Quantum Cost for Sending Entanglement

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(Received 20 March 2012; published 18 June 2012)

Establishing quantum entanglement between two distant parties is an essential step of many protocols in quantum information processing. One possibility for providing long-distance entanglement is to create an entangled composite state within a lab and then physically send one subsystem to a distant lab. However, is this the "cheapest" way? Here, we investigate the minimal "cost" that is necessary for establishing a certain amount of entanglement between two distant parties. We prove that this cost is intrinsically quantum, and is specified by quantum correlations. Our results provide an optimal protocol for entanglement distribution and show that quantum correlations are the essential resource for this task.

DOI: 10.1103/PhysRevLett.108.250501 PACS numbers: 03.67.Hk

Imagine that one wants to send a letter in the old-fashioned way. The postage cost that the sender has to invest depends on the amount of the transmitted substance, quantified by the weight of the letter. If the receiver had already provided some prepaid envelope, the sender may have to add an appropriate stamp if he or she wants to send a heavier letter. Naturally, the allowed weight of the letter is smaller or equal to a limit which is linked to the total postage.

Now, imagine that a sender wants to send quantum entanglement to a receiver. How does the cost that the sender has to invest depend on the amount of entanglement sent, quantified by some entanglement measure? Is this cost reduced when sender and receiver already shared some preestablished entanglement? And what is the nature of this cost—can one pay in classical quantities, or does one have to invest a quantum cost?

One might be tempted to consider these questions and their answers as obvious matters. However, quantum mechanics has often surprised us with puzzling features: counterintuitively, as shown in [1], separable states (i.e., states without entanglement) can be used to distribute entanglement. What is then the resource that makes this process possible and enables entanglement distribution without actually sending an entangled state?

In order to address this question in a well defined and quantitative way we will consider the following setting, see Fig. 1: the sender is called Alice (A), and the distant receiver Bob (B). Each of them has a quantum particle in his or her possession. In addition, they have a third quantum particle or ancilla (C) available, which is at the beginning located in Alice's lab, and then sent (via a noiseless quantum channel) to Bob's lab. This is a general model for any interaction: one can consider the particle C as the intermediate particle that realises the global interaction between A and B. A similar scenario was also considered in a different context in [2,3].

Initially, the total joint quantum state may or may not carry entanglement. In the following, we will be only interested in bipartite entanglement; i.e., two out of the three particles A, B, and C are grouped together. We quantify the initial entanglement between AC and B as $E^{AC|B}$, and the final entanglement, after sending C to Bob, as $E^{A|BC}$. As a quantifier of entanglement we will first use the relative entropy of entanglement, which is a well established and widely studied measure of entanglement for mixed states [4,5]. It is defined as the minimal relative entropy $S(\rho \| \sigma) = \mathrm{Tr}[\rho \log\rho] - \mathrm{Tr}[\rho \log\sigma]$ between the given state ρ^{XY} for two parties X and Y and the set of separable states $\mathcal{S}$:

$$E^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in\mathcal{S}} S(\rho^{XY} \| \sigma^{XY}). \tag{1}$$

Besides the fact that the relative entropy plays a crucial role in quantum information theory [6], the significance of

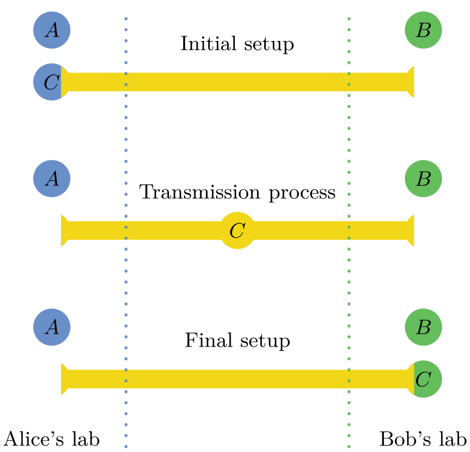

FIG. 1 (color online). Entanglement distribution between Alice and Bob. The upper figure shows the initial setup before the transmission: Alice holds the particles A and C, while Bob is in possession of the particle B. The middle figure shows the transmission process: Alice uses a quantum channel to send C to Bob. The final situation is shown in the lower figure. See also main text.

the relative entropy of entanglement is also provided by its close relation to the distillable entanglement [7].

In a naive approach to our original question, namely determining in a quantitative way the cost for sending a certain amount of entanglement, a natural conjecture would be the inequality $Q^{C|AB} \geq E^{A|BC} - E^{AC|B}$, where Q denotes a yet undefined kind of correlations. This inequality can be interpreted as follows: if initially Alice and Bob share some preestablished entanglement, quantified by $E^{AC|B}$, and wish to achieve final entanglement of $E^{A|BC}$ between them, the ancilla C, sent from Alice to Bob, needs to carry at least an amount of correlations given by the difference of final and initial entanglement. This inequality quantifies the intuition, that entanglement distribution does not come for free, but always requires to invest some correlations. In other words, $Q^{C|AB}$ could be interpreted as the "cost" for sending the entanglement $E^{A|BC} - E^{AC|B}$. Quite surprisingly, it is not the entanglement between C and AB, which plays a crucial role here: as was demonstrated in [1], all steps of the protocol can be successfully implemented without any entanglement between C and the rest of the system. In other words, if some inequality of the conjectured form exists, the quantity Q cannot be a measure of entanglement. However, does the fact that entanglement distribution is possible via separable states mean that the "cost" for this protocol is of classical nature? As we will show in the following, this is not the case: the cost for sending entanglement is of quantum nature.

Even separable states, which by definition can be prepared locally with the help of classical communication, can carry quantum properties; i.e., they can be quantum correlated. A composite quantum state is called strictly classically correlated if its correlations can be described by a joint probability distribution for classical variables of the subsystems [8]. If this is not the case, quantum correlations are manifest in the state. Recently, there has been much interest in characterising quantum correlations [9–15], in interpreting their occurrence in quantum information protocols [16–20], and in particular in determining their role in quantum algorithms [21–25], see also the feature article [26] and the comprehensive review [27]. In the following we will quantify the amount of quantum correlations according to the thermodynamical approach presented in [12,28]. There the authors provided the notion of the information deficit: it quantifies the amount of information which cannot be localised by classical communication between two parties. If only one-way classical communication from party X to party Y is allowed, this leads to the one-way information deficit:

$$\Delta^{X|Y}(\rho^{XY}) = \min_{\{\Pi_i^X\}} S\left(\rho^{XY} \parallel \sum_i \Pi_i^X \rho^{XY} \Pi_i^X\right), \quad (2)$$

where the minimization is done over all local von Neumann measurements $\{\Pi_i^X\}$ on subsystem X.

We will show in the following that the measure defined in Eq. (2) quantifies the cost discussed above, thus revealing the fundamental role of quantum correlations as a resource for the distribution of entanglement:

$$\Delta^{C|AB} \geq E^{A|BC} - E^{AC|B}, \quad (3)$$

where the entanglement measure $E^{X|Y}$ was defined in Eq. (1). This inequality is our central result; we will discuss its meaning and implications below. We point out that this inequality holds for any dimension of the three subsystems, see Fig. 2 for illustration. The main idea of the proof of Eq. (3) is sketched in Fig. 3. We name the state σ to be the closest separable state to ρ, i.e., $E^{AC|B}(\rho) = S(\rho \parallel \sigma)$. We then consider the local measurement $\{\Pi_i^C\}$ on particle C that minimizes the relative entropy of the resulting state ρ' with respect to the original ρ, i.e., $\rho' = \sum_i \Pi_i^C \rho \Pi_i^C$ such that $\Delta^{C|AB}(\rho) = S(\rho \parallel \rho')$. In Fig. 3 we also show the state $\sigma' = \sum_i \Pi_i^C \sigma \Pi_i^C$, which results from the application of the same measurement on the state σ. It is crucial to note that the three states ρ, ρ' and σ' lie on a straight line, as shown in Fig. 3:

$$S(\rho \| \sigma') = S(\rho \| \rho') + S(\rho' \| \sigma'). \quad (4)$$

For proving this equality it is enough to show the relations $\mathrm{Tr}[\rho \log \rho'] = \mathrm{Tr}[\rho' \log \rho']$ and $\mathrm{Tr}[\rho \log \sigma'] = \mathrm{Tr}[\rho' \log \sigma']$, then Eq. (4) immediately follows. These two equalities can be shown in a straight-forward way, by using the idempotent property of the projectors, the cyclic invariance of the trace, and the fact that the projectors Π_i^C sum up to the identity.

The final ingredient in the proof of Eq. (3) is the fact that the relative entropy does not increase under quantum operations [4,29,30]: $S(\Lambda(\rho) \parallel \Lambda(\sigma)) \leq S(\rho \parallel \sigma)$, and thus $S(\rho' \parallel \sigma') \leq S(\rho \parallel \sigma)$. Inserting this into Eq. (4) implies the inequality $S(\rho \parallel \sigma') \leq \Delta^{C|AB}(\rho) + E^{AC|B}(\rho)$. To complete the proof of Eq. (3), we notice that the state σ' is a tripartite fully separable state, and thus gives an upper bound on the entanglement $E^{A|BC}(\rho) \leq S(\rho \parallel \sigma')$.

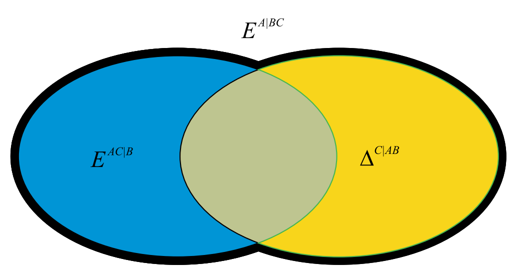

FIG. 2 (color online). Illustration of the main result: The size of the left area represents the entanglement between AC and B, while the size of the right area represents the quantum correlations between C and AB. The total area, enclosed by the black curve, represents the entanglement between A and BC. One can read off the main result: $E^{A|BC} \leq E^{AC|B} + \Delta^{C|AB}$.

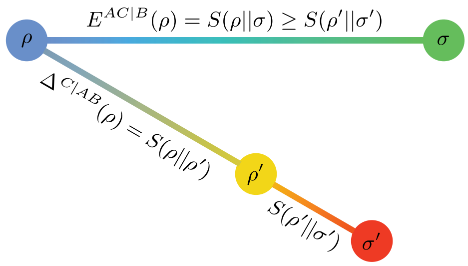

FIG. 3 (color online). Proof of the main result in Eq. (3): The separable state σ is the closest separable state to the given state ρ. The measured state $\rho' = \sum_i \Pi_i^C \rho \Pi_i^C$ is defined such that $\Delta^{C|AB}(\rho) = S(\rho||\rho')$. Application of the same measurement on σ gives the state $\sigma' = \sum_i \Pi_i^C \sigma \Pi_i^C$. The states ρ, ρ', and σ' lie on a straight line; for details see main text.

The techniques presented above can also be applied to a more general measure of entanglement, where the relative entropy $S(\rho_1 \parallel \rho_2)$ is replaced—both for the entanglement measure and the quantum correlation measure—by a general distance $D(\rho_1, \rho_2)$. We only demand that D has the following two properties: (1) D does not increase under any quantum operation, (2) D satisfies the triangle inequality. Then Eq. (4) becomes an inequality: $D(\rho, \sigma') \leq D(\rho, \rho') + D(\rho', \sigma')$, and the proof of Eq. (3) follows from the same arguments as above. Well-known and frequently used examples for distances that fulfil these two properties [31] are, e.g., the trace distance, defined as $D_t(\rho_1, \rho_2) = \frac{1}{2}\,\mathrm{tr}|\rho_1 - \rho_2|$ and the Bures distance [32], defined as $D_B(\rho_1, \rho_2) = 2(1 - \sqrt{F(\rho_1, \rho_2)})$, with $F(\rho_1, \rho_2) = (\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})^2$.

Let us point out that our main result in inequality (3) can be alternatively seen as a restricting link between the correlation properties of the three possible bipartite splits of a tripartite quantum state in any dimension: the entanglement across one of the bipartite splits cannot be larger than the sum of the entanglement across one of the other splits plus the quantum correlations across the remaining split. Thus, the inequality (3) may be interpreted as a type of "monogamy" relation between three entangled parties. This inequality also holds for all permutations of the parties. By permuting the systems A and B in Eq. (3), we obtain the generally valid inequality

$$E^{AC|B} - \Delta^{C|AB} \leq E^{A|BC} \leq E^{AC|B} + \Delta^{C|AB}. \tag{5}$$

This inequality tells us, that the entanglement between A and BC is not independent from the entanglement between AC and B. In particular, in the case of vanishing quantum correlations, i.e. $\Delta^{C|AB} = 0$, we immediately see that these two quantities are equal: $E^{A|BC} = E^{AC|B}$. We also note that for those situations, where $\Delta^{C|AB} = E^{C|AB}$, this happens, e.g., for the relative entropy when the state under consideration is pure, one arrives, using all permutations of inequality (3), at the triangle inequality $|E^{B|AC} - E^{C|AB}| \leq E^{A|BC} \leq E^{B|AC} + E^{C|AB}$. However, we stress again that this symmetric inequality is a special case of the general inequality (5), and is valid only for certain classes of states.

We are now in position to answer the question posed in the first paragraph of this paper: what is the cheapest way for distributing entanglement? In order to answer this question in full generality, we consider the most general distribution protocol, which may contain n uses of the quantum channel together with local operations and classical communication between Alice and Bob. The amount of entanglement sent in this process of entanglement growing cannot be larger than the total cost in the protocol:

$$E_{\mathrm{final}} - E_{\mathrm{initial}} \leq \sum_{i=1}^{n} \Delta_i, \tag{6}$$

where E_{initial} and E_{final} is the amount of entanglement between Alice and Bob before and after the protocol, and Δ_i is the amount of quantum correlations between the sent particle and the remaining system in the ith application of the quantum channel.

In order to prove Eq. (6), we first consider a protocol where the quantum channel is used once from Alice to Bob and once in the other direction, i.e., $n = 2$. Suppose that Alice and Bob start with a state ρ_1, the initial entanglement is $E_{\mathrm{initial}} = E^{AC|B}(\rho_1)$. After sending the particle C to Bob the entanglement between the two parties is given by $E^{A|BC}(\rho_1)$, and the cost for this process is given by $\Delta^{C|AB}(\rho_1)$. Now Alice and Bob locally act on their subsystems, and may additionally communicate classically with each other, thus arriving at the final state ρ_2 with the entanglement $E^{A|BC}(\rho_2)$. In the final step of this single-round protocol Bob sends the particle C back to Alice, and the final entanglement is $E_{\mathrm{final}} = E^{AC|B}(\rho_2)$. The corresponding cost for this final step is given by $\Delta^{C|AB}(\rho_2)$. We will now show that the amount of entanglement sent in the total process cannot be larger than the total cost:

$$E_{\mathrm{final}} - E_{\mathrm{initial}} \leq \Delta^{C|AB}(\rho_1) + \Delta^{C|AB}(\rho_2). \tag{7}$$

This inequality can be seen by applying inequality (3) to the two states ρ_1 and ρ_2 independently, and considering the sum of the both inequalities: $E^{AC|B}(\rho_2) - E^{A|BC}(\rho_2) + E^{A|BC}(\rho_1) - E^{AC|B}(\rho_1) \leq \Delta^{C|AB}(\rho_2) + \Delta^{C|AB}(\rho_1)$. Note that the entanglement $E^{A|BC}(\rho_2)$ is not larger than $E^{A|BC}(\rho_1)$, since the state ρ_2 results from the state ρ_1 after application of local operations and classical communication. This proves the desired inequality (7). To prove the general expression in Eq. (6), we now suppose that the quantum channel is used n times, where n can be even or odd. We can define the states $\rho_1, \ldots, \rho_n$ in an analogous way as above. Using the same argumentation we arrive at Eq. (6).

The result in Eq. (6) can now be used to find the most "economic" way to distribute entanglement. If Alice and

Bob are told to send a fixed amount of entanglement $E = E_{\mathrm{final}} - E_{\mathrm{initial}}$, they can achieve this in the most economic way by choosing a protocol such that the inequality (6) becomes an equality. One possibility to achieve this is the well-known "trivial" one: Alice locally prepares a pure state $|\psi\rangle^{AC}$ with entanglement $E = E^{A|C}$, and sends the particle C to Bob. However, this is not the only possibility: the inequality (6) can also be satisfied without sending entanglement, see the example below. If one considers entanglement to be an expensive resource, one may thus be able to distribute entanglement in a "cheaper" way by sending quantum correlations without entanglement.

The results presented in this Letter provide new powerful tools to understand and quantify entanglement as well as quantum correlations. In this paragraph we 1will demonstrate how Eq. (3) can be used to evaluate the entanglement and the one-way information deficit in the specific state η, which was used in [1] to show that entanglement distribution with separable states is possible:

$$\eta = \frac{1}{3}|\Psi_{\mathrm{GHZ}}\rangle\langle\Psi_{\mathrm{GHZ}}| + \sum_{i,j,k=0}^{1} \beta_{ijk}\Pi_{ijk} \tag{8}$$

with $|\Psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, $\Pi_{ijk} = |ijk\rangle\langle ijk|$, and all β's are zero apart from $\beta_{001} = \beta_{010} = \beta_{101} = \beta_{110} = \frac{1}{6}$. It was shown in [1] that the entanglement is zero between two different cuts: $E^{AC|B} = E^{AB|C} = 0$. As an application of Eq. (3) we will now prove that the remaining two quantities are equal: $E^{A|BC}(\eta) = \Delta^{C|AB}(\eta) = \frac{1}{3}$. This can be seen by considering the relative entropy between η and the state $\eta' = \sum_i \Pi_i^C \eta \Pi_i^C$ with orthogonal projectors $\Pi_i^C = |i\rangle\langle i|^C$ in the computational basis. It can be verified by inspection that $S(\eta \parallel \eta') = \frac{1}{3}$, and thus $\Delta^{C|AB}(\eta)$ is not larger than $\frac{1}{3}$. On the other hand, the entanglement $E^{A|BC}(\eta)$ is bounded from below by $\frac{1}{3}$. This follows from the two facts that the state η can be used to distil Bell states with probability $\frac{1}{3}$ [1], and that the relative entropy of entanglement is not smaller than the distillable entanglement [7]. In this example, quantum correlations provide the most economic and cheapest resource for entanglement distribution.

In conclusion, we have identified quantum correlations as the key resource for entanglement distribution. They quantify the quantum cost that one has to invest for increasing the entanglement between two distant parties. Explicitly, we proved that the entanglement between two parties cannot grow more than the amount of quantum correlations which the particle carries that mediates the interaction between the two parties. Our result is completely general and is valid regardless of the particular realization of the protocol. Thus it provides a fundamental connection between quantum entanglement on one side and quantum correlations on the other side. Since the study of quantum correlations is believed to be important for understanding the power of quantum computers, our results may find applications far beyond the scope of this work.

We thank Otfried Gühne for helpful discussions. A. S. thanks Kavan Modi, Hugo Cable and Vlatko Vedral for organizing the Quantum Discord Workshop in Singapore, where he had enlightening discussions with Gerardo Adesso, Wojciech H. Zurek, and Animesh Datta. We acknowledge financial support from Deutsche Forschungsgemeinschaft (DFG) and ELES.

Note added.— During the completion of this work we became aware of independent related work by T. K. Chuan *et al.* in [33]. There, the authors derive similar results, and also provide alternative examples for entanglement distribution with separable states.

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[1] T. S. Cubitt, F. Verstraete, W. Dür, and J. I. Cirac, Phys. Rev. Lett. **91**, 037902 (2003).
[2] I. Devetak and J. Yard, Phys. Rev. Lett. **100**, 230501 (2008).
[3] J. Yard and I. Devetak, IEEE Trans. Inf. Theory **55**, 5339 (2009).
[4] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[6] V. Vedral, Rev. Mod. Phys. **74**, 197 (2002).
[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **84**, 2014 (2000).
[8] M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **100**, 090502 (2008).
[9] W. H. Zurek, Ann. Phys. (Leipzig) **9**, 855 (2000).
[10] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[11] L. Henderson and V. Vedral, J. Phys. A: Math. Gen. **34**, 6899 (2001).
[12] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).
[13] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010).
[14] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. Acín, Phys. Rev. A **81**, 052318 (2010).
[15] B. Dakić, V. Vedral, and Č. Brukner, Phys. Rev. Lett. **105**, 190502 (2010).
[16] W. H. Zurek, Phys. Rev. A **67**, 012320 (2003).
[17] V. Madhok and A. Datta, Phys. Rev. A **83**, 032323 (2011).
[18] D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter, Phys. Rev. A **83**, 032324 (2011).
[19] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **106**, 160401 (2011).
[20] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, Phys. Rev. Lett. **106**, 220403 (2011).

[21] E. Knill and R. Laflamme, Phys. Rev. Lett. **81**, 5672 (1998).
[22] D. P. DiVincenzo, M. Horodecki, D. W. Leung, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **92**, 067902 (2004).
[23] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **100**, 050502 (2008).
[24] B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, Phys. Rev. Lett. **101**, 200501 (2008).
[25] A. Datta, Ph.D. thesis, University of New Mexico, 2008, arXiv:0807.4490.
[26] Z. Merali, Nature **474**, 24 (2011).
[27] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, arXiv:1112.6238.
[28] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A **71**, 062307 (2005).
[29] G. Lindblad, Commun. Math. Phys. **40**, 147 (1975).
[30] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).
[31] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[32] A. Uhlmann, in *Quantum Groups and Related Topics: Proceedings of the First Max Born Symposium*, edited by R. Gielerak, J. Lukierski, and Z. Popowicz (Kluwer Academic Publishers, Dordrecht, 1992), pp 267.
[33] T. K. Chuan, J. Maillard, K. Modi, T. Paterek, M. Paternostro, and M. Piani, arXiv:1203.1268.