\Delta W(t,a,x), \\ W(t,0,x) = g(u(t,x)) + (1-\kappa)W(t,r,x), \end{cases} \tag{1.3}$$

with initial condition, for $0 < a < r$ and $x \in \mathbb{R}^n$,

$$u(0,x) = u_0(x), \quad W(0,a,x) = \phi(a,x).$$

Let us note that the variable a measures the 'age' of an individual in the w-phase. Next, the dynamics of the u-phase is influenced only by the w-individuals only of age $a = r$.

Lemma 1.1. *The system (1.3) is equivalent to the system (1.1).*

Proof. First, according to (1.47) in [35], the solution to the second equation in (1.3) has the form

$$W(t,a,x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \Gamma_d(a, x-y) W(t-a, 0, y)\mathrm{d}y, & t-a>0, \\ \displaystyle\int_{\mathbb{R}^n} \Gamma_d(t, x-y) \phi(a-t, y)\mathrm{d}y, & a-t \geq 0. \end{cases} \tag{1.4}$$

Let $w(t,x) := W(t,0,x)$ denote the density of newly born individuals in the w-phase in position $x \in \mathbb{R}^n$ at time $t \geq 0$. Substituting (1.4) into the third equation in (1.3), we have for $t > r$,

$$w(t,x) = g(u(t,x)) + (1-\kappa)\int_{\mathbb{R}^n} \Gamma_d(r, x-y) w(t-r, y)\mathrm{d}y. \tag{1.5}$$

We make a translation of the initial conditions so as to define them on the interval $[-r, 0]$. We have thus shown that, for $t > 0$, $x \in \mathbb{R}^n$, the system (1.1) is equivalent to the system

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &=& D\Delta u(t,x) - f(u(t,x)) + h\left(\kappa \displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) w(t-r, y)\mathrm{d}y\right), \\ w(t,x) &=& g(u(t,x)) + (1-\kappa)\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) w(t-r, y)\mathrm{d}y, \end{cases} \tag{1.6}$$

supplemented by the set of initial conditions of the form

$$u(0,x) = u_0(x) \quad \text{and} \quad w(\theta, x) = w_0(\theta, x), \quad \text{for } \theta \in [-r, 0].$$

Thus, the system (1.6) is identical to the system (1.1), hence equivalent to the system (1.3). The lemma is proved. □

In (1.1), the term $\kappa \displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) w(t-r, y)\mathrm{d}y$ represents the fraction of the population w at the end of the w-phase which enters the u-phase with a rate κ at time t. The other fraction, $1-\kappa$, returns to the beginning of the w-phase to perform another turn in this compartment. Due to the diffusion and the stay duration r in the w-phase, the population at time t and position x was at the entry of this phase at time $t-r$ and at any position y of $\mathbb{R}^n$. The heat kernel $\Gamma_d(t, x-y)$ allows to define the new position x of the population w. The function h which determines the entry (or the transformation) of the population w into u is generally a linear function as in the case of the hematopoiesis model (1.2). The function h can sometimes be more complicated for certain biological population. In the case of megakaryopoiesis, for example, which is the process of production and regulation of platelets, it is not cell division that allows the production of platelets but fragmentation of each megakaryocyte in many platelets that are regulated by a nonlinear function.

The system (1.1) can be reduced by multiplying the second equation by κ and making the change of variables $v = \kappa w$, to obtain

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &=& D\Delta u(t,x) - f(u(t,x)) + h\left(\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) v(t-r, y)\mathrm{d}y\right), \\ v(t,x) &=& g_\kappa(u(t,x)) + (1-\kappa)\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r, x-y) v(t-r, y)\mathrm{d}y, \end{cases} \tag{1.7}$$

with

$$g_\kappa(u) := \kappa g(u), \quad u \geq 0 \text{ and } 0 \leq \kappa \leq 1.$$

In [2], we studied the existence of monotone traveling waves for the system (1.2) by reducing the problem to the existence of an admissible pair of upper and lower solutions, and by using a monotone iteration technique (see also [11]). In [3], we studied the existence of traveling wave fronts of a differential-difference diffusive Kermack-McKendrick SIR epidemic model with age-structured protection phase (with limited duration), due for example to vaccination or drugs with temporary immunity. The system analyzed in [3] is different from (1.7) and does not use the above mentioned techniques.

Throughout this paper we assume that the functions f, g and h, defined on $\mathbb{R}^+$, are sufficiently regular and satisfy the following assumptions.

(H1) $f(0) = g(0) = h(0) = 0$ and there exists a unique $u^\star > 0$ such that $f(u^\star) = h(g(u^\star))$.
(H2) $f(u) \geq 0$, $g(u) \geq 0$, for $u \in (0, u^\star)$, and $h(v) \geq 0$ for $v \in (0, v^\star)$, with $v^\star = g(u^\star)$.
(H3) $g'(u) \geq 0$, for $u \in [0, u^\star]$ and $h'(v) \geq 0$, for $v \in [0, v^\star]$.
(H4) $f(u) \geq f'(0)u$, $g(u) \leq g'(0)u$, for $u \in [0, u^\star]$ and $h(v) \leq h'(0)v$, for $v \in [0, v^\star]$.
(H5) $f'(0) < h'(0)g'(0)$ and $f'(u^\star) > h'(v^\star)g'(u^\star)$.

The assumptions (H1)–(H5) are the classical conditions for the existence of traveling waves for the system (1.7) in the non-degenerate monostable case. Under natural conditions imposed on the functions δ and β, and the parameters K, γ and r, the hypotheses (H1)–(H5) are satisfied by the hematopoiesis model (1.2) (see, *e.g.* [2]). In the case where $\kappa = 1$, the system (1.7) is equivalent to the following delayed reaction-diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) - f(u(t,x)) + h\left(\int_{\mathbb{R}^n} \Gamma_d(r, x-y) g(u(t-r,y))\mathrm{d}y\right). \tag{1.8}$$

Using heat kernel properties, we note that

$$\lim_{d \mapsto 0^+} \int_{\mathbb{R}^n} \Gamma_d(r, x-y) v(t-r,y)\mathrm{d}y = v(t-r,x).$$

Then, if $d \to 0^+$ the system (1.7) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &=& D\Delta u(t,x) - f(u(t,x)) + h\left(v(t-r,x)\right), \\ v(t,x) &=& g_\kappa(u(t,x)) + (1-\kappa)v(t-r,x). \end{cases}$$

In particular, if $\kappa = 1$ and $d \to 0^+$ we obtain

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) - f(u(t,x)) + h(g(u(t-r,x))).$$

Likewise, if $r \to 0^+$ we have

$$\lim_{r \mapsto 0^+} \int_{\mathbb{R}^n} \Gamma_d(r, x-y) v(t-r,y)\mathrm{d}y = v(t,x).$$

Then, the system (1.7) is reduced to

$$\frac{\partial u}{\partial t}(t,x) = D\Delta u(t,x) + F(u(t,x)), \tag{1.9}$$

with the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$F(u) = -f(u) + h(g(u)).$$

Note that this last system doesn't depend on d and κ. We have from (H1) and (H5)

$$F(0) = F(u^\star) = 0 \quad \text{and} \quad F'(0) = -f'(0) + h'(0)g'(0) > 0.$$

Furthermore, F satisfies

$$F(u) > 0, \text{ for } u \in (0, u^\star) \text{ and } F(u) < 0, \text{ for } u > u^\star,$$

and

$$F(u) \le -f'(0)u + h'(0)g(u) \le -f'(0)u + h'(0)g'(0)u = F'(0)u, \text{ for } u \in [0, u^\star].$$

We conclude that under the hypotheses (H1)–(H5), the function F satisfies the classical Fisher-KPP conditions (see, *e.g.*, [5, 13, 21, 27, 32]). Then, the system (1.9) has traveling wave front solutions satisfying

$$\begin{cases} c\phi' = D\phi'' + F(\phi), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = u^\star, \end{cases}$$

with speed c if and only if

$$c \ge c^* := 2\sqrt{DF'(0)} = 2\sqrt{D(h'(0)g'(0) - f'(0))}.$$

If $c < c^*$, there is no traveling fronts.

In the next section, we start by some preliminary results about the solutions of the system (1.7). The Section 3 is devoted to prove the existence of planar monotone traveling wave fronts for speeds greater or equal to a threshold $c^\star$. The Section 4 is dedicated to a detailed analysis of the critical speed wave $c^\star$ - we give some informations on its location and its asymptotic behavior with respect to the parameters of the model.

2. Preliminaries

Recall that $X := BUC(\mathbb{R}^n, \mathbb{R})$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$ and $X^+ := \{\phi \in X : \phi(x) \ge 0, \text{ for all } x \in \mathbb{R}^n\}$. The space X is a Banach lattice under the partial ordering induced by the closed cone X^+.

Let, for $\mathcal{D} > 0$, $T_\mathcal{D}(t) : X \to X$ be a bounded linear operator defined, for $t > 0$ and $w_0 \in X$, by

$$(T_\mathcal{D}(t)w_0)(x) := \int_{\mathbb{R}^n} \Gamma_\mathcal{D}(t, x - y)w_0(y)\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Then, we get (see, *e.g.*, [37]) an analytic semigroup $(T_\mathcal{D}(t))_{t\ge0}$ on X such that $T_\mathcal{D}(t)X^+ \subset X^+$, for all $t \ge 0$. Consider the Banach space $C := C([-r, 0], X)$ of continuous functions from $[-r, 0]$ into X with the classical supremum norm $\|\cdot\|_C$ and the closed cone $C^+ := C([-r, 0], X^+)$. We identify the functions $u, v \colon [0, b) \times \mathbb{R}^n \to \mathbb{R}$, $b > 0$, (respectively, $v_0 \colon [-r, 0] \times \mathbb{R}^n \to \mathbb{R}$) as functions from $[0, b)$ (respectively, $[-r, 0]$) into X, by putting $u(t)(x) = u(t, x)$ and $v(t)(x) = v(t, x)$ (respectively, $v_0(\theta)(x) = v_0(\theta, x)$). Using the same notations, we consider the functions f, g, $h \colon X \to X$ defined, for $u \in X$, by

$$f(u)(x) = f(u(x)), \quad g(u)(x) = g(u(x)) \quad \text{and} \quad h(u)(x) = h(u(x)), \quad x \in \mathbb{R}^n.$$

The functions f, g and h satisfy $f(X^+) \subseteq X^+$, $g(X^+) \subseteq X^+$ and $h(X^+) \subseteq X^+$. The system (1.7) reads, for $t > 0$,

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} & = & D\Delta u(t) - f(u(t)) + h(T_d(r)v(t-r)), \\ v(t) & = & g_\kappa(u(t)) + (1-\kappa)T_d(r)v(t-r), \end{cases} \tag{2.1}$$

with initial conditions

$$u(0) = u_0 \in X^+ \quad \text{and} \quad v(\theta) = v_0(\theta), \;\; \theta \in [-r, 0], \;\; v_0 \in C^+. \tag{2.2}$$

An abstract integral formulation of the system (2.1) (in terms of the semigroup $T_D(t)$) is

$$\begin{cases} u(t) & = & T_D(t)u_0 - \displaystyle\int_0^t T_D(t-s)f(u(s))\mathrm{d}s \\ & & \qquad\qquad + \displaystyle\int_0^t T_D(t-s)h(T_d(r)v(s-r))\mathrm{d}s, \\ v(t) & = & g_\kappa(u(t)) + (1-\kappa)T_d(r)v(t-r). \end{cases} \tag{2.3}$$

The solutions of (2.3) are called mild solutions of (2.1) (and equivalently (1.7)). Since the semigroup $T_D(t)$ is analytic, such mild solutions are also classical solutions of (2.1) (and equivalently (1.7)) (see [37], Cor. 2.5, page 50). The systems (2.1) (or (1.7)) and (2.3) are equivalent. The existence and uniqueness of solutions can be established by developing, as in [2], a comparison principle for upper and lower solutions.

3. Planar traveling wave fronts

A planar traveling wave solution of (1.7) is a special solution of the form

$$(u(t)(x), v(t)(x)) = (\phi(x \cdot \mathbf{e} + ct), \psi(x \cdot \mathbf{e} + ct)),$$

where $\phi, \psi \in C^2(\mathbb{R}, \mathbb{R}^+)$, $c > 0$ is a constant corresponding to the wave speed and $\mathbf{e}$ is a unit vector of the basis of $\mathbb{R}^n$ (see [26, 32, 37]). Without loss of generality, we can assume that $\mathbf{e} = (1, 0, ..., 0)$. Thus, $x \cdot \mathbf{e} = x_1$. We put $z = x_1 + ct$ and we substitute $(\phi(z), \psi(z))$ to $(u(t)(x), v(t)(x))$ in (2.1). We obtain the wave system

$$\begin{cases} c\phi'(z) = D\phi''(z) - f(\phi(z)) + h\left(\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r,y)\psi(z - y_1 - cr)\mathrm{d}y\right), \\ \psi(z) = g_\kappa(\phi(z)) + (1-\kappa)\displaystyle\int_{\mathbb{R}^n} \Gamma_d(r,y)\psi(z - y_1 - cr)\mathrm{d}y. \end{cases} \tag{3.1}$$

Let us note that we have

$$\Gamma_d(r,y) = \frac{1}{(4d\pi r)^{n/2}} \exp\left(-\frac{|y|^2}{4dr}\right) = \prod_{i=1}^n \frac{1}{(4d\pi r)^{1/2}} \exp\left(-\frac{y_i^2}{4dr}\right) := \prod_{i=1}^n \Gamma_d^1(r, y_i),$$

and that

$$\Gamma_d^1(r,z) = \frac{1}{(4d\pi r)^{1/2}} \exp\left(-\frac{z^2}{4dr}\right), \quad z \in \mathbb{R},$$

satisfies the identity

$$\int_{-\infty}^{+\infty} \Gamma_d^1(r,z)\mathrm{d}z = 1. \tag{3.2}$$

As a consequence, the system (3.1) is equivalent to

$$\begin{cases} c\phi'(z) = D\phi''(z) - f(\phi(z)) + h\left[(T_d^1(r)\psi)(z-cr)\right], \\ \psi(z) = g_\kappa(\phi(z)) + (1-\kappa)\left(T_d^1(r)\psi\right)(z-cr), \end{cases} \tag{3.3}$$

with

$$\left(T_d^1(r)\chi\right)(z-cr) = \int_{-\infty}^{+\infty} \Gamma_d^1(r,y_1)\chi(z-y_1-cr)dy_1, \quad \chi \in X,$$

and $X = BUC(\mathbb{R},\mathbb{R})$ the Banach space of all bounded and uniformly continuous functions. If for some constant $c > 0$ the system (3.3) has a monotone solution (ϕ,ψ) defined on $\mathbb{R}$, subject to the following asymptotic conditions

$$\phi(-\infty) = \psi(-\infty) = 0, \quad \phi(+\infty) = u^\star \quad \text{and} \quad \psi(+\infty) = v^\star, \tag{3.4}$$

where $(u^\star, v^\star)$ is the constant given by the assumptions (H1)–(H2), then the solution $(u(t)(x), v(t)(x)) = (\phi(x \cdot \mathbf{e} + ct), \psi(x \cdot \mathbf{e} + ct))$ of (3.3)–(3.4) is called a planar traveling wave front with wave speed $c > 0$.

Let $A \colon X \to X$ be the bounded linear operator defined by

$$\left(A\psi\right)(z) = (1-\kappa)\left(T_d^1(r)\psi\right)(z-cr), \quad z \in \mathbb{R}. \tag{3.5}$$

Then, the second equation of (3.3) can be written as $(Id - A)(\psi)(z) = g_\kappa(\phi(z))$, for $z \in \mathbb{R}$. We have the following result.

Lemma 3.1. *The operator $Id - A$ is invertible and its inverse is given by*

$$(Id - A)^{-1}(\chi) = \xi * \chi, \quad \chi \in X,$$

where

$$\xi(z) = \sum_{k=0}^{+\infty} \xi_k(z), \quad z \in \mathbb{R}, \tag{3.6}$$

$$\xi_k(z) = \frac{(1-\kappa)^k}{2(kd\pi r)^{1/2}} \exp\left(-\frac{(z-kcr)^2}{4kdr}\right) = (1-\kappa)^k \Gamma_d^1(kr, z-kcr), \tag{3.7}$$

and

$$(\xi * \chi)(z) = \int_{-\infty}^{+\infty} \xi(z-y)\chi(y)\mathrm{d}y. \tag{3.8}$$

Proof. The proof of this result is similar to the proof of Lemma 1 in [2]. ∎

The system (3.3) can be reduced to the following single equation

$$c\phi'(z) = D\phi''(z) - f(\phi(z)) + h\left[T_d^1(r)\left(\xi * g_\kappa(\phi)\right)(z - cr)\right], \tag{3.9}$$

with the asymptotic conditions

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = u^\star. \tag{3.10}$$

The second component ψ of the system (3.3) is given by the expression

$$\psi(z) = (\xi * g_\kappa(\phi))(z), \quad z \in \mathbb{R}. \tag{3.11}$$

It is clear that if (ϕ, ψ) is a monotone solution of (3.3)–(3.4), then ϕ is a monotone solution of (3.9)–(3.10). On the other side, if ϕ is a monotone solution of (3.9)–(3.10), then $(\phi, \xi * g_\kappa(\phi))$ is a monotone solution of (3.3)–(3.4).

Our objective now is to show the existence of monotone solutions of (3.9)–(3.10). They are traveling wave fronts for the coupled reaction-diffusion and difference equation (1.7). We consider the function $F : X \mapsto X$, defined by

$$F(\chi)(z) = h\left[T_d^1(r)\left(\xi * g_\kappa(\chi)\right)(z)\right], \quad \chi \in X,\ z \in \mathbb{R}. \tag{3.12}$$

Then, the problem (3.9)–(3.10) becomes

$$\begin{cases} c\phi'(z) = D\phi''(z) - f(\phi(z)) + F(\phi)(z - cr), \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = u^\star. \end{cases} \tag{3.13}$$

The techniques developed in this part for the existence of traveling waves are inspired by the papers [30, 38], where they used the method of upper and lower solutions, and monotonic iterative scheme. In particular, they have shown that the monotonic sequences obtained converge to the traveling wave. This technique reduces the problem of the existence of traveling waves to the existence of an upper and lower solution of (3.13).

Let us define the following profile

$$\Lambda = \left\{ \begin{array}{ll} & \phi \in X^+ : \\ (i) & \phi(z) \ \text{ is non-decreasing in } \mathbb{R}, \\ (ii) & \lim\limits_{z \to -\infty} \phi(z) = 0,\ \lim\limits_{z \to +\infty} \phi(z) = u^\star \end{array} \right\}.$$

We introduce the notion of upper and lower solutions of (3.13).

Definition 3.2. A continuous function $\overline{\phi} \in C(\mathbb{R}, \mathbb{R}^+)$ is called an upper solution of (3.13) if $\overline{\phi}'$ and $\overline{\phi}''$ exist a.e. (almost everywhere), they are essentially bounded on $\mathbb{R}$, and satisfy

$$c\overline{\phi}'(z) \geq D\overline{\phi}''(z) - f(\overline{\phi}(z)) + F(\overline{\phi})(z - cr), \quad \text{a.e. in } \mathbb{R}.$$

A lower solution $\underline{\phi}$ of (3.13) is defined in a similar way but it satisfies the above differential inequality in reversed order.

The next theorem proves, under the hypothesis of existence of an upper and a lower solution, the existence of non-decreasing solutions of (3.13), and then the existence of planar waves for the system (1.7).

Theorem 3.3. *Suppose that there exist an upper solution $\overline{\phi} \in \Lambda$ and a lower solution $\underline{\phi}$ (which is not necessarily in Λ) of (3.13) and that they satisfy the following properties*

(i) $0 \le \underline{\phi}(z) \le \overline{\phi}(z) \le u^\star$, *for all* $z \in \mathbb{R}$,
(ii) $\underline{\phi} \not\equiv 0$,
(iii) $\sup_{s\le z} \underline{\phi}(s) \le \overline{\phi}(z)$, $\overline{\phi}'(z_+) \le \overline{\phi}'(z_-)$ *and* $\underline{\phi}'(z_+) \ge \underline{\phi}'(z_-)$, *for all* $z \in \mathbb{R}$.

Then, there exists a non-decreasing solution of (3.13).

Proof. The proof is similar to the proof of the corresponding result in [38, 39]. □

Now, we have to construct suitable upper $\overline{\phi}$ and lower $\underline{\phi}$ solutions of (3.13). Let consider the transcendental characteristic function

$$\lambda \mapsto \Delta_c(\lambda),$$

for the linearized problem of (3.9) about the zero solution. Then, we have

$$\Delta_c(\lambda) = -D\lambda^2 + c\lambda + f'(0) - \frac{h'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2 - cr\lambda}}{1-(1-\kappa)e^{dr\lambda^2-cr\lambda}}. \tag{3.14}$$

$\Delta_c(\lambda)$ is well defined for $\lambda \in (\lambda^-(c), \lambda^+(c))$, with $\lambda^-(c) < 0$ and $\lambda^+(c) > 0$ given by

$$\lambda^{\pm}(c) := \frac{c}{2d}\left(1 \pm \sqrt{1 + \frac{4d}{rc^2}\ln\left(\frac{1}{1-\kappa}\right)}\right). \tag{3.15}$$

We put

$$\begin{cases} p(\lambda) = -D\lambda^2 + c\lambda + f'(0), \\ q(\lambda) = 1-(1-\kappa)e^{\mathrm{d}r\lambda^2 - cr\lambda}, \\ l(\lambda) = h'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2-cr\lambda}. \end{cases} \tag{3.16}$$

Then, (3.14) can be written as

$$\Delta_c(\lambda) = p(\lambda) - \frac{l(\lambda)}{q(\lambda)}, \qquad \lambda \in \left(\lambda^-(c), \lambda^+(c)\right).$$

We have

$$\lim_{\lambda\to\lambda^+(c)} \Delta_c(\lambda) = -\infty,$$

and thanks to the hypothesis (H5),

$$\Delta_c(0) = f'(0) - \frac{h'(0)g'_\kappa(0)}{\kappa} = f'(0) - h'(0)g'(0) < 0.$$

Now, we consider $\lambda \in [0, \lambda^+(c))$. Then, we have

$$\frac{\partial}{\partial\lambda}\Delta_c(\lambda) = \frac{(c-2D\lambda)q^2(\lambda) + (c-2d\lambda)rl(\lambda)}{q^2(\lambda)}. \tag{3.17}$$

We put

$$\underline{\lambda}(c) := \frac{c}{2}\min\left\{\frac{1}{D},\frac{1}{d}\right\} \quad \text{and} \quad \overline{\lambda}(c) := \min\left\{\frac{c}{2}\max\left\{\frac{1}{D},\frac{1}{d}\right\},\lambda^+(c)\right\}.$$

It is easy to prove that $0<\underline{\lambda}(c)\le\overline{\lambda}(c)\le\lambda^+(c)$ and

$$\left(\frac{\partial}{\partial\lambda}\Delta_c(\lambda)\big|_{\lambda=\underline{\lambda}(c)}\right)\left(\frac{\partial}{\partial\lambda}\Delta_c(\lambda)\big|_{\lambda\to\overline{\lambda}(c)}\right)<0.$$

Furthermore, the second derivative of the function $\lambda\mapsto\Delta_c(\lambda)$ is given, for all $\lambda\in[0,\lambda^+(c))$, by

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}\Delta_c(\lambda) \quad &= \quad -\frac{2Dq^4(\lambda)+2drl(\lambda)q^2(\lambda)+(c-2d\lambda)^2r^2l(\lambda)q^2(\lambda)}{q^4(\lambda)} \\ &\quad\quad -\frac{2r^2l(\lambda)q(\lambda)(c-2d\lambda)^2(1-q(\lambda))}{q^4(\lambda)}<0.\end{aligned} \tag{3.18}$$

First, let suppose that $D\neq d$. Then, for each $c>0$ there exists a unique $\lambda^\star(c)\in\big(\underline{\lambda}(c),\overline{\lambda}(c)\big)$ such that

$$\frac{\partial}{\partial\lambda}\Delta_c(\lambda)\big|_{\lambda=\lambda^\star(c)}=0.$$

Consider the function $c\mapsto\Delta_c\left(\lambda^\star(c)\right)$ defined for $c>0$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\left[\Delta_c\left(\lambda^\star(c)\right)\right] \quad &= \quad \frac{\partial}{\partial c}\Delta_c\left(\lambda^\star(c)\right)+\left[\frac{\partial}{\partial\lambda}\Delta_c\left(\lambda(c)\right)\big|_{\lambda=\lambda^\star(c)}\right]\times\left[\frac{\mathrm{d}}{\mathrm{d}c}\lambda^\star(c)\right], \\ &= \quad \frac{\partial}{\partial c}\Delta_c\left(\lambda^\star(c)\right).\end{aligned}$$

Moreover,

$$\lim_{c\to0}\Delta_c\left(\lambda^\star(c)\right)=-D\lambda^2+f'(0)-\frac{h'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2}}{1-(1-\kappa)e^{dr\lambda^2}}<0,$$

and

$$\Delta_c\left(\lambda^\star(c)\right)\ge\Delta_c\left(\underline{\lambda}(c)\right) \quad \text{with} \quad \lim_{c\to+\infty}\Delta_c\left(\underline{\lambda}(c)\right)=+\infty.$$

We conclude that there exists a unique $c^\star>0$ such that

$$\begin{cases}\Delta_{c^\star}(\lambda^\star(c^\star))=0,\\ \dfrac{\partial}{\partial\lambda}\Delta_{c^\star}(\lambda)\big|_{\lambda=\lambda^\star(c^\star)}=0.\end{cases} \tag{3.19}$$

Next, if we suppose that $D=d$, then we have $\underline{\lambda}(c)=\overline{\lambda}(c)=\dfrac{c}{2d}<\lambda^+(c)$. In this case, $\lambda^\star(c)=\dfrac{c}{2d}$. Furthermore, the unique $c^\star>0$ that satisfies (3.19) is given explicitly by

$$c^\star=2\sqrt{dG^{-1}\left(h'(0)g'_\kappa(0)\right)},$$

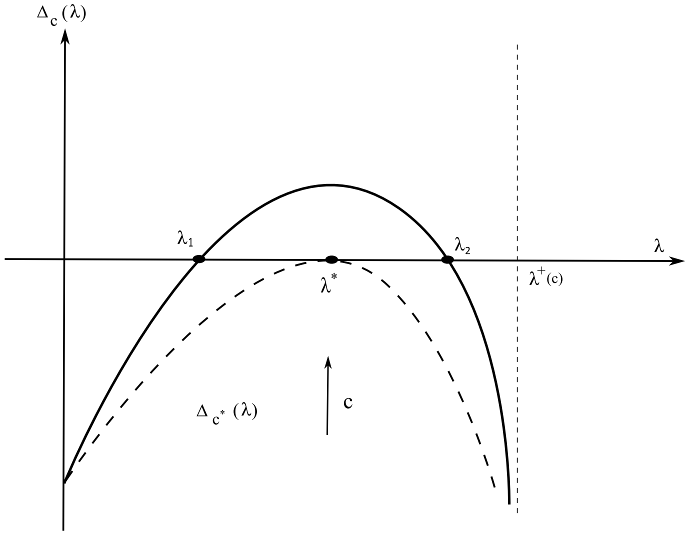

FIGURE 2. Representation of $\lambda \mapsto \Delta_c(\lambda)$.

where the function G is given, for $x \geq 0$, by

$$G(x) = (x + f'(0))\left(e^{rx} + \kappa - 1\right). \tag{3.20}$$

We proved the following result (see Fig. 2).

Lemma 3.4. *For each $c > 0$, there exists a unique $\lambda^\star(c) \in \left[\underline{\lambda}(c), \overline{\lambda}(c)\right]$ and a unique $c^\star > 0$ such that*

(i) $\Delta_{c^\star}(\lambda^\star(c^\star)) = \dfrac{\partial}{\partial\lambda}\Delta_{c^\star}(\lambda)\big|_{\lambda=\lambda^\star(c^\star)} = 0$,
(ii) if $c > c^\star$, there exist two real roots, $\lambda_1(c)$ and $\lambda_2(c)$ of the equation $\Delta_c(\lambda) = 0$, such that $0 < \lambda_1(c) < \lambda^\star(c) < \lambda_2(c) < \lambda^+(c)$ and $\Delta_c(\lambda) > 0$, for all $\lambda \in (\lambda_1(c), \lambda_2(c))$,
(iii) if $0 < c < c^\star$, $\Delta_c(\lambda) < 0$, for all $\lambda \in (0, \lambda^+(c))$.

In particular, if $D = d$ then

$$\lambda^\star(c) = \frac{c}{2d}, \quad c > 0 \quad \text{and} \quad c^\star = 2\sqrt{dG^{-1}\left(h'(0)g'_\kappa(0)\right)},$$

where the function G is given by (3.20).

Now, we are in the position to construct upper and lower solutions for equation (3.13). We fix $c > c^\star$ and we put $\lambda_1 := \lambda_1(c)$, $\lambda_2 := \lambda_2(c)$, where $c^\star$, $\lambda_1(c)$ and $\lambda_2(c)$ given by Lemma 3.4.

Lemma 3.5. *Let $c > c^\star$ be fixed and $\lambda_1 := \lambda_1(c)$. Then, the function $\overline{\phi}\colon \mathbb{R} \to \mathbb{R}^+$ defined by*

$$\overline{\phi}(z) = \min\{u^\star, e^{\lambda_1 z}\}, \quad z \in \mathbb{R},$$

is an upper solution of (3.13) belonging to Λ.

Proof. As $\lambda_1 > 0$, then

$$\overline{\phi}(z) = \begin{cases} u^\star, & z \geq z_1, \\ e^{\lambda_1 z}, & z < z_1, \end{cases} \tag{3.21}$$

for $z_1 := \dfrac{1}{\lambda_1} \ln (u^\star)$. It is clear that $\overline{\phi} \in \Lambda$. First, suppose that $z \in [z_1, +\infty)$. Then, $\overline{\phi}(z) = u^\star$ and $\overline{\phi}'(z) = \overline{\phi}''(z) = 0$. As the functions g and h are increasing on $[0, u^\star]$, we have

$$F\left(\overline{\phi}\right)(z - cr) \leq h\left[T_d^1(r)\left(\xi * g_\kappa(u^\star)\right)(z - cr)\right] = h\left(\frac{g_\kappa(u^\star)}{\kappa}\right) = h(g(u^\star)).$$

Then, we obtain

$$c\overline{\phi}'(z) - D\overline{\phi}''(z) + f(\overline{\phi}(z)) - F\left(\overline{\phi}\right)(z - cr) \geq f(u^\star) - h(g(u^\star)) = 0.$$

Next, suppose that $z \in (-\infty, z_1)$. Then, $\overline{\phi}(z) = e^{\lambda_1 z}$, and consequently

$$c\overline{\phi}'(z) - D\overline{\phi}''(z) = \left(c\lambda_1 - D\lambda_1^2\right) e^{\lambda_1 z}.$$

Thanks to (H4), we have

$$f(\overline{\phi}(z)) \geq f'(0)\overline{\phi}(z) = f'(0)e^{\lambda_1 z}.$$

Furthermore, again by (H4),

$$\begin{aligned} F\left(\overline{\phi}\right)(z - cr) &\leq h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi * \overline{\phi}\right)(z - cr)\right], \\ &\leq h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi * e^{\lambda_1 \cdot}\right)(z - cr)\right], \\ &= \frac{h'(0)g_\kappa'(0)e^{\mathrm{d}r\lambda_1^2 + (z - cr)\lambda_1}}{1 - (1 - \kappa)e^{dr\lambda_1^2 - cr\lambda_1}}. \end{aligned}$$

Hence

$$c\overline{\phi}'(z) - D\overline{\phi}''(z) + f(\overline{\phi}(z)) - F\left(\overline{\phi}\right)(z - cr) \geq \Delta_c(\lambda_1)e^{\lambda_1 z} = 0.$$

□

Figure 3 shows the profile of an upper solution (see Lem. 3.5) and a lower solution (see Lem. 3.6).

Lemma 3.6. *Let $c > c^\star$ be fixed, $\lambda_1 := \lambda_1(c)$ and $\lambda_2 := \lambda_2(c)$. Then, the following properties hold.*

1. *There exists $M > 1$ such that the function $\underline{\phi}\colon \mathbb{R} \to \mathbb{R}^+$ defined by*

$$\underline{\phi}(z) = \max\{0, e^{\lambda_1 z} - Me^{\omega\lambda_1 z}\}, \quad z \in \mathbb{R},$$

   with $\omega \in (1, \min\{2, \lambda_2/\lambda_1\})$ is a lower solution of (3.13).
2. *$\underline{\phi}(z) \leq \overline{\phi}(z)$, for all $z \in \mathbb{R}$,*
3. *$\sup_{s \leq z} \underline{\phi}(s) \leq \overline{\phi}(z)$, $\overline{\phi}'(z_+) \leq \overline{\phi}'(z_-)$ and $\underline{\phi}'(z_+) \geq \underline{\phi}'(z_-)$, for all $z \in \mathbb{R}$.*

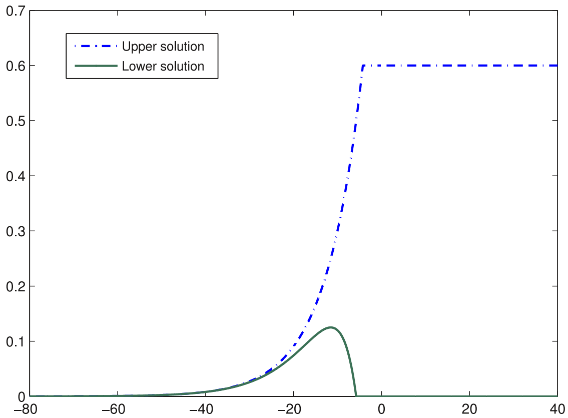

FIGURE 3. The shapes of upper and lower solutions.

Proof. We will construct a lower solution $\underline{\phi}$ of the form

$$\underline{\phi}(z) = \begin{cases} e^{\lambda_1 z} - Me^{\omega\lambda_1 z}, & z < z_2, \\ 0, & z \geq z_2, \end{cases}$$

with

$$z_2 := \frac{1}{(\omega - 1)\lambda_1} \ln\left(\frac{1}{M}\right) \quad \text{and} \quad M > 1.$$

Then, $z_2 < 0$. Remark that to have $\underline{\phi} \leq \overline{\phi}$, it suffices to choose

$$M > (u^\star)^{1-\omega}.$$

Let $z \in [z_2, +\infty)$. Then, $\underline{\phi}(z) = 0$. Thus,

$$c\underline{\phi}'(z) - D\underline{\phi}''(z) + f(\underline{\phi}(z)) - F\left(\underline{\phi}\right)(z - cr) = -F\left(\underline{\phi}\right)(z - cr).$$

As the function $\underline{\phi}$ is nonnegative on $\mathbb{R}$, then the function $F\left(\underline{\phi}\right)$ is also nonnegative on $\mathbb{R}$. We conclude that

$$c\underline{\phi}'(z) - D\underline{\phi}''(z) + f(\underline{\phi}(z)) - F\left(\underline{\phi}\right)(z - cr) \leq 0, \quad \text{for all } z \in [z_2, +\infty).$$

Now, let $z \in (-\infty, z_2)$. We have $\underline{\phi}(z) = e^{\lambda_1 z} - Me^{\omega\lambda_1 z}$. Then,

$$c\underline{\phi}'(z) - D\underline{\phi}''(z) = c\lambda_1 e^{\lambda_1 z} - cM\omega\lambda_1 e^{\omega\lambda_1 z} - D\lambda_1^2 e^{\lambda_1 z} + DM\omega^2\lambda_1^2 e^{\omega\lambda_1 z}.$$

Thanks to $\Delta_c(\lambda_1)=0$, we obtain

$$\begin{aligned} c\underline{\phi}'(z)-D\underline{\phi}''(z) &= \frac{l(\lambda_1)}{q(\lambda_1)}e^{\lambda_1 z}-f'(0)e^{\lambda_1 z}-\Delta_c(\omega\lambda_1)Me^{\omega\lambda_1 z}\\ &\quad -\frac{l(\omega\lambda_1)}{q(\omega\lambda_1)}Me^{\omega\lambda_1 z}+f'(0)Me^{\omega\lambda_1 z}. \end{aligned}$$

Furthermore, as $0<\lambda_1<\lambda_2<\lambda^+(c)$ and $\omega\in(1,\min\{2,\lambda_2/\lambda_1\})$, we have

$$1-(1-\kappa)e^{dr\lambda_1^2-cr\lambda_1}>0,\quad \Delta_c(\omega\lambda_1)>0\quad\text{and}\quad 1-(1-\kappa)e^{dr\omega^2\lambda_1^2-cr\omega\lambda_1}>0. \tag{3.22}$$

Consequently,

$$\frac{l(\lambda_1)}{q(\lambda_1)}e^{\lambda_1 z}-\frac{l(\omega\lambda_1)}{q(\omega\lambda_1)}Me^{\omega\lambda_1 z}\le h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\underline{\phi}\right)(z-cr)\right].$$

Let $\nu\in(\omega-1,\min\{2,\lambda_2/\lambda_1\}-1)$. It is clear that $0<\nu<1$. Recall that under the assumption that f is a C^2-function between 0 and the uniform equilibrium $u^\star$, there exists $\alpha_1>0$ such that

$$f(u)-f'(0)u\le\alpha_1 u^{\nu+1},\quad \text{for all }\ u\in[0,u^\star].$$

Then, we have

$$f(\underline{\phi}(z))-f'(0)\underline{\phi}(z)\quad\le\quad\alpha_1\underline{\phi}^{(\nu+1)}(z),$$

with

$$\underline{\phi}(z)=e^{\lambda_1 z}-Me^{\omega\lambda_1 z},\quad z\in(-\infty,z_2).$$

We conclude that

$$\begin{aligned} c\underline{\phi}'(z)-D\underline{\phi}''(z)+f(\underline{\phi}(z)) &\le -\Delta_c(\omega\lambda_1)Me^{\omega\lambda_1 z}+\alpha_1\underline{\phi}^{(\nu+1)}(z)\\ &\quad +h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\underline{\phi}\right)(z-cr)\right]. \end{aligned}$$

To proceed, let us examine the expression

$$h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\underline{\phi}\right)(z-cr)\right]-F\left(\underline{\phi}\right)(z-cr).$$

Under the assumption that g and h are C^2-functions between 0 and the uniform equilibria $u^\star$ and $v^\star$, there exists $\alpha_2>0$ such that

$$\begin{cases} g_\kappa'(0)u-g_\kappa(u)\le\alpha_2 u^{\nu+1}, & \text{for all }\ u\in[0,u^\star],\\ h'(0)v-h(v)\le\alpha_2 v^{\nu+1}, & \text{for all }\ v\in[0,v^\star]. \end{cases}$$

This implies

$$\begin{aligned} &h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\underline{\phi}\right)\right](\cdot)-h\left[\left(T_d^1(r)\left(\xi*g_\kappa(\underline{\phi})\right)\right)(\cdot)\right]\\ &\qquad\le\alpha_2 h'(0)\left[T_d^1(r)\left(\xi*\underline{\phi}^{\nu+1}\right)\right](\cdot)+\alpha_2\left(\left[T_d^1(r)\left(\xi*g_\kappa(\underline{\phi})\right)\right](\cdot)\right)^{\nu+1}. \end{aligned}$$

We use the assumption $g(u) \leq g'(0)u$, for all $u \in [0, u^\star]$, to get

$$\begin{aligned}
&h'(0)g'_\kappa(0)\left[T_d^1(r)\left(\xi * \underline{\phi}\right)\right](z-cr) - F\left(\underline{\phi}\right)(z-cr) \\
&\qquad \leq \alpha_2 h'(0)\left[T_d^1(r)\left(\xi * \underline{\phi}^{\nu+1}\right)\right](z-cr) + \alpha_2 (g'_\kappa(0))^{\nu+1}\left(\left[T_d^1(r)\left(\xi * \underline{\phi}\right)\right](z-cr)\right)^{\nu+1}.
\end{aligned}$$

It is not difficult to see that

$$\underline{\phi}(z) \leq e^{\lambda_1 z}, \quad \text{for all } \ z \in \mathbb{R}.$$

Consequently,

$$\begin{aligned}
&c\underline{\phi}'(z) - D\underline{\phi}''(z) + f(\underline{\phi}(z)) - F\left(\underline{\phi}\right)(z-cr) \\
&\qquad \leq -\Delta_c(\omega\lambda_1)Me^{\omega\lambda_1 z} + \alpha_1 e^{(\nu+1)\lambda_1 z} + \alpha_2 h'(0)T_d^1(r)\left[\xi * e^{(\nu+1)\lambda_1 \cdot}\right](z-cr) \\
&\qquad\quad + \alpha_2 (g'_\kappa(0))^{\nu+1}\left[T_d^1(r)(\xi * e^{\lambda_1 \cdot})(z-cr)\right]^{\nu+1}.
\end{aligned}$$

As

$$\begin{aligned}
\left(\xi * e^{\lambda_1 \cdot}\right)(z-cr) &= \int_{\mathbb{R}} \xi(z-cr-y)e^{\lambda_1 y}\mathrm{d}y, \\
&= e^{-\lambda_1 cr}\sum_{k=0}^{+\infty}(1-\kappa)^k \int_{\mathbb{R}} \Gamma_d^1\left(kr, z-y-kcr\right)e^{\lambda_1 y}\mathrm{d}y, \\
&= \frac{e^{-\lambda_1 cr}}{1-(1-\kappa)e^{dr\lambda_1^2 - cr\lambda_1}}e^{\lambda_1 z}
\end{aligned}$$

then

$$T_d^1(r)\left[\xi * e^{(\nu+1)\lambda_1 \cdot}\right](z-cr) = \frac{e^{(\nu+1)^2\lambda_1^2 dr - (\nu+1)\lambda_1 cr}}{1-(1-\kappa)e^{dr(\nu+1)^2\lambda_1^2 - cr(\nu+1)\lambda_1}}e^{(\nu+1)\lambda_1 z} := K_1 e^{(\nu+1)\lambda_1 z},$$

and

$$\left[T_d^1(r)\left(\xi * e^{\lambda_1 \cdot}\right)(z-cr)\right]^{\nu+1} = \left[\frac{e^{(\nu+1)\lambda_1^2 dr - \lambda_1 cr}}{1-(1-\kappa)e^{dr\lambda_1^2 - cr\lambda_1}}e^{\lambda_1 z}\right]^{\nu+1} := K_2 e^{(\nu+1)\lambda_1 z}.$$

Consequently,

$$\begin{aligned}
&c\underline{\phi}'(z) - D\underline{\phi}''(z) + f(\underline{\phi}(z)) - F\left(\underline{\phi}\right)(z-cr) \\
&\leq -\Delta_c(\omega\lambda_1)Me^{\omega\lambda_1 z} + \alpha_1 e^{(\nu+1)\lambda_1 z} + \alpha_2 h'(0)K_1 e^{(\nu+1)\lambda_1 z} + \alpha_2 (g'_\kappa(0))^{\nu+1}K_2 e^{(\nu+1)\lambda_1 z}.
\end{aligned}$$

So, we obtain

$$c\underline{\phi}'(z) - D\underline{\phi}''(z) + f(\underline{\phi}(z)) - F\left(\underline{\phi}\right)(z-cr) \leq e^{\omega\lambda_1 z}\left[-\Delta_c(\omega\lambda_1)M + \alpha e^{(\nu+1-\omega)\lambda_1 z}\right],$$

with

$$\alpha := \max\{\ \alpha_1, \alpha_2 h'(0)K_1, \alpha_2 (g'_\kappa(0))^{(\nu+1)}K_2\}.$$

Remember that $\nu+1-\omega>0$, $\lambda_1>0$ and $z_2<0$. Thus, $e^{(\nu+1-\omega)\lambda_1 z}<1$ for all $z<z_2$, hence, for $M>\alpha\Delta_c(\omega\lambda_1)^{-1}$, we obtain,

$$c\underline{\phi}'(z)-D\underline{\phi}''(z)+f(\underline{\phi}(z))-F\left(\underline{\phi}\right)(z-cr)<0, \quad \text{for all } z<z_2.$$

In fact,

$$M>\max\{\,1,(u^\star)^{1-\omega},\alpha\Delta_c(\omega\lambda_1)^{-1}\}$$

satisfies all the established above conditions for the parameter M. It follows that $\underline{\phi}$ is a lower solution and it satisfies $\underline{\phi}(z)\leq\overline{\phi}(z)$, for all $z\in\mathbb{R}$. Finally, it is clear that $\sup_{s\leq z}\underline{\phi}(s)\leq\overline{\phi}(z)$, $\overline{\phi}'(z_+)\leq\overline{\phi}'(z_-)$ and $\underline{\phi}'(z_+)\geq\underline{\phi}'(z_-)$, for all $z\in\mathbb{R}$. □

We have just proved that the lower and upper solutions $\underline{\phi}$ and $\overline{\phi}$ satisfy the three conditions (i), (ii) and (iii) of Theorem 3.3. Then, we can formulate the following theorem for the existence of monotone traveling wave fronts for the system (1.7).

Theorem 3.7. *For every $c\geq c^\star$, System (1.7) has monotone traveling wave fronts connecting $(0,0)$ to the positive equilibrium $(u^\star,v^\star)$.*

Proof. For $c>c^\star$, the proof of the theorem follows directly from Theorem 3.3, Lemma 3.5 and Lemma 3.6. For $c=c^\star$, we can proceed as in [2, 10, 31, 41, 42] by using a limit argument. □

Let us now consider the case $c\in(0,c^\star)$. We get in this case the non-existence of monotone traveling wave. To show this result, we need some properties of traveling wave fronts.

Lemma 3.8. *Let $c\in(0,c^\star)$, ϕ be a solution of (3.9) connecting 0 to $u^\star$ and $z\in\mathbb{R}$. We have*

1. $\displaystyle\int_{-\infty}^{z}\left|\left[T_d^1(r)\left(\xi*\phi\right)\right](y-cr)-\phi(y)\right|\mathrm{d}y<+\infty,$
2. $\displaystyle\varphi(z):=\int_{-\infty}^{z}\phi(y)\mathrm{d}y<+\infty,$
3. $\displaystyle\int_{-\infty}^{z}\left[T_d^1(r)\left(\xi*\phi\right)\right](y-cr)\mathrm{d}y=\left(\left(\Gamma_d^1*\xi\right)*\varphi\right)(z-cr),$
4. $\displaystyle\int_{-\infty}^{z}\left|\left(\left(\Gamma_d^1*\xi\right)*\varphi\right)(y-cr)-\varphi(y)\right|\mathrm{d}y<+\infty.$

Moreover, there exists a positive constant $\mu_0<\lambda^+(c)$ such that $\sup_{z\in\mathbb{R}}\left[e^{-\mu_0 z}\phi(z)\right]<+\infty$.

Except some few manipulations in the nonlinearity, the proof of the above lemma is the same as Proposition 7 and Theorem 5 of [2] and we omit the details in this paper. Then, we obtain the following theorem.

Theorem 3.9. *For any $c\in(0,c^\star)$, there is no monotone traveling wave fronts of the system (1.7).*

Proof. Assume that $c\in(0,c^\star)$ and consider ϕ such that

$$c\phi'(z)=D\phi''(z)-f(\phi(z))+h\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right], \tag{3.23}$$

with

$$\phi(-\infty)=0, \quad \phi(+\infty)=u^*.$$

From Lemma 3.8, the following Laplace transform is well defined

$$\mathcal{L}(\lambda)(\phi)=\int_{-\infty}^{+\infty}e^{-\lambda z}\phi(z)\mathrm{d}z,\qquad \text{for}\quad 0<\mathrm{Re}(\lambda)<\mu_0,$$

where μ_0 is a positive constant such that $\mu_0<\lambda^+(c)$. Considering the equation (3.23), we obtain

$$\begin{aligned}&-c\phi'(z)+D\phi''(z)-f'(0)\phi(z)+h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\phi\right)(z-cr)\right]\\&\qquad=f(\phi(z))-f'(0)\phi(z)-h\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right]+h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\phi\right)(z-cr)\right].\end{aligned}\tag{3.24}$$

Consider $\lambda\in\mathbb{C}$ such that $0<\mathrm{Re}(\lambda)<\mu_0$. The Fubini theorem implies that

$$\int_{\mathbb{R}}e^{-\lambda z}\left[T_d^1(r)\left(\xi*\phi\right)\right](z-cr)\mathrm{d}z=\int_{\mathbb{R}}\phi(z)\left[T_d^1(r)\left(\xi*e^{\lambda\cdot}\right)\right](-z-cr)\mathrm{d}z.$$

Then, we get the following formula

$$\int_{\mathbb{R}}e^{-\lambda z}\left[T_d^1(r)\left(\xi*\phi\right)\right](z-cr)\mathrm{d}z=\frac{e^{\mathrm{d}r\lambda^2-cr\lambda}}{1-(1-\kappa)e^{dr\lambda^2-cr\lambda}}\mathcal{L}(\lambda)(\phi).$$

The Laplace transform applied to equation (3.24) yields to

$$\begin{aligned}-\Delta_c(\lambda)\mathcal{L}(\lambda)(\phi)=&\int_{-\infty}^{+\infty}e^{-\lambda z}\left[f(\phi(z))-f'(0)\phi(z)\right.\\&\left.-h\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right]+h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\phi\right)(z-cr)\right]\right]\mathrm{d}z.\end{aligned}\tag{3.25}$$

Recall that $f(0)=g_\kappa(0)=h(0)=0$ and $\lim\limits_{z\to-\infty}\phi(z)=0$. As $z\to-\infty$, we obtain from the Taylor's development that

$$f(\phi(z))-f'(0)\phi(z)=O(\phi^2(z)).$$

Then, by the hypothesis (H4), we get

$$\begin{aligned}&-h\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right]+h'(0)g_\kappa'(0)\left[T_d^1(r)\left(\xi*\phi\right)(z-cr)\right]\\&\quad=h'(0)\left[T_d^1(r)\left(\xi*\left[g_\kappa'(0)\phi-g_\kappa(\phi)\right]\right)(z-cr)\right]-h\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right]\\&\qquad+h'(0)\left[T_d^1(r)\left(\xi*g_\kappa(\phi)\right)(z-cr)\right],\\&\quad=O(\left[T_d^1(r)\left(\xi*\phi^2\right)\right](z-cr))+O(\left[\left[T_d^1(r)\left(\xi*\phi\right)\right](z-cr)\right]^2).\end{aligned}$$

Using the fact that,

$$\phi(z)=O(e^{\mu_0 z})\quad\text{as}\quad z\to-\infty,$$

we conclude that there exists $M>0$ such that

$$\phi^2(z)\le Me^{2\mu_0 z},\quad\left[T_d^1(r)\left(\xi*\phi^2\right)\right](z-cr)\le Me^{2\mu_0 z}\quad\text{and}\quad\left[\left[T_d^1(r)\left(\xi*\phi\right)\right](z-cr)\right]^2\le Me^{2\mu_0 z},\quad\text{as }z\to-\infty.$$

As a result, the right hand side of (3.25) is well defined for all $\lambda \in \mathbb{C}$ such that $0 < \text{Re}(\lambda) < 2\mu_0$. We rewrite the equation (3.25) in the following form

$$\begin{aligned} \mathcal{L}(\lambda)(\phi) \quad = \quad & \frac{-1}{\Delta_c(\lambda)} \int_{-\infty}^{+\infty} e^{-\lambda z} \left[f(\phi(z)) - f'(0)\phi(z) \right. \\ & \left. -h \left[T_d^1(r) \left(\xi * g_\kappa(\phi) \right)(z - cr) \right] + h'(0) g_\kappa'(0) \left[T_d^1(r) \left(\xi * \phi \right)(z - cr) \right] \right] \mathrm{d}z. \end{aligned} \tag{3.26}$$

Since $0 < c < c^\star$, then $\Delta_c(\lambda)$ has no real root in the interval $(0, \lambda^+(c))$. Thus, $\lambda \mapsto \mathcal{L}(\lambda)(\phi)$ has no real singularity in the interval $[0, \min\{2\mu_0, \lambda^+(c)\})$. From Theorem 5b, page 58 of [36] and since $\phi \geq 0$, $\lambda \mapsto \mathcal{L}(\lambda)(\phi)$ is analytic for $0 < \text{Re}(\lambda) < \min\{2\mu_0, \lambda^+(c)\}$. Suppose that $2\mu_0 < \lambda^+(c)$. Since $\lambda \mapsto \mathcal{L}(\lambda)(\phi)$ is defined for $0 < \text{Re}\lambda < 2\mu_0$, it implies that $\phi(z) = O(e^{2\mu_0 z})$ as $z \to -\infty$ (see page 39 of [36]). We can repeat the above arguments to get that $\lambda \mapsto \mathcal{L}(\lambda)(\phi)$ is defined for $0 < \text{Re}(\lambda) < \lambda^+(c)$ and the same for the expression (3.25).

We will now show that there is no traveling wave for $0 < c < c^\star$. Recall that

$$\lim_{\lambda \to \lambda^+(c)} \Delta_c(\lambda) = -\infty.$$

Thus, there exists $A > 0$ such that for all $\lambda \in \mathbb{R}$ with $A < \lambda < \lambda^+(c)$, we have

$$\begin{aligned} & \Delta_c(\lambda)\phi(z) + f(\phi(z)) - f'(0)\phi(z) \\ & \qquad -h \left[T_d^1(r) \left(\xi * g_\kappa(\phi) \right)(z - cr) \right] + h'(0) g_\kappa'(0) \left[T_d^1(r) \left(\xi * \phi \right)(z - cr) \right] < 0. \end{aligned}$$

We multiply this inequality by $e^{-\lambda z}, \lambda \in \mathbb{R}$, to obtain

$$\begin{aligned} & e^{-\lambda z} \left[\Delta_c(\lambda)\phi(z) + f(\phi(z)) - f'(0)\phi(z) \right. \\ & \qquad \left. -h \left[T_d^1(r) \left(\xi * g_\kappa(\phi) \right)(z - cr) \right] + h'(0) g_\kappa'(0) \left[T_d^1(r) \left(\xi * \phi \right)(z - cr) \right] \right] < 0. \end{aligned}$$

This yields to

$$\begin{aligned} -\Delta_c(\lambda)\mathcal{L}(\lambda)(\phi) > & \int_{-\infty}^{+\infty} e^{-\lambda z} \left[f(\phi(z)) - f'(0)\phi(z) \right. \\ & \left. -h \left[T_d^1(r) \left(\xi * g_\kappa(\phi) \right)(z - cr) \right] + h'(0) g_\kappa'(0) \left[T_d^1(r) \left(\xi * \phi \right)(z - cr) \right] \right] \mathrm{d}z. \end{aligned}$$

This is a contradiction with the expression (3.25). Therefore, there is no monotone traveling wave fronts of the system (1.7). □

4. Minimal wave speed estimations and asymptotic behavior

The purpose of this section is to analyze the minimal wave speed $c^\star$ in terms of the parameters of the system and to give some properties. We start with the monotony of $c^\star$ with respect to the parameters d, D and κ.

Proposition 4.1. *The minimal wave speed $c^\star$ is an increasing function with respect to d and D, and a decreasing function with respect to κ.*

Proof. Let μ denote one of the parameters d, D or κ. We consider $c^\star := c^\star(\mu)$ as a function of μ and we define the function $\mu \mapsto \Delta_{c^\star(\mu)}(\mu, \lambda^\star(\mu))$, where $\lambda^\star(\mu) := \lambda^\star(c^\star(\mu))$ is given by Lemma 3.4. Then, we have

$$\Delta_{c^\star(\mu)}(\mu, \lambda^\star(\mu)) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} \Delta_{c^\star(\mu)}(\mu, \lambda) \big|_{\lambda = \lambda^\star(\mu)} = 0. \tag{4.1}$$

Using the implicit function theorem for the first equation, we obtain

$$\frac{\partial}{\partial\mu}\Delta_{c^\star(\mu)}(\mu,\lambda^\star(\mu))+\frac{d}{d\mu}\lambda^\star(\mu)\frac{\partial}{\partial\lambda}\Delta_{c^\star(\mu)}(\mu,\lambda)\big|_{\lambda=\lambda^\star(\mu)} + \frac{\mathrm{d}}{\mathrm{d}\mu}c^\star(\mu)\frac{\partial}{\partial c}\Delta_c(\mu,\lambda^\star(\mu))\big|_{c=c^\star(\mu)}=0.$$

The second equation of (4.1) implies that

$$\frac{\partial}{\partial\mu}\Delta_{c^\star(\mu)}(\mu,\lambda^\star(\mu))+\frac{\mathrm{d}}{\mathrm{d}\mu}c^\star(\mu)\frac{\partial}{\partial c}\Delta_c(\mu,\lambda^\star(\mu))\big|_{c=c^\star(\mu)}=0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}c^\star(\mu)=-\frac{\frac{\partial}{\partial\mu}\Delta_{c^\star(\mu)}(\mu,\lambda^\star(\mu))}{\frac{\partial}{\partial c}\Delta_c(\mu,\lambda^\star(\mu))\big|_{c=c^\star(\mu)}}.$$

Remember that

$$\Delta_c(\lambda)=p_c(\lambda)-\frac{l_c(\lambda)}{q_c(\lambda)},\quad \lambda\in[0,\lambda^+(c)),$$

where p_c, l_c and q_c are given by the right hand sides of (3.16) and $\lambda^+(c)$ by (3.15). Then, we have

$$\frac{\partial}{\partial c}\Delta_c(\lambda)=\frac{\partial}{\partial c}p_c(\lambda)-\frac{q_c(\lambda)\frac{\partial}{\partial c}l_c(\lambda)-l_c(\lambda)\frac{\partial}{\partial c}q_c(\lambda)}{q_c^2(\lambda)}.$$

By (3.16), we obtain

$$\frac{\partial}{\partial c}p_c(\lambda)=\lambda,\quad \frac{\partial}{\partial c}q_c(\lambda)=r(1-\kappa)\lambda e^{\mathrm{d}r\lambda^2-cr\lambda}\quad\text{and}\quad \frac{\partial}{\partial c}l_c(\lambda)=-r\lambda l_c(\lambda).$$

This implies that $\frac{\partial}{\partial c}\Delta_c(\lambda)>0$ and in particular

$$\frac{\partial}{\partial c}\Delta_c(\mu,\lambda^\star(\mu))\big|_{c=c^\star(\mu)}>0,\quad \text{for all } \mu.$$

Consequently, $\frac{\mathrm{d}}{\mathrm{d}\mu}c^\star(\mu)$ has the same sign as the function $-\frac{\partial}{\partial\mu}\Delta_{c^\star(\mu)}(\mu,\lambda^\star(\mu))$. By calculating the derivatives, we obtain the following expressions.

- If $\mu=d$, we have $-\frac{\partial}{\partial d}\Delta_c(d,\lambda)=\dfrac{r\lambda^2 l_c(\lambda)\left(q_c(\lambda)+(1-\kappa)e^{\mathrm{d}r\lambda^2-cr\lambda}\right)}{q_c^2(\lambda)}>0.$
- If $\mu=D$, we have $-\frac{\partial}{\partial D}\Delta_c(D,\lambda)=\lambda^2>0.$
- If $\mu=\kappa$, we have $-\frac{\partial}{\partial\kappa}\Delta_c(\kappa,\lambda)=-\dfrac{l_c(\lambda)e^{\mathrm{d}r\lambda^2-cr\lambda}}{q_c^2(\lambda)}<0.$

We conclude that the minimal wave speed $c^\star$ is an increasing function with respect to the parameters d and D and a decreasing function with respect to the parameter κ. □

The variation of the minimal wave speed $c^\star$ in terms of the delay r is more complicated to investigate. We will only consider the special case $D = d$ (see Lem. 2.5 of [28] for a particular case).

Proposition 4.2. *Assume that $D = d$. The minimal wave speed $c^\star$ is a decreasing function with respect to the delay r.*

Proof. For $D = d$, by Lemma 3.4, we have

$$\lambda^\star(c) = \frac{c}{2d} < \lambda^+(c), \quad \text{for all } c > 0. \tag{4.2}$$

Let us consider the minimal wave speed $c^\star := c^\star(r)$ as a function of $r > 0$. As in the proof of Proposition 4.1, we have

$$\frac{\partial}{\partial r}\Delta_{c^\star(r)}(r, \lambda^\star(r)) + \frac{\mathrm{d}}{\mathrm{d}r}c^\star(r)\left[\frac{\partial}{\partial c}\Delta_c(r, \lambda^\star(r))\big|_{c=c^\star(r)}\right] = 0,$$

with

$$\frac{\partial}{\partial c}\Delta_c(r, \lambda^\star(r))\big|_{c=c^\star(r)} > 0.$$

Note that

$$\frac{\partial}{\partial r}\Delta_{c^\star}(r, \lambda^\star) = -\frac{(d\lambda^\star - c^\star)\,\lambda^\star l_{c^\star}(\lambda^\star)}{q^2_{c^\star}(\lambda^\star)},$$

and, according to (4.2)

$$c^\star - d\lambda^\star = \frac{c^\star}{2} > 0.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}r}c^\star(r) < 0.$$

This proves the claim of the lemma. □

We focus now on some estimates of the minimal wave speed $c^\star$. We put

$$P := \frac{h'(0)g'_\kappa(0)}{\kappa} - f'(0) = h'(0)g'(0) - f'(0) \quad \text{and} \quad Q_\kappa := \frac{h'(0)g'_\kappa(0)}{f'(0)} + (1-\kappa) = \kappa\frac{P}{f'(0)} + 1, \tag{4.3}$$

which depend only on κ, $f'(0)$, $g'(0)$ and $h'(0)$. The condition (H5) implies that $P > 0$ and $Q_\kappa > 1$.

Theorem 4.3. *The following estimates hold.*

– *If $0 < d \leq \frac{1}{2}D$, we have*

$$0 < c^\star < 2\sqrt{D}\min\left\{\sqrt{P}, \sqrt{\frac{D}{4(D-d)}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}\right\}. \tag{4.4}$$

– *If* $\frac{1}{2}D \le d \le D$*, we have*

$$0 < c^\star < 2\sqrt{D}\min\left\{\sqrt{P}, \sqrt{\frac{d}{D}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}\right\}. \tag{4.5}$$

– *If* $D \le d < 2D$*, we have*

$$0 < c^\star < 2\sqrt{d}\min\left\{\sqrt{\frac{d}{2d-D}}\sqrt{P}, \sqrt{\frac{1}{r}\ln(Q_\kappa)}\right\}. \tag{4.6}$$

– *If* $d > 2D$*, we have*

$$0 < c^\star < 2\min\left\{\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P}, \sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}, \sqrt{D}\sqrt{\frac{D}{d-2D}}\sqrt{\frac{1}{r}\ln(1/(1-\kappa))}\right\}. \tag{4.7}$$

Proof. As $P > 0$ and $Q_\kappa > 1$, all the thresholds of $c^\star$ in Theorem 4.3 are well defined. Recall that the function $\lambda \mapsto \Delta_c(\lambda)$ is given for $c > 0$ fixed and $\lambda \in [0, \lambda^+(c))$, with

$$\lambda^+(c) := \frac{c}{2d}\left(1 + \sqrt{1 + \frac{4d}{rc^2}\ln\left(\frac{1}{1-\kappa}\right)}\right),$$

by

$$\Delta_c(\lambda) = p_c(\lambda) - \frac{l_c(\lambda)}{q_c(\lambda)} + f'(0) = p_c(\lambda) - j_c(\lambda),$$

where

$$\begin{cases} p_c(\lambda) = -D\lambda^2 + c\lambda, \\ q_c(\lambda) = 1 - (1-\kappa)e^{dr\lambda^2 - cr\lambda}, \\ l_c(\lambda) = h'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2 - cr\lambda}, \end{cases}$$

and

$$j_c(\lambda) = \frac{l_c(\lambda)}{q_c(\lambda)} - f'(0) = \frac{h'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2 - cr\lambda}}{1 - (1-\kappa)e^{dr\lambda^2 - cr\lambda}} - f'(0).$$

The derivatives of p_c and j_c are given by

$$p'_c(\lambda) = -(2D\lambda - c) \quad \text{and} \quad j'_c(\lambda) = (2d\lambda - c)\frac{rh'(0)g'_\kappa(0)e^{\mathrm{d}r\lambda^2 - \lambda cr}}{(1 - (1-\kappa)e^{\mathrm{d}r\lambda^2 - cr\lambda})^2}.$$

The second derivatives satisfy $p_c''(\lambda) < 0$ and $j_c''(\lambda) > 0$ for all $\lambda \in [0, \lambda^+(c))$. The maximum of p_c and the minimum of j_c are given, respectively for $\lambda_D^\star = \dfrac{c}{2D}$ and $\lambda_d^\star = \dfrac{c}{2d}$, by

$$p_c(\lambda_D^\star) = \frac{c^2}{4D} \qquad \text{and} \qquad j_c(\lambda_d^\star) = \frac{h'(0)g_\kappa'(0)e^{-\frac{rc^2}{4d}}}{1-(1-\kappa)e^{-\frac{rc^2}{4d}}} - f'(0) = S\left(\frac{rc^2}{4d}\right),$$

where the function S is defined for $x \in (-\ln(1/(1-\kappa)), +\infty)$, by

$$S : x \mapsto \frac{h'(0)g_\kappa'(0)e^{-x}}{1-(1-\kappa)e^{-x}} - f'(0).$$

S is decreasing on $(-\ln(1/(1-\kappa)), +\infty)$ and its maximum on the interval $[0, +\infty)$ is to $S(0) = P > 0$, with P given by (4.3). Furthermore, $S(x) > 0$ if and only if $x < \ln(Q_\kappa)$, where $Q_\kappa > 1$ is given by (4.3). We also have

$$p_c(\lambda_d^\star) = \frac{c^2}{4d^2}(2d - D) \quad \text{and} \quad j_c(\lambda_D^\star) = S\left(\frac{rc^2}{4D^2}(2D - d)\right).$$

When the parameter c takes the value $c^\star$, we obtain $\lambda_D^\star := \dfrac{c^\star}{2D}$ and $\lambda_d^\star := \dfrac{c^\star}{2d}$.

First, consider the case $D = d$. Then, $\lambda_D^\star = \lambda_d^\star := \lambda^\star$ and $p_{c^\star}(\lambda^\star) = j_{c^\star}(\lambda^\star)$ (see Fig. 4). This is equivalent to

$$S\left(\frac{rc^{\star 2}}{4D}\right) = \frac{c^{\star 2}}{4D}. \tag{4.8}$$

From the properties of the function S, we deduce that

$$0 < \frac{c^{\star 2}}{4D} < P \quad \text{and} \quad 0 < \frac{rc^{\star 2}}{4D} < \ln(Q_\kappa).$$

This is equivalent to

$$0 < c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad 0 < c^\star < 2\sqrt{D}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

This proves (4.5) and (4.6) for $d = D$. Let us fix $D > 0$ and suppose that $0 < d < D$. Then, the critical points of $p_{c^\star}$ and $j_{c^\star}$ are given, see Figure 4, by

$$\lambda_D^\star = \frac{c^\star}{2D} < \lambda_d^\star = \frac{c^\star}{2d}.$$

Thanks to Proposition 4.1, $d \mapsto c^\star(d)$ is an increasing function. Let $\underline{c}^\star := c^\star(D)$ be the minimal wave speed for $d = D$. Then, for all $0 < d < D$, we have $c^\star := c^\star(d) < \underline{c}^\star$. Using the result of the last paragraph, we obtain for $d < D$

$$0 < c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad 0 < c^\star < 2\sqrt{D}\sqrt{\frac{1}{r}\ln(Q_\kappa)}. \tag{4.9}$$

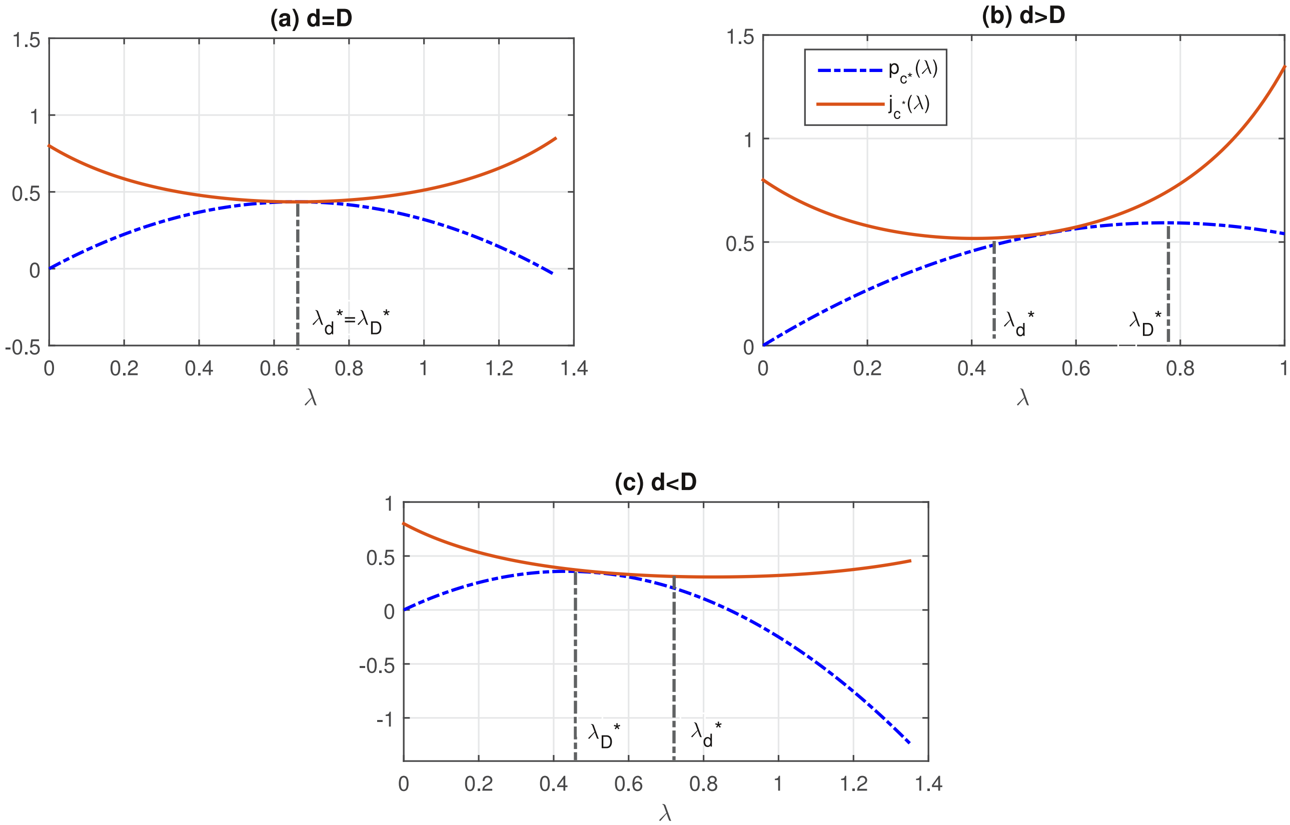

FIGURE 4. The curves of the functions p_c and j_c are drawn for (a) $D = d$, (b) $D < d$ and (c) $D > d$. Fixed parameters are : $\kappa = 0.8$, $r = 1$, $f'(0) = 0.1$, $g'_\kappa(0) = 0.9$ and $h'(0) = 0.8$. The functions are plotted with the parameters : (a) $d = D = 1$, $c = c^\star = 1.32$, (b) $d = 1.9 > D = 1$, $c = c^\star = 1.54$, and (c) $d = 1 < D = 1.9$, $c = c^\star = 1.65$.

We can use a similar argument for the case $d > D$, see Figure 4, to prove

$$0 < c^\star < 2\sqrt{d}\sqrt{P} \quad \text{and} \quad 0 < c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}. \tag{4.10}$$

Our next objective is to improve the estimations (4.9) and (4.10). From the properties of $p_{c^\star}$ and $j_{c^\star}$, we have (see Fig. 4)

$$p_{c^\star}\left(\lambda_D^\star\right) < j_{c^\star}\left(\lambda_D^\star\right) \quad \text{and} \quad p_{c^\star}\left(\lambda_d^\star\right) < j_{c^\star}\left(\lambda_d^\star\right). \tag{4.11}$$

The first inequality of (4.11) is equivalent to

$$\frac{c^{\star 2}}{4D} < S\left(\frac{rc^{\star 2}}{4D^2}(2D - d)\right) \tag{4.12}$$

and the second is equivalent to

$$\frac{c^{\star 2}}{4d^2}(2d - D) < S\left(\frac{rc^{\star 2}}{4d}\right). \tag{4.13}$$

We will treat the inequalities (4.12) and (4.13) separately. Suppose that $d < 2D$. Then, (4.12) implies

$$\frac{c^{\star 2}}{4D} < P \quad \text{and} \quad \frac{rc^{\star 2}}{4D^2}(2D - d) < \ln(Q_\kappa).$$

We conclude that for $d < 2D$, we have

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{D}\sqrt{\frac{D}{2D - d}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}. \tag{4.14}$$

The second inequality of (4.14) is 'better' than that of (4.9) for $d < D$. Indeed, we have

$$\frac{D}{2D - d} < 1 \quad \text{if and only if} \quad d < D.$$

However, if $D < d < 2D$ we have

$$d < \frac{D^2}{2D - d}.$$

Then, the second inequality of (4.10) gives a smaller lower bound for $c^\star$ than (4.14). To further improve the second inequality of (4.14), we introduce the real number

$$\widehat{\lambda}_D := \frac{c^\star}{D} \in (0, \lambda^+(c^\star)).$$

We have

$$0 = p_{c^\star}(\widehat{\lambda}_D) < j_{c^\star}(\widehat{\lambda}_D).$$

This is equivalent to

$$S\left(\frac{rc^{\star 2}}{D}\left(1 - \frac{d}{D}\right)\right) > 0. \tag{4.15}$$

Suppose that $d < D$. Then, (4.15) implies

$$\frac{rc^{\star 2}}{D}\left(1 - \frac{d}{D}\right) < \ln(Q_\kappa)$$

which is equivalent to

$$c^\star < 2\sqrt{D}\sqrt{\frac{D}{4(D - d)}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}. \tag{4.16}$$

We have

$$\frac{D}{4(D-d)} < \frac{D}{2D-d} \quad \text{if and only if} \quad d < \frac{2}{3}D.$$

We conclude that for $d < \frac{2}{3}D$ the the inequality (4.16) is better than the second inequality of (4.14). We summarize the obtained inequalities as follows.

– If $D \le d < 2D$, we have from (4.10),

$$c^\star < 2\sqrt{d}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

– If $\frac{2}{3}D \le d \le D$, we have from (4.9) and (4.14),

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{D}\sqrt{\frac{D}{2D-d}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

– If $d \le \frac{2}{3}D$, we have from (4.9) and (4.16),

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{D}\sqrt{\frac{D}{4(D-d)}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

Our objective now is to explore the inequality (4.13). We first remark that if $d \le \frac{1}{2}D$, then (4.13) is always satisfied and does not bring about any improvement.

Suppose that $\frac{1}{2}D < d$. Using the same techniques as before, we obtain from (4.13), the inequalities

$$c^\star < 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}. \tag{4.17}$$

We remark that

$$\frac{d}{2d-D} < 1 \quad \text{if and only if} \quad D < d.$$

Then, (4.17) improves the inequalities (4.10), for $D < d$.

We now suppose that $\frac{1}{2}D < d < D$. We have to consider two sub-cases:

$$\frac{1}{2}D < d < \frac{2}{3}D \quad \text{and} \quad \frac{2}{3}D < d < D.$$

If we suppose $\frac{1}{2}D < d < \frac{2}{3}D$, then

$$D < \frac{d^2}{2d-D} \quad \text{and} \quad d < \frac{D^2}{4(D-d)}.$$

In this case, we obtain the estimations

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

If we suppose $\frac{2}{3}D < d < D$, we also obtain

$$D < \frac{d^2}{2d-D} \quad \text{and} \quad d < \frac{D^2}{2D-d}.$$

Then, we again obtain the estimates

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

We sum up the 'best' inequalities obtained as follows.

– If $0 < d \leq \frac{1}{2}D$, we have

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{D}\sqrt{\frac{D}{4(D-d)}}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

This proves (4.4).

– If $\frac{1}{2}D \leq d \leq D$, we have

$$c^\star < 2\sqrt{D}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

This gives the estimate (4.5).

– If $D \leq d < 2D$, we have

$$c^\star < 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)}.$$

Then, we obtain the estimate (4.6).

To finish the proof of the theorem, we have to consider the case $d > 2D$. Then, on one side we have

$$c^\star < 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P} \quad \text{and} \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)},$$

and on the other side the inequality (4.12) gives that

$$c^\star < 2\sqrt{D}\sqrt{\frac{D}{d-2D}}\sqrt{\frac{1}{r}\ln(1/(1-\kappa))}.$$

Then, for $d > 2D$ we have the inequalities

$$c^\star < 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P}, \quad c^\star < 2\sqrt{d}\sqrt{\frac{1}{r}\ln(Q_\kappa)} \quad \text{and} \quad c^\star < 2\sqrt{D}\sqrt{\frac{D}{d-2D}}\sqrt{\frac{1}{r}\ln(1/(1-\kappa))}.$$

This gives the estimate (4.7).

□

Using the estimates proved in Theorem 4.3, we can derive, in particular, the following limits of the minimum wave speed $c^\star$.

Proposition 4.4. *We have the following limits.*

1. $c^\star \to 0$ *as* $(d, D) \to (0^+, 0^+)$.
2. $c^\star = O(\sqrt{D}) \to +\infty$ *as* $D \to +\infty$.
3. $c^\star \to 0$ *as* $r \to +\infty$.
4. *For all* $D > 0$ *and* $d > 0$, *we have*

$$2\sqrt{mP} \le \liminf_{r\to 0^+} c^\star(r) \le \limsup_{r\to 0^+} c^\star(r) \le 2\sqrt{MP}, \tag{4.18}$$

with $m := \min\{D, d\}$ *and* $M := \max\{D, d\}$.

5. *If* $d \ge D$, *we have*

$$2\sqrt{DP} \le \liminf_{r\to 0^+} c^\star(r) \le \limsup_{r\to 0^+} c^\star(r) \le 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P}. \tag{4.19}$$

Proof. Let us first note that the upper bounds in the estimates (4.18) and (4.19) are independent on $\kappa \in (0, 1)$. Indeed, as it was noted in the introduction, when $r \to 0^+$ the system (1.7) is independent on κ. The first three points can be directly obtained from the estimates established in Theorem 4.3. It remains to prove the limits when $r \to 0^+$. We start by assuming that $D = d$. Then, we have from (4.8)

$$S\left(\frac{rc^{\star 2}(r)}{4d}\right) = \frac{c^{\star 2}(r)}{4d}.$$

From Proposition 4.2, we know that $r \mapsto c^\star(r)$ is decreasing. As a consequence, $\lim_{r\to 0^+} c^\star(r)$ exists and

$$\lim_{r\to 0^+} c^\star(r) = 2\sqrt{dS(0)} = 2\sqrt{dP}.$$

Let d be fixed. We denote by $c^\star_d$ and $c^\star_D$ the minimal wave speeds in the cases $D = d$ and $d < D$, respectively. Using Proposition 4.1 and the estimates (4.4)–(4.5), we obtain

$$0 < c^\star_d(r) < c^\star_D(r) < \ 2\sqrt{D}\min\left\{\sqrt{P}, \sqrt{\frac{1}{r}\ln(Q_\kappa)}\right\}.$$

Let us suppose that $0 < \kappa < 1$ and choose $r_0 := \ln(Q_\kappa)/P > 0$. Then, for all $0 < r < r_0$, we have

$$P < \frac{1}{r}\ln(Q_\kappa).$$

Consequently, for $d < D$

$$0 < c_d^\star(r) < c_D^\star(r) < 2\sqrt{DP}. \tag{4.20}$$

Thus, if $r \to 0^+$ in (4.20) and as $\lim\limits_{r\to 0^+} c_d^\star(r) = 2\sqrt{dP}$, we obtain for $d < D$

$$2\sqrt{dP} \le \liminf_{r\to 0^+} c^\star(r) \le \limsup_{r\to 0^+} c^\star(r) \le 2\sqrt{DP}.$$

We can use the same idea to prove that if $d > D$, we have

$$2\sqrt{DP} \le \liminf_{r\to 0^+} c^\star(r) \le \limsup_{r\to 0^+} c^\star(r) \le 2\sqrt{dP}.$$

We thus proved the estimate (4.18). Our final objective is to show the estimate (4.19). Let $d \ge D$. We denote by $c_D^\star$ and $c_d^\star$ the minimal wave speeds in the cases $d = D$ and $d > D$, respectively. Then, from Proposition 4.1 and Theorem 4.3 we conclude that for $r > 0$ small enough

$$0 < c_D^\star(r) < c_d^\star(r) < \ 2\sqrt{d}\sqrt{\frac{d}{2d-D}}\sqrt{P}.$$

Thus, if $r \to 0^+$ we obtain, for $d \ge D$, the estimate (4.19). □

5. Discussion

We have studied the existence and properties of traveling wave fronts for a relatively general models represented by a coupled system of reaction-diffusion and difference equations (renewal equation) in n-dimensional space, with nonlocal dispersal terms and an implicit time delay (system (1.1)). This system can be derived from the reaction–diffusion system with age structure (1.3), describing the interaction between two phases (compartments) of a population, of which at least one has a limited duration. The dynamics of many populations with individuals remaining temporarily in a state of quiescence, dormancy or slowed inactivity can be analysed by using our two-compartment model (1.1) (see, *e.g.* [16, 17]). For a prey population, the compartment with a limited duration may represent a refuge where the population is protected from predators. For predators, this can be the resting phase where the predator is not hunting. In epidemiology, this may represent a period of temporary immunity due to vaccination or taking medications. The assumptions (H1)–(H5) are the usual conditions so that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(u) = -f(u) + h(g(u))$, that gives the stationary steady states, satisfies the classical Fisher-KPP conditions

$$F(0) = F(u^\star) = 0,\ F(u) > 0,\ \text{for } u \in (0, u^\star),\ F(u) < 0,\ \text{for } u > u^\star \ \text{ and }\ F(u) \le F'(0)u,\ \text{for } u \in [0, u^\star].$$

The existence of planar monotone traveling wave fronts has been proved by the method of upper and lower solutions. An upper solution gives rise to a traveling wave as the limit of monotone iterations and a lower solution allows us to connect the trivial equilibrium to the positive uniform one. Using the characteristic equation, we have proved that there is a minimum wave speed $c^\star > 0$. We showed that the existence of traveling waves is valid not only for $c > c^\star$ but also for $c = c^\star$. In addition, we analyzed the minimum wave speed $c^\star$ as a function of the system parameters and established some estimates of $c^\star$ in different scenarios satisfied by the two diffusion coefficients. The investigations of traveling wave fronts for reaction-diffusion models with nonlocal dispersal terms and time delays are attracting more and more attention (see, *e.g.*, [12, 15, 22–24, 31, 33, 34] and references therein). Some mathematical tools were developed to make the link between the traveling wave solutions and the spreading speeds for these systems, such as integral equation approach by Thieme and Zhao

[31], and monotone semiflows method by Liang and Zhao [23] and Fang and Zhao [12]. Despite the fact that the system (1.1) generates a monotone semiflow, the method developed by Liang and Zhao [23] cannot be applied. This is because the solution map associated with the reaction-diffusion and difference system (1.1) is not compact with respect to compact open topology (see, [2]).

Note that the stability of the traveling waves has not been investigated here. Another point, we preferred to start with the monostable case and leave the bistable case for future work. Indeed, the bistable case requires other mathematical tools which are more complex for this type of models. Finally, the condition (H3) was crucial in the analysis of our model (1.1). However, we believe that this assumption could be weakened. All these issues will be the subject of future studies.

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