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Non-standard interface conditions in flexure of mixture unified gradient Nanobeams

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ABSTRACT

Structural schemes of applicative interests in Engineering Science frequently encounter the intricate phenomenon of discontinuity. The present study intends to address the discontinuity in the flexure of elastic nanobeam by adopting an abstract variational scheme. The mixture unified gradient theory of elasticity is invoked to realize the size-effects at the ultra-small scale. The consistent form of the interface conditions, stemming from the established stationary variational principle, is meticulously set forth. The boundary-value problem of equilibrium is properly closed and the analytical solution of the transverse displacement field of the elastic nanobeam is addressed. As an alternative approach, the eigenfunction expansion method is also utilized to scrutinize the efficacy of the presented variational formulation in tackling the flexure of elastic nanobeams with discontinuity. The flexural characteristic of mixture unified gradient beams with diverse kinematic constraints is numerically illustrated and thoroughly discussed. The anticipated nanoscopic features of the characteristic length-scale parameters are confirmed. The demonstrated numerical results can advantageously serve as a benchmark for the analysis and design of pioneering ultra-sensitive nano-sensors. The established variationally consistent size-dependent framework paves the way ahead in nanomechanics and inspires further research contributing to fracture mechanics of ultra-small scale elastic beams.

1. Introduction

With the rapid and monumental developments in technology in recent decades, the prevailing trend is to downsize devices to reduce energy consumption, among other objectives. Thanks to the micro- and nano-fabrication techniques, rapid and cost-effective manufacturing of micro- and nano-electromechanical systems (MEMS and NEMS) is now easier than ever (Elishakoff et al., 2012; Mirkin & Rogers, 2001). Despite their diminutive scale, these intelligent systems exhibit multitasking capabilities with exceptional precision. They have found critical applications in numerous scientific, technological, and industrial fields, including healthcare, environmental monitoring, energy, electronics, and optics (Kalinin et al., 2004; Kalinin et al., 2005; Karapetian et al., 2005; Roukes, 2001). For example, a micro-gripper with significant gripping force, constructed from a shape memory alloy, is presented as details are addressed in Lee et al. (1996). This miniaturized tool can manipulate tissues during endoscopic surgical procedures. Another illustration is the ultra-sensitive micro- and nano-mechanical mass sensor utilized for evaluating the hydrogen storage capacity of carbon nanotube bundles (Ono et al., 2003), or detecting the presence of volatile organic compounds in the air (Kilinc et al., 2014). Current

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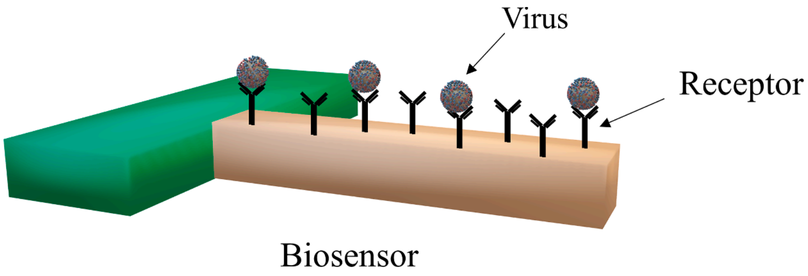

Fig. 1. Configuration of a biosensor with viruses attached at different locations.

trends in small-scale structures involve the design of graded and porous materials (Civalek et al., 2023; Wang et al., 2023), or employing material architectures like sandwich (Milić et al., 2024) and multilayered (Vattré & Pan, 2021) configurations.

Despite their simplicity in geometry and manufacturing, micro- and nanobeams are essential building blocks of MEMS and NEMS. Therefore, understanding the mechanical response of micro- and nanobeams is crucial for the design and advancement of small-scale pioneering devices. Conventional beam theories based on classical continuum mechanics often fail to accurately predict the mechanical response of micro- and nanobeams. The main reason is that such classical beam theories do not account for the size-effect, a phenomenon experimentally proven for micro- and nanobeams, as demonstrated in Lam et al. (2003); Tang and Alici (2011). To address this shortcoming in theories based on classical continuum mechanics, several generalized and non-classical theories have been formulated within the Engineering Science community to model the size-dependent mechanical behavior at small scales. Examples of such pioneering non-classical models include micropolar theory, surface elasticity theory, theories based on fractional calculus, couple stress theory, modified couple stress theory, strain gradient theory, modified strain gradient theory, and Eringen's nonlocal elasticity. These models are thoroughly discussed, and their applications to small-scale structures are reviewed in depth in the seminal articles (Askes & Aifantis, 2011; Thai et al., 2017; Shodja & Moosavian, 2020). The representative instances of the implementation of non-classical models are as follows; the surface elasticity theory is used in Hu et al. (2021), Eremeyev and Sharma (2019), Mikhasev et al. (2023) to study different mechanical problems at small scales. The strain and stress gradient theories are, also, applied for capturing the size-dependent mechanical behavior of miniaturized structures (Barretta et al., 2019; Eremeyev et al., 2021; F. Hache et al., 2019; Yadav & Gupta, 2024). A three-dimensional nonlinear beam model based on the micropolar continuum theory is developed in Dadgar-Rad et al. (2024). The dynamics and instability of small-scale beams and plates are studied in F. Hache et al. (2019) adopting the nonlocal theory of elasticity. The rotary inertia and more general microinertia considering one-dimensional models of microstructured bars and beams are discussed in Eremeyev and Elishakoff (2024). Structural theories based on these non-classical continuum mechanics models are generally preferable to atomistic models such as molecular and lattice dynamics (Kushch, 2023; Sharma & Eremeyev, 2019), due to their simplicity in implementation and calculations.

The above-mentioned non-classical continuum mechanics-based theories have their advantages and limitations. For instance, theories based on the strain gradient theory of elasticity can only capture the stiffening behavior of structures as the size of nanostructures decreases. On the other hand, structural theories based on Eringen's nonlocal integral elasticity theory, solely, predict a softening behavior for structures with small-scale dimensions. Therefore, integrating the ideas of these non-classical continuum mechanics-based theories to broaden their applicability range has been a hot topic of recent research in this field. The nonlocal strain gradient theory (Lim et al., 2015), which unifies the nonlocal elasticity and the strain gradient theory, is one outcome of this line of inquiry. Alternative instances are as follows; the nonlocal modified gradient theory (Faghidian, 2021), the higher-order nonlocal gradient theory (S.A. Faghidian et al., 2022), the nonlocal surface elasticity (Li et al., 2020), and the mixture stress gradient theory (S.A. Faghidian et al., 2022). More recently, the mixture unified gradient theory of elasticity has been introduced in the most general intrinsic form as a stationary variational principle (S.A. Faghidian et al., 2023). This theory integrates the stress gradient model, the strain gradient model, and the classical theory of elasticity. Consequently, the mixture unified gradient theory can efficaciously capture both stiffening and softening behaviors in small-scale structures. It has been effectively applied to realize the size-effect in a variety of structural problems in nanomechanics including free vibrations of nanobeams (Faghidian & Tounsi, 2022), mechanics of nanobars (S.A. Faghidian et al., 2022), wave propagations in nanobeams (Faghidian & Elishakoff, 2022), linear and nonlinear flexure mechanics of nanobeams (S.A. Faghidian et al., 2023; S.A. Faghidian et al., 2023), and random vibrations of nanobeams (S.A. Faghidian & Elishakoff, 2023).

The mixture unified gradient theory of elasticity, in its original intrinsic form, applies to problems where the field variables are continuous functions defined on smooth orientable surfaces in the absence of topographical defects (S.A. Faghidian et al., 2023). Structural problems of empirical interest frequently incorporate discontinuity within the domain. Such discontinuity in the field variables arises from various factors such as concentrated loadings, kinematic constraints, cracks, defects, or attached particles. The latter is particularly relevant in cases of technological importance, such as ultra-sensitive micro- or nano-mechanical mass sensors. In these instances, the mass sensor typically takes the form of a micro- or nano-cantilever, as schematically depicted in Fig. 1. In practical applications, biological entities may land on the sensor at various locations, resulting in the discontinuity of the field variables. To address problems involving discontinuous field variables, the domain should be decomposed into different segments, and suitable continuity conditions should be applied at the interfaces between these segments to solve the corresponding boundary-vale problem. Such continuity conditions associated with the stress-driven nonlocal theory of elasticity have been recently derived and utilized to

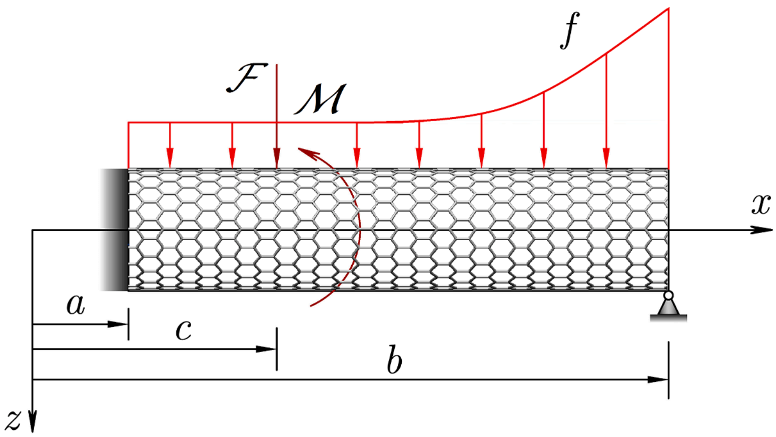

Fig. 2. Coordinate system and configuration of a nano-sized beam.

solve various structural problems (Darban, 2023; Darban et al., 2022; Darban et al., 2023; Darban et al., 2020). Nevertheless, interface (continuity) conditions associated with the mixture unified gradient theory are currently absent from the state of the art in literature and will be rigorously examined in the present research to broaden the range of applicability of the mixture unified gradient theory.

In the present study, a set of non-standard interface conditions associated with the mixture unified gradient theory is determined by adopting an abstract variational scheme. As a result, the flexure problem in micro- and nanobeams subject to concentrated loadings and couples at the interior parts of the domain can be advantageously formulated within the context of the mixture unified gradient theory of elasticity. This paper addressed the fundamental challenge in the literature to extend the applicability of the mixture unified gradient theory to the study of small-scale structures with discontinuous field variables. Consequently, it is anticipated that the present work paves the way ahead in nanomechanics and inspires further research contributing to advancements in studies related to fracture mechanics of micro- and nanobeams, ultra-sensitive mechanical mass sensors, and structural responses of small-scale cracked beams.

The article is structured as follows. The consistent variational formulation of the flexure mechanics of nanobeams with discontinuity is presented in Sect. 2. The associated boundary-value problem of equilibrium is properly equipped with the interface conditions stemming from the introduced abstract variational scheme. Sect. 3 is devoted to presenting the practical application of interface conditions in flexural analysis of the deflection of nanobeams with boundary conditions of applicative interest, i.e. elastic nanobeam with clamped-free, simple supports, fixed-ends, and fixed-simple ends subject to concentrated loads. To provide a benchmark for the verification of the efficiency of the established variationally consistent formulation, a series solution founded on the eigenfunction expansion approach is, also, generated in this section, which does not require the imposition of the derived non-standard interface conditions. The introduced abstract variational formulation consistent with the mixture unified gradient theory is, subsequently, employed to explore the size-effect in the flexure of nanobeams subject to concentrated loadings. Sect. 3 is, further, enriched with a demonstration and discussion of the numerical results associated with the nanoscopic effects of the characteristic length-scale parameters on the flexural response of elastic nanobeams. Effects of the line of action of the concentrated force on the flexure of the mixture unified gradient elastic beam are also discussed. Concluding remarks are drawn in Sect. 4.

2. Variational principle for nanobeam with discontinuity

The interface conditions in the flexure mechanics of nanobeams are meticulously investigated within the framework of the mixture unified gradient theory of elasticity. To this end, let us consider a homogeneous prismatic elastic beam of length $(b-a)$ with a symmetric cross-section $\mathbb{A}$ as demonstrated in Fig. 2. In its undeformed state, the beam is referred to orthogonal co-ordinates (x,y,z), with an x-axis coinciding with the beam centroid axis and the z-axis coinciding with the symmetry axis of $\mathbb{A}$. The beam is constrained at the ends $x = a$ and $x = b$, in such a way as to impede any rigid motion of the beam. The beam is subjected to a transverse load per unit length f and to some concentrated forces and couples, to be specified later on. In the elastostatic analysis, inertia forces are considered to be insignificant. The elastic material is characterized by the elastic modulus E. The shear deformation is overlooked within the framework of the Euler-Bernoulli beam model (S.A. Faghidian & Elishakoff, 2023); this is achieved by taking the displacement field of the beam as

$$u_1 = -z\partial_x w(x), u_2 = 0, u_3 = w(x) \tag{1}$$

with w denoting the transverse displacement of the centroid points of the beam. The ensuing axial strain field component ε, described in terms of the beam flexural curvature $\chi = \partial_{xx} w$, reads as

$$\varepsilon(x) = -z\chi(x) = -z\partial_{xx} w(x) \tag{2}$$

Stimulated by the intrinsic form of the stationary variational principle associated with the mixture unified gradient theory (S.A. Faghidian et al., 2023; S.A. Faghidian et al., 2023), the variational functional $\mathbb{F}$, for any virtual kinetic test field having compact support on the beam ends, is introduced as

$$\begin{aligned}\mathbb{F} = \int_a^b \Bigg(& M_0(x)\chi(x) + \frac{1}{2I_E}(M_0(x))^2 + \frac{\ell_c^2}{2I_E}(\partial_x M_0(x))^2 + M_1(x)\partial_x\chi(x) \\ & + \frac{1}{2I_E(\alpha\ell_c^2+\ell_s^2)}(M_1(x))^2 + \frac{\ell_c^2}{2I_E(\alpha\ell_c^2+\ell_s^2)}(\partial_x M_1(x))^2 + f(x)w(x) \Bigg) dx\end{aligned} \tag{3}$$

where the resultant moments M_0 and M_1 are, respectively, defined as the dual fields of the flexural curvature χ and its first-order derivative along the centroid axis $\partial_x\chi$. The stress gradient characteristic length ℓ_c and the strain gradient length-scale parameter ℓ_s are introduced to address the significance of the corresponding gradient theory of elasticity. The effect of the classical elasticity theory is integrated into the mixture unified gradient theory via the mixture parameter $0 \le \alpha \le 1$. The flexural stiffness I_E is defined by the second moment of elastic area weighted with the scalar field of elastic modulus.

It is well-known within the context of the stationary variational principles that all the kinematic and kinetic field variables can be treated independently of each other. Performing the first variation of the functional $\mathbb{F}$, followed by the integration by part for the kinetic field variables, leads to

$$\begin{aligned}\delta\mathbb{F} = \int_a^b & (M_0(x)\delta\chi(x) + M_1(x)\partial_x\delta\chi(x) + f(x)\delta w(x) \\ & + \delta M_0(x)\left(\chi(x) + \frac{1}{I_E}M_0(x) - \frac{\ell_c^2}{I_E}\partial_{xx}M_0(x)\right) \\ & + \delta M_1(x)\left(\partial_x\chi(x) + \frac{1}{I_E(\alpha\ell_c^2+\ell_s^2)}M_1(x) - \frac{\ell_c^2}{I_E(\alpha\ell_c^2+\ell_s^2)}\partial_{xx}M_1(x)\right)\Bigg) dx \\ & + \frac{\ell_c^2}{I_E}(\partial_x M_0 \delta M_0)\Big|_a^b + \frac{\ell_c^2}{I_E(\alpha\ell_c^2+\ell_s^2)}(\partial_x M_1 \delta M_1)\Big|_a^b\end{aligned} \tag{4}$$

The boundary congruence conditions can be released in view of the *heuristic* assumption on the virtual kinetic test field variables to have compact support on the domain boundary.

A scrutiny of the literature reveals that the field variables in problems of practical interest in nanomechanics are usually discontinuous. The discontinuity typically arises from factors such as concentrated loads or couples within the interior parts of the domain. Accordingly, the flexure mechanics of nanobeams within the framework of the mixture unified gradient theory is examined here for piecewise regular field variables. Let us assume that the elastostatic field variables are discontinuous at some interior points of the nanobeam; for the sake of simplicity, let us assume there exists an abscissa $a < c < b$ where a certain kind of discontinuity is addressed. Let us, furthermore, assume that the elastic nanobeam is subject to a concentrated force $\mathscr{F}$ and a concentrated couple $\mathscr{M}$ at the abscissa $x = c$, see Fig. 2. The first variation of the functional $\mathbb{F}$ is, subsequently, written as

$$\begin{aligned}\delta\mathbb{F} = & \int_a^{c^-} (M_0(x)\delta\chi(x) + M_1(x)\partial_x\delta\chi(x) + f(x)\delta w(x))dx \\ & + \int_{c^+}^b (M_0(x)\delta\chi(x) + M_1(x)\partial_x\delta\chi(x) + f(x)\delta w(x))dx \\ & + \int_a^b \Bigg(\delta M_0(x)\left(\chi(x) + \frac{1}{I_E}M_0(x) - \frac{\ell_c^2}{I_E}\partial_{xx}M_0(x)\right) \\ & + \delta M_1(x)\left(\partial_x\chi(x) + \frac{1}{I_E(\alpha\ell_c^2+\ell_s^2)}M_1(x) - \frac{\ell_c^2}{I_E(\alpha\ell_c^2+\ell_s^2)}\partial_{xx}M_1(x)\right)\Bigg) dx \\ & + \mathscr{F}\delta w|_{x=c} - \mathscr{M}\partial_x\delta w|_{x=c}\end{aligned} \tag{5}$$

While the field variables are assumed to be smooth functions in the intervals $[a, c[$ and $]c,b]$, they can be discontinuous at the abscissa $x = c$. Performing the integration by parts for kinematic field variables while adopting the kinematic compatibility condition, one can write

$$
\begin{aligned}
\delta\mathbb{F} = &\int_a^{c^-} ((\partial_{xx}M_0(x) - \partial_{xxx}M_1(x) + f(x)))\delta w(x)dx \\
&+ \int_{c^+}^{b} ((\partial_{xx}M_0(x) - \partial_{xxx}M_1(x) + f(x)))\delta w(x)dx \\
&+ \int_a^b \left(\delta M_0(x)\left(\chi(x) + \frac{1}{I_E}M_0(x) - \frac{\ell_c^2}{I_E}\partial_{xx}M_0(x)\right)\right. \\
&\left. +\delta M_1(x)\left(\partial_x\chi(x) + \frac{1}{I_E(\alpha\ell_c^2 + \ell_s^2)}M_1(x) - \frac{\ell_c^2}{I_E(\alpha\ell_c^2 + \ell_s^2)}\partial_{xx}M_1(x)\right)\right)dx \\
&-(\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_a^{c^-} + (M_0(x) - \partial_x M_1(x))\partial_x\delta w|_a^{c^-} + M_1(x)\partial_{xx}\delta w|_a^{c^-} \\
&-(\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_{c^+}^{b} + (M_0(x) - \partial_x M_1(x))\partial_x\delta w|_{c^+}^{b} + M_1(x)\partial_{xx}\delta w|_{c^+}^{b} \\
&+\mathscr{F}\delta w|_{x=c} - \mathscr{M}\partial_x\delta w|_{x=c}
\end{aligned}
\tag{6}
$$

Prescribing the stationarity of the variational functional $\delta\mathbb{F} = 0$, while assuming arbitrary variations of the flexural curvature at the beam ends, leads to the differential and boundary conditions of equilibrium for the mixture unified gradient elastic beam

$$
\begin{cases}
\partial_{xx}M_0(x) - \partial_{xxx}M_1(x) + f(x) = 0 \\
(\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_{x=a} = (\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_{x=b} = 0 \\
(M_0(x) - \partial_x M_1(x))\partial_x\delta w|_{x=a} = (M_0(x) - \partial_x M_1(x))\partial_x\delta w|_{x=b} = 0 \\
M_1(x)|_{x=a} = M_1(x)|_{x=b} = 0
\end{cases}
\tag{7}
$$

equipped with the interface conditions at the abscissa $x = c$ expressed as

$$
\begin{cases}
(\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_{x=c^-} - (\partial_x M_0(x) - \partial_{xx}M_1(x))\delta w|_{x=c^+} = \mathscr{F}\delta w|_{x=c} \\
(M_0(x) - \partial_x M_1(x))\partial_x\delta w|_{x=c^-} - (M_0(x) - \partial_x M_1(x))\partial_x\delta w|_{x=c^+} = \mathscr{M}\partial_x\delta w|_{x=c} \\
M_1(x)\partial_{xx}\delta w|_{x=c^-} = M_1(x)\partial_{xx}\delta w|_{x=c^+}
\end{cases}
\tag{8}
$$

Additionally in view of the stationarity of the established variational principle, the desired constitutive laws of the resultant moments M_0 and M_1 are cast as differential relations

$$
\begin{cases}
M_0(x) - \ell_c^2\partial_{xx}M_0(x) = -I_E\chi(x) \\
M_1(x) - \ell_c^2\partial_{xx}M_1(x) = -I_E(\alpha\ell_c^2 + \ell_s^2)\partial_x\chi(x)
\end{cases}
\tag{9}
$$

To further simplify the boundary-value problem of equilibrium, it is a common choice within the context of the gradient theory of elasticity to introduce the total flexural moment M as

$$
M(x) = M_0(x) - \partial_x M_1(x)
\tag{10}
$$

The differential and boundary conditions of equilibrium for the elastic beam consistent with the mixture unified gradient theory, thus, simplifies as

$$
\begin{cases}
\partial_{xx}M(x) + f(x) = 0 \\
\partial_x M(x)\delta w|_{x=a} = \partial_x M(x)\delta w|_{x=b} = 0 \\
M(x)\partial_x\delta w|_{x=a} = M(x)\partial_x\delta w|_{x=b} = 0 \\
M_1(x)|_{x=a} = M_1(x)|_{x=b} = 0
\end{cases}
\tag{11}
$$

endowed with the interface conditions at the abscissa $x = c$ as

$$
\begin{cases}
\partial_x M(x)\delta w|_{x=c^-} - \partial_x M(x)\delta w|_{x=c^+} = \mathscr{F}\delta w|_{x=c} \\
M(x)\partial_x\delta w|_{x=c^-} - M(x)\partial_x\delta w|_{x=c^+} = \mathscr{M}\partial_x\delta w|_{x=c} \\
M_1(x)\partial_{xx}\delta w|_{x=c^-} = M_1(x)\partial_{xx}\delta w|_{x=c^+}
\end{cases}
\tag{12}
$$

Lastly, the sought constitutive law of the total flexural moment M, in view of Eq. (9) and Eq. (10), can be cast in the form

$$
M(x) - \ell_c^2\partial_{xx}M(x) = -I_E(\chi(x) - (\alpha\ell_c^2 + \ell_s^2)\partial_{xx}\chi(x))
\tag{13}
$$

The constitutive model of the resultant fields associated with the mixture unified gradient beam is of higher-order in comparison with the classical beam model. The structural problem consistent with the mixture unified gradient theory is, accordingly, closed by prescribing four classical (kinematic or natural) boundary conditions along with two extra non-standard boundary conditions. The classical boundary conditions are expressed in terms of the kinematic field variables, i.e. transverse displacement w and rotation field of cross-section $\partial_x w$, in addition to kinetic field variables, i.e. total flexural moment M and shear resultant $\partial_x M$. The extra non-standard boundary conditions are, merely, described in terms of the resultant moment M_1 as the resultant moment M_1 needs to vanish at both beam ends.

The interface conditions of the elastic nanobeam comprise four well-known classical conditions, described in terms of kinematic and kinetic field variables, along with two extra non-standard conditions, expressed in terms of the resultant moment M_1 and the flexural curvature χ. Remarkably, the interface conditions are detected in the presented abstract scheme based on a consistent variational formulation without the necessity of imposing fictitious assumptions on the continuity of higher-order kinematic field variables (Pinnola et al., 2022).

Furthermore, accurate determination of the explicit mathematical formula of the resultant moment M_1 is of utmost importance in the appropriate prediction of the structural characteristics of the elastic nanobeam. An erroneous prescription of the non-standard boundary conditions will lead to misleading implications of the size-dependent response of nano-structures (Pinnola et al., 2022). Following some straightforward mathematics, the explicit mathematical expression of the resultant moment M_1 within the framework of the mixture unified gradient theory can be determined as (S.A. Faghidian et al., 2023)

$$M_1(x) = -\frac{(\alpha\ell_c^2+\ell_s^2)\ell_c^4}{\ell_c^2(1-\alpha)-\ell_s^2}\partial_x f(x) - I_E(\alpha\ell_c^2+\ell_s^2)\partial_x\chi(x) + I_E\frac{(\alpha\ell_c^2+\ell_s^2)^2\ell_c^2}{\ell_c^2(1-\alpha)-\ell_s^2}\partial_{xxx}\chi(x) \tag{14}$$

- Remark I

In the absence of the discontinuous field variables, the constitutive law and non-standard boundary conditions associated with the mixture unified gradient beam model coincide with the micromorphic beam theory developed by Challamel and Wang (Challamel & Wang, 2008; Zhang et al., 2010). The constitutive model of the moment-curvature differential law within either frameworks of the mixture unified gradient theory (S.A. Faghidian et al., 2023) or the micromorphic beam theory (Challamel & Wang, 2008) consistent with the kinematics of the Euler-Bernoulli beam is given by

$$M(x) - \ell_c^2\partial_{xx}M(x) = -I_E\left(\chi(x) - \ell_a^2\partial_{xx}\chi(x)\right) \tag{15}$$

with $\ell_a = \sqrt{\alpha\ell_c^2+\ell_s^2}$ designating the intrinsic characteristic parameter introduced within the context of the micromorphic beam theory by Challamel and Wang (Challamel & Wang, 2008). Moreover, the extra non-standard boundary conditions of the mixture unified gradient beam, i.e. vanishing the resultant moment M_1 at both beam ends $M_1(a) = M_1(b) = 0$, can be also *equivalently* expressed as (Challamel & Wang, 2008; Zhang et al., 2010)

$$\left(l_c^2\partial_x M(x) - EI\ell_a^2\partial_x\chi(x)\right)\big|_{x=a} = \left(l_c^2\partial_x M(x) - EI\ell_a^2\partial_x\chi(x)\right)\big|_{x=b} = 0 \tag{16}$$

- Remark II

Stimulated by the micromorphic beam theory developed by Challamel and Wang (Challamel & Wang, 2008; Zhang et al., 2010), the interface conditions of the elastic nanobeam within the framework of the mixture unified gradient theory can be further simplified. Let us consider a mixture unified gradient elastic beam which is, merely, subject to a concentrated force $\mathscr{F}$ at the abscissa $x = c$. In view of Remark I, the interface conditions can be presented as

$$\begin{gathered}
\partial_x M(x)\delta w|_{x=c^-} - \partial_x M(x)\delta w|_{x=c^+} = \mathscr{F}\delta w|_{x=c} \\
M(x)\partial_x\delta w|_{x=c^-} - M(x)\partial_x\delta w|_{x=c^+} = 0 \\
\partial_{xx}w(x)|_{x=c^-} = \partial_{xx}w(x)|_{x=c^+} \\
M_1(x)|_{x=c^-} - M_1(x)|_{x=c^+} = \left(l_c^2\partial_x M(x) - EI\ell_a^2\partial_x\chi(x)\right)\big|_{x=c^-} - \left(l_c^2\partial_x M(x) - EI\ell_a^2\partial_x\chi(x)\right)\big|_{x=c^-} \\
= l_c^2\mathscr{F} - EI\ell_a^2\left(\partial_x\chi(x)|_{x=c^-} - \partial_x\chi(x)|_{x=c^+}\right) = 0
\end{gathered} \tag{17}$$

An abstract variational scheme is established to rigorously examine the discontinuity in the flexure mechanics of nanobeams. The appropriate form of the interface conditions is consistently detected based on the introduced stationary variational principle. The efficacy of the proposed approach is evinced in Section 3 by studying the flexure of the mixture unified gradient elastic beam subject to discontinuous load conditions of practical interest in nanomechanics.

3. Flexure of nanobeams with discontinuity

3.1. Analytical solution

A variety of mathematical strategies exist in the literature to deal with structural problems. While the governing equations are mainly described in terms of kinematic field variables, implementation of the series form solution of the kinetic field variables (S. Ali Faghidian, 2017; S. Ali Faghidian, 2017) and autonomous series solution of kinematic and kinetic field variables (S.A. Faghidian et al., 2022; S.A. Faghidian et al., 2023) are, also, invoked in nanomechanics. As a different school of thought, the governing differential equations of lower-order should be successively integrated in lieu of directly solving the governing equations described in terms of the kinematic field variable. This efficient approach is first adopted in the interval $[a, c[$ to detect the transverse displacement of the beam. Integrating the differential condition of equilibrium Eq. $(11)_1$ results in the equilibrated flexural moment M in terms of integration constants $\mathbb{A}_1, \mathbb{A}_2$

$$M(x) = -\int_a^x \int_a^\varsigma f(\xi) d\xi d\varsigma + \mathbb{A}_1 x + \mathbb{A}_2 \tag{18}$$

The flexural curvature χ is subsequently determined via integration of the constitutive differential equation Eq. (13) up to integration constants $\mathbb{A}_3, \mathbb{A}_4$

$$\begin{aligned}\chi(x) &= \mathbb{A}_3 \exp\left(\frac{x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right) + \mathbb{A}_4 \exp\left(-\frac{x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right) \\ &+ \frac{1}{2I_E\sqrt{\alpha\ell_c^2 + \ell_s^2}} \int_a^x \exp\left(\frac{x-\xi}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right)\left(M(\xi) - \ell_c^2 \partial_{\xi\xi} M(\xi)\right) d\xi \\ &- \frac{1}{2I_E\sqrt{\alpha\ell_c^2 + \ell_s^2}} \int_a^x \exp\left(\frac{\xi - x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right)\left(M(\xi) - \ell_c^2 \partial_{\xi\xi} M(\xi)\right) d\xi\end{aligned} \tag{19}$$

Integrating the differential condition of kinematic compatibility, lastly, provides the transverse displacement of the nanobeam up to integration constants $\mathbb{A}_5, \mathbb{A}_6$

$$w(x) = \int_a^x \int_a^\varsigma \chi(\xi) d\xi d\varsigma + \mathbb{A}_5 x + \mathbb{A}_6 \tag{20}$$

As a result, the transverse displacement field of the elastic nanobeam in the interval $[a, c[$ is detected in terms of six unknown integration constants $\{\mathbb{A}_1, \ldots, \mathbb{A}_6\}$. Following the analogous mathematical approach, the transverse displacement of the elastic nanobeam can be, also, determined for the interval $]c, b]$ in terms of integration constants $\{\mathbb{B}_1, \ldots, \mathbb{B}_6\}$. To detect twelve unknown integration constants $\mathbb{A}_j (j = 1..6)$ and $\mathbb{B}_j (j = 1..6)$, the transverse displacements of the elastic nanobeam should, correspondingly, fulfill six boundary conditions, i.e. four classical and two extra non-standard boundary conditions Eq. $(11)_{2-4}$, along with six interface conditions, i.e. four classical and two extra non-standard conditions Eq. (12). The exact analytical solution of the transverse displacement of the mixture unified gradient elastic beam subject to a loading discontinuity is, therefore, derived. Evidence of the well-posedness of the established abstract variational framework in capturing the nanoscopic characteristics of elastic nanobeams subject to discontinuous loads is elucidated by rigorous examination of the elastostatic flexural response of structural schemes of applicative interest in nanomechanics.

3.2. Eigenfunction expansions approach

A different line of thought, departing from the conceived analytical solution approach, is achieved by substituting the discontinuous loading with its series solution form. To this end, the flexural moment M is represented by the subsequent eigenfunction expansion

$$M(x) = \sum_{m=1}^{N} \mathbb{C}_m \mathbb{M}_m(x) \tag{21}$$

with $\mathbb{M}_m$denoting the corresponding coordinate function that is determined contingent on the natural boundary conditions of the elastic nanobeam where the coefficients $\mathbb{C}_m$ are known constants and N represents the degree of development of the series.

Accordingly, solving the constitutive differential equation Eq. (13) for the coordinate function $\mathbb{M}_m$ provides the mth term of the flexural curvature in terms of integration constants $\mathbb{k}_1^m, \mathbb{k}_2^m$

$$\begin{aligned}\chi_m(x) &= \mathbb{k}_1^m \exp\left(\frac{x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right) + \mathbb{k}_2^m \exp\left(-\frac{x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right) \\ &+ \frac{1}{2I_E\sqrt{\alpha\ell_c^2 + \ell_s^2}} \int_a^x \exp\left(\frac{x-\xi}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right)\left(\mathbb{M}_m(\xi) - \ell_c^2 \partial_{\xi\xi} \mathbb{M}_m(\xi)\right) d\xi \\ &- \frac{1}{2I_E\sqrt{\alpha\ell_c^2 + \ell_s^2}} \int_a^x \exp\left(\frac{\xi - x}{\sqrt{\alpha\ell_c^2 + \ell_s^2}}\right)\left(\mathbb{M}_m(\xi) - \ell_c^2 \partial_{\xi\xi} \mathbb{M}_m(\xi)\right) d\xi\end{aligned} \tag{22}$$

The mth term of the transverse displacement of the mixture unified gradient elastic beam w_m can be, subsequently, determined by integration of the differential condition of the kinematic compatibility up to integration constants $\mathbb{k}_3^m, \mathbb{k}_4^m$

$$w_m(x) = \int_a^x \int_a^\varsigma \chi_m(\xi) d\xi d\varsigma + \mathbb{k}_3^m x + \mathbb{k}_4^m \tag{23}$$

As the natural boundary conditions of the elastic nanobeam are taken into consideration in the eigenfunction expansion of the

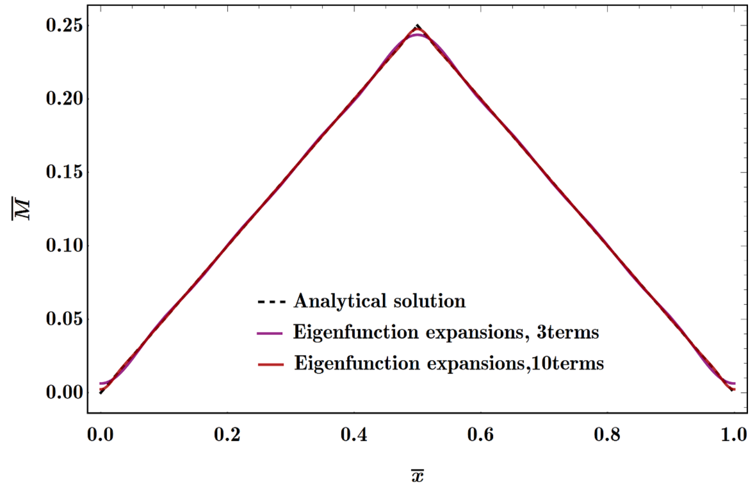

Fig. 3. Flexural moment of a simply supported nanobeam subject to a transverse force at the midspan, analytical solution vs. eigenfunction expansion.

flexural moment, the unknown integration constants $\{\mathbb{k}_1^m, \mathbb{k}_2^m, \mathbb{k}_3^m, \mathbb{k}_4^m\}$ are determined by prescribing the kinematic boundary conditions along with extra non-standard boundary conditions of the mixture unified gradient elastic beam. The linear superposition principle, lastly, gives

$$\mathbb{w}(x) = \sum_{m=1}^{N} \mathbb{C}_m w_m(x) \tag{24}$$

The series solution form of the transverse displacement of the elastic nanobeam is derived by applying the eigenfunction expansion of the flexural moment. Notably, since the coordinate function $\mathbb{M}_m$is assumed to be smooth, the interface conditions of the elastic nanobeam do not necessitate to be imposed. Nevertheless, the existence of the *Gibbs*-type phenomenon is foreseeable as a function with discontinuous gradients is approximated by a set of smooth coordinate functions.

Last but not least, the application of the eigenfunction expansions approach is mainly restricted to the statically determinate structures where the series expansion form of the flexural moment M is assessable. Like other approximate methods founded on series solutions, the convergence analysis should be performed as well. Therefore, the eigenfunction expansion approach can be practically utilized as a proper counterpart for the comparison sake.

3.3. Numerical results and discussions

To rigorously analyze the efficacy of the introduced abstract variational framework, the flexure of elastic nanobeams subject to a concentrated force $\mathscr{F}$ along the beam at abscissa $x = c < (b - a)$ with diverse boundary conditions is examined. Different kinematic constraints of practical interest in nanomechanics, i.e. nanobeams with clamped-free, simple supports, fixed-ends, and fixed-simple ends, are considered. Nanoscopic effects of the characteristic length-scale parameters on the flexural response of mixture unified gradient elastic beams are thoroughly studied. Furthermore, effects of the line of action of the concentrated force on the flexural response of the elastic nanobeam are also discussed. For the sake of consistency, non-dimensional parameters such as the axial abscissa $\overline{x}$, the stress gradient characteristic parameter ζ, the strain gradient characteristic parameter η, the flexural moment $\overline{M}$, and the transverse displacement function $\overline{w}$ are introduced as

$$\overline{x} = \frac{x-a}{b-a}, \zeta = \frac{\ell_c}{b-a}, \eta = \frac{\ell_s}{b-a}, \overline{M} = \frac{1}{\mathscr{F}(b-a)} M, \overline{w} = \frac{I_E}{\mathscr{F}(b-a)^3} w \tag{25}$$

Let us first consider a mixture unified gradient elastic beam with simple support ends which is subject to a concentrated transverse force at the midspan, i.e. abscissa $\overline{x} = \overline{c} = 1/2$. Applying the introduced analytical approach as discussed in Section 3.1, the exact analytical solution of the transverse displacement of the elastic nanobeam w is addressed by means of integrating differential equations of lower-order.

Since a simply supported nanobeam represents a statically determinate structure, the eigenfunction expansion of the flexural moment M can be represented by the subsequent Fourier cosine series form

$$M(x) = -\frac{\mathscr{F}}{2}\left(\frac{(b-a)}{4} - 2\frac{(b-a)}{\pi^2}\sum_{m=1}^{N}\frac{1}{(2m-1)^2}\cos\left((2m-1)\frac{2\pi x}{(b-a)}\right)\right) \tag{26}$$

Utilizing the eigenfunction expansions approach, the series solution form of the transverse displacement of the elastic nanobeam is

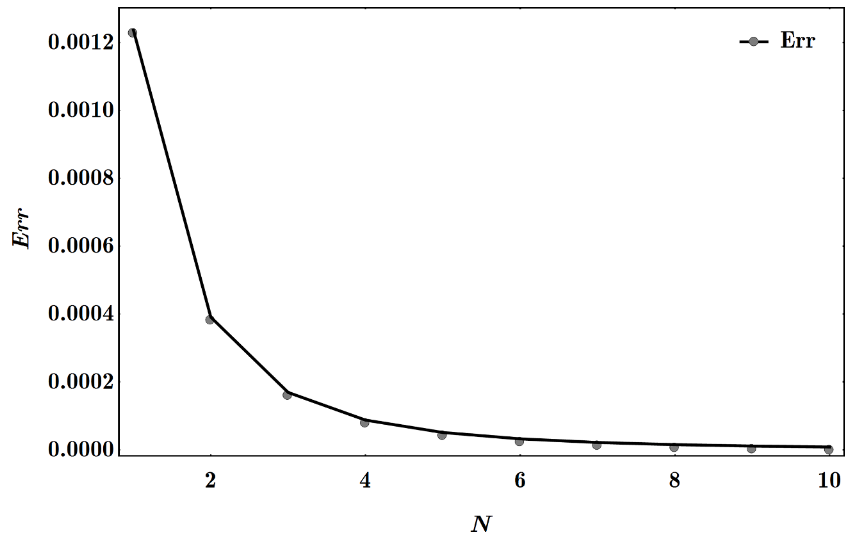

Fig. 4. Convergence of the eigenfunction expansion approach for a simply supported nanobeam subject to a transverse force at the midspan.

derived. Prior to studying the nanoscopic effects of the characteristic length-scale parameters, the comparison of the flexural response of the mixture unified elastic beam based on the established analytical approach and the approximated series solution method is of interest. The associated relative error is, accordingly, defined as

$$Err = \left| \frac{\overline{)\mathbb{w}}(\overline{x} = 1/2) - \overline{w}(\overline{x} = 1/2)}{\overline{w}(\overline{x} = 1/2)} \right| \tag{27}$$

The exact flexural moment of a simply supported nanobeam, obtained based on the introduced analytical solution, is first compared with the Fourier cosine series form for $N = 3$ and $N = 10$ terms in Fig. 3. The series solution of the flexural moment with $N = 10$ terms, predictably, exhibits an excellent agreement with the exact analytical solution result. The convergence analysis is, furthermore, performed by studying the relative error *Err* in the determination of the transverse deformation of the mixture unified gradient elastic beam versus the degree of development of the series form associated with $\overline{)\mathbb{w}}$. For prescribed values of the gradient characteristic parameters as $\zeta = 1/3, \eta = 1/5$ and vanishing mixture parameter, the corresponding variation of the relative error *Err* in terms of N is illustrated in Fig. 4. The numerical results demonstrate that the application of merely $N = 10$ terms in the series solution form of $\overline{)\mathbb{w}}$leads to an excellent agreement with the analytical solution of the transverse displacement of the elastic nanobeam w. The comparison made in Figs. 3 and 4, notably, demonstrates that the established abstract variational formulation can consistently realize the discontinuity in the flexure mechanics of mixture unified gradient elastic beams. The introduced boundary-value problem enriched with the variationally consistent interface conditions can be, therefore, efficiently implemented to realize the nanoscopic flexural response of elastic nanobeams with discontinuity.

Before conducting a comprehensive analysis of the size-effects associated with the characteristic length-scale parameters, let us consider a mixture unified gradient elastic beam with clamped-free ends subject to a concentrated transverse force applied along the beam at abscissa $\overline{x} = \overline{c} < 1$. Employing the established analytical approach, the exact analytical solution of the non-dimensional transverse displacement of the elastic nanobeam at the line of action of the concentrated force $\mathscr{F}$ is determined as

$$\begin{aligned} \overline{w}(\overline{c}) &= \frac{1}{3}\overline{c}^3 - \overline{c}\left((-1+\alpha)\zeta^2 + \eta^2\right) \\ &+ \frac{1}{2\left(-1+\exp\left(\frac{2}{\ell_a}\right)\right)}\exp\left(-\frac{2\overline{c}}{\ell_a}\right)\left(-1+\exp\left(\frac{\overline{c}}{\ell_a}\right)\right)\ell_a\left((-1+\alpha)\zeta^2+\eta^2\right) \\ &\times\left(-\exp\left(\frac{2}{\ell_a}\right) - 3\exp\left(\frac{2\overline{c}}{\ell_a}\right) + \exp\left(\frac{3\overline{c}}{\ell_a}\right) + 3\exp\left(\frac{2+\overline{c}}{\ell_a}\right)\right) \end{aligned} \tag{28}$$

where $\ell_a = \sqrt{\alpha\ell_c^2 + \ell_s^2}$ represents the intrinsic characteristic parameter introduced within the framework of the micromorphic beam theory developed by Challamel and Wang (Challamel & Wang, 2008).

The determination of the asymptotic behavior of the transverse displacement of the elastic nanobeam at the line of action of the concentrated force is of interest. As $\overline{c}$ approaches unity, the problem of tip-loaded cantilever mixture unified gradient elastic beam is addressed, and the flexural response of the elastic nanobeam is simplified as

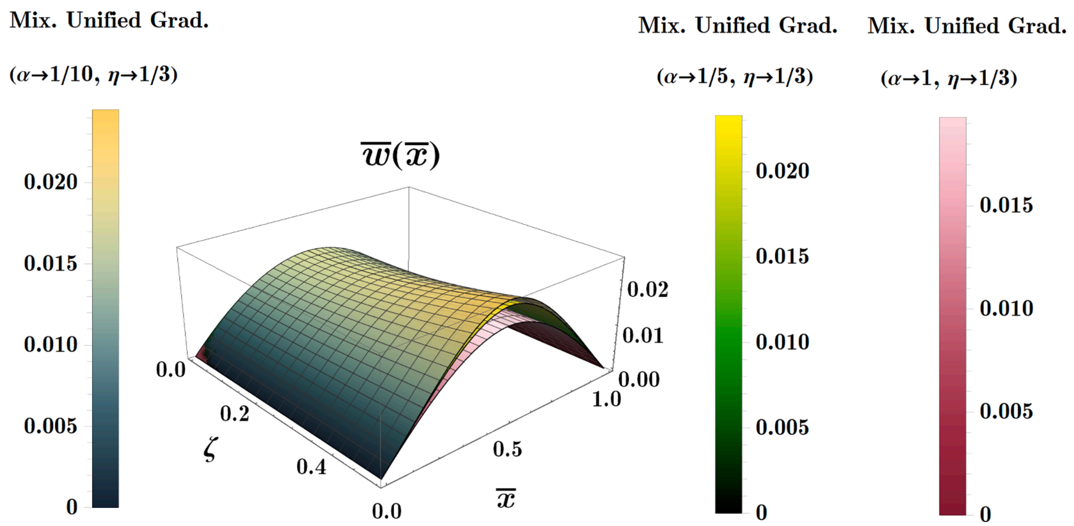

Fig. 5. A mixture unified gradient beam with simple supports subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and ζ for fixed $\eta = 1\,/3$ and $\alpha = 1/10, 1/5, 1$.

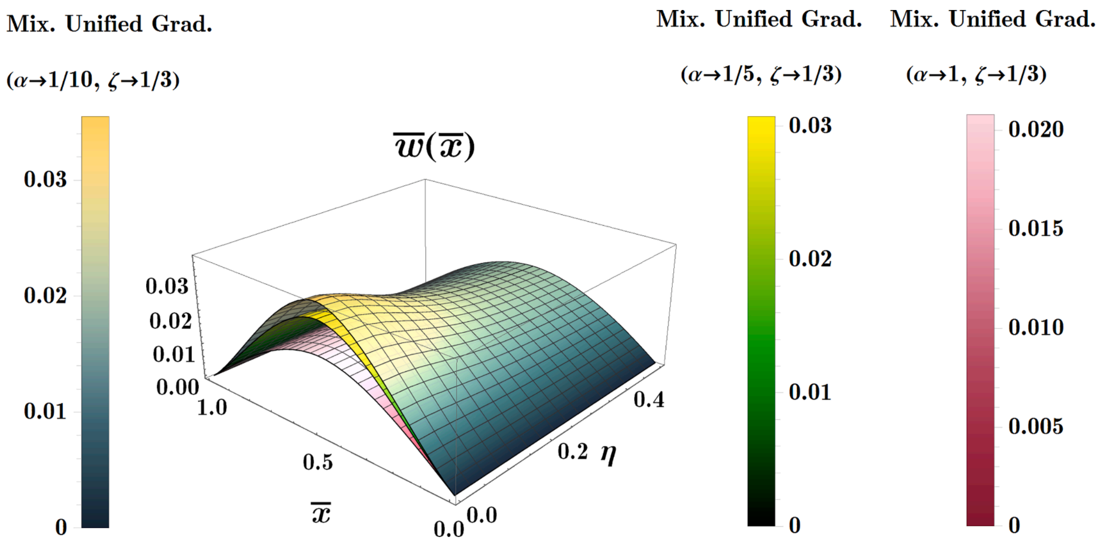

Fig. 6. A mixture unified gradient beam with simple supports subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and η for fixed $\zeta = 1\,/3$ and $\alpha = 1/10, 1/5, 1$.

$$\overline{w}(1) = \frac{1}{3} - (-1+\alpha)\zeta^2 - \eta^2 + \frac{\ell_a((-1+\alpha)\zeta^2+\eta^2)}{2\left(-1+\exp\left(\frac{2}{\ell_a}\right)\right)}\exp\left(-\frac{2}{\ell_a}\right)\left(-1+\exp\left(\frac{1}{\ell_a}\right)\right)\left(-4\exp\left(\frac{2}{\ell_a}\right)+4\exp\left(\frac{3}{\ell_a}\right)\right) \tag{29}$$

which literally coincides with the maximum transverse displacement of the elastic nanobeam consistent with the micromorphic beam framework developed in Zhang et al. (2010).

Nano-scale effects of the gradient characteristic parameters on the flexure of the elastic nanobeam with simple supports, fixed-ends, and fixed-simple ends are studied in Figs. 5 through 10. The nanobeams are considered to be subject to a concentrated force at the midspan $\overline{x} = \overline{c} = 1/2$. The mixture unified gradient theory of elasticity is invoked to realize the ultra-small scale size-effects. The spatial variation of the transverse displacement of the nano-sized beam with simple support ends versus the centroid axis $\overline{x}$ is examined in Fig. 5 and 6 in terms of the stress gradient characteristic parameter ζ and the strain gradient characteristic parameter η, respectively. The flexural response of mixture unified gradient elastic beams with fixed-ends in terms of gradient characteristic parameters is studied in Fig. 7 and 8. Likewise, the size-effects on the flexure of elastic nanobeams with fixed-simple ends are examined in Fig. 9 and 10. In

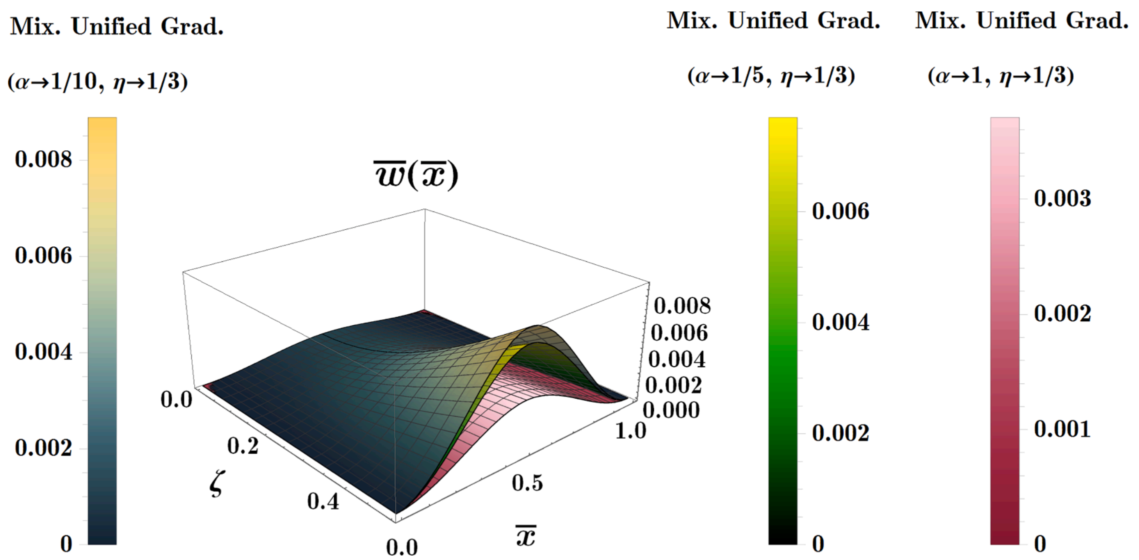

Fig. 7. A mixture unified gradient beam with fixed-ends subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and ζ for fixed $\eta = 1/3$ and $\alpha = 1/10, 1/5, 1$.

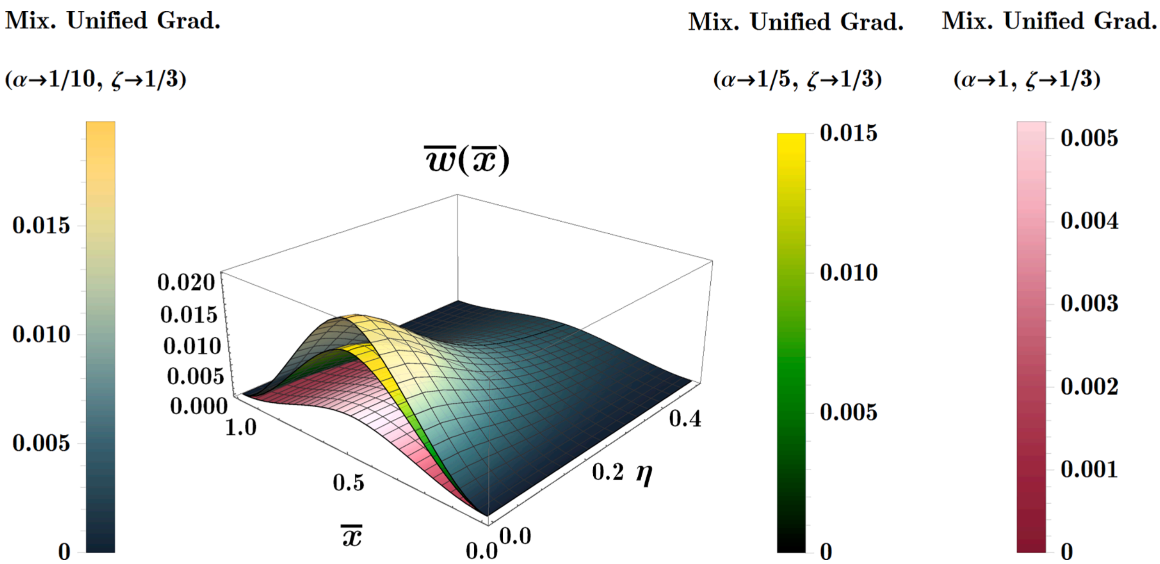

Fig. 8. A mixture unified gradient beam with fixed-ends subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and η for fixed $\zeta = 1/3$ and $\alpha = 1/10, 1/5, 1$.

Figs. 5, 7, and 9, where the nano-scale effects of the stress gradient characteristic parameter are studied, the stress gradient characteristic parameter ζ is considered to range in the interval $]0, 1/2]$ for the given value of the strain gradient characteristic parameter $\eta = 1/3$, while three values of the mixture parameter as $\alpha = 1/10, 1/5, 1$ are prescribed. Similarly, in Figs. 6, 8, and 10, the ranging interval of the strain gradient characteristic parameter is considered as $]0, 1/2]$ for the given value of the stress gradient characteristic parameter $\zeta = 1/3$ as three values of the mixture parameter $\alpha = 1/10, 1/5, 1$ are prescribed. In all the numerical illustrations, the centroid axis $\overline{x}$ is, inevitably, ranging in the interval $[0, 1]$.

It is remarkably deduced from the illustrated numerical results associated with the flexural response of the mixture unified gradient elastic beam that the transverse displacement of the nanobeam is increased by increasing the stress gradient characteristic parameter ζ, i.e. a larger value of ζ involves a larger transverse displacement for prescribed strain gradient characteristic parameter and the mixture parameter. A softening flexural response in terms of the stress gradient characteristic parameter is, therefore, confirmed within the framework of the mixture unified gradient theory. In contrast, the strain gradient characteristic parameter η has the effect of decreasing the transverse displacement of the mixture unified gradient elastic beam, i.e. a larger value of η involves a smaller transverse displacement for prescribed stress gradient and mixture parameters. A stiffening flexural response in terms of the strain gradient

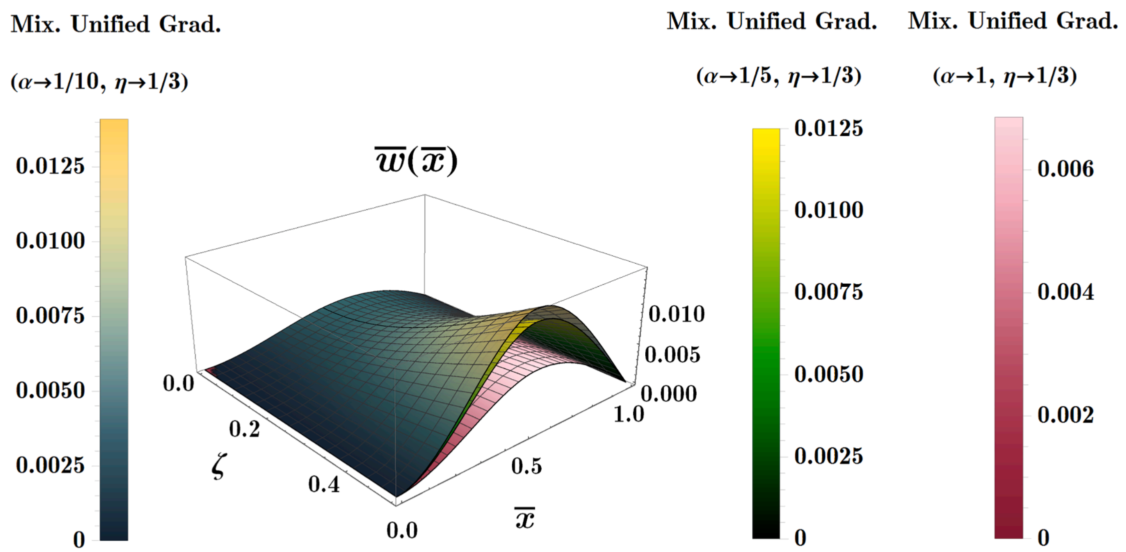

Fig. 9. A mixture unified gradient beam with fixed-simple ends subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and ζ for fixed $\eta = 1/3$ and $\alpha = 1/10, 1/5, 1$.

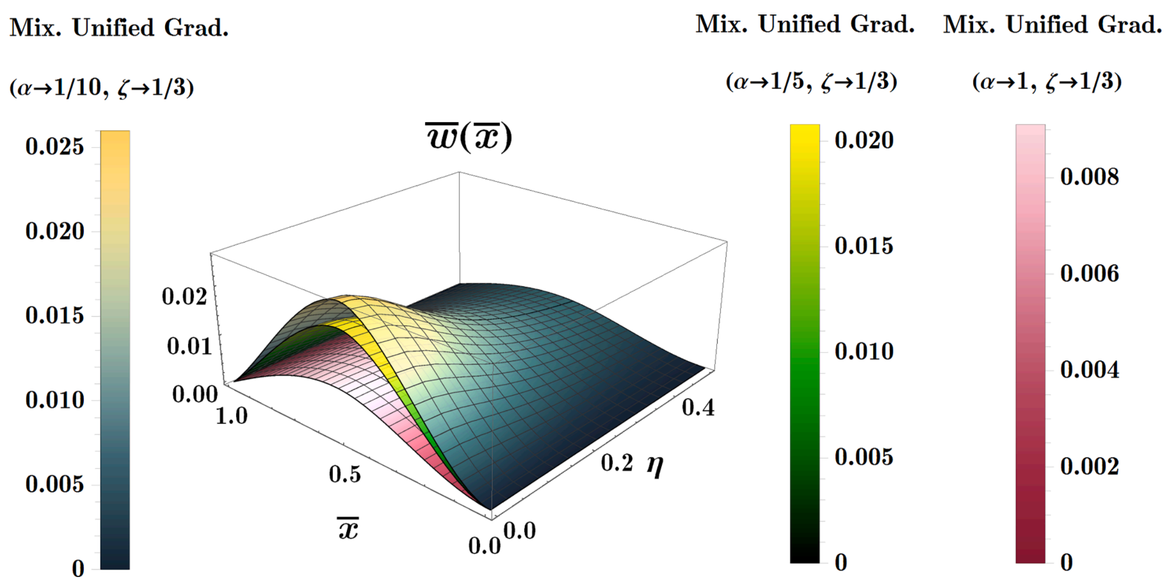

Fig. 10. A mixture unified gradient beam with fixed-simple ends subject to a transverse force at the midspan: $\overline{w}$ versus $\overline{x}$ and η for fixed $\zeta = 1/3$ and $\alpha = 1/10, 1/5, 1$.

characteristic parameter is, accordingly, realized for the mixture unified gradient elasticity framework. As the mixture parameter α continuously varies from zero to unity, the ultra-small size-effects of the stress gradient theory are reduced and reinstated with the effects of the classical elasticity theory. With increasing the mixture parameter α, the transverse displacement of the mixture unified gradient beam is realized to be decreased for prescribed values of the gradient characteristic parameters. A stiffening flexural response in terms of the mixture parameter α is, therefore, confirmed for the mixture unified gradient theory. The classical flexural characteristic of the mixture unified gradient elastic beam is, predictably, retrieved as either the mixture parameter approaches unity in the absence of the strain gradient characteristic parameter or as the gradient characteristic parameters vanish.

Numerical data associated with the mid-span transverse displacement of the elastic nanobeam consistent with the mixture unified gradient theory is summarized in Tables 1, 2, and 3 for the simple supports, fixed-ends, and fixed-simple ends conditions, correspondingly. For the sake of clarity, numerical results of the transverse displacement of the nano-sized beam for the prescribed value of the mixture parameter $\alpha = 1/2$ are not demonstrated in the numerical illustrations, nevertheless, they are tabulated in Tables 1, 2, and 3.

Subsequent to the analysis of the size-effects of the gradient characteristic parameters on the flexural response of the elastic

Table 1
Maximum deflection of a mixture unified gradient beam with simple supports subjected to a transverse point force at the midspan.

$\overline{w}_{\max}$												
	$\alpha \to 1/10$						$\alpha \to 1/5$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.02083	0.01932	0.01762	0.01675	0.01633	0.01610	0.02083	0.01932	0.01762	0.01675	0.01633	0.01610
0.1	0.02280	0.02069	0.01838	0.01719	0.01660	0.01629	0.02248	0.02055	0.01834	0.01718	0.01660	0.01628
0.2	0.02756	0.02435	0.02053	0.01847	0.01742	0.01684	0.02599	0.02352	0.02026	0.01838	0.01738	0.01682
0.3	0.03348	0.02930	0.02372	0.02046	0.01871	0.01773	0.02963	0.02695	0.02284	0.02013	0.01858	0.01767
0.4	0.03930	0.03459	0.02753	0.02298	0.02042	0.01894	0.03257	0.03003	0.02556	0.02218	0.02007	0.01876
0.5	0.04440	0.03956	0.03155	0.02586	0.02245	0.02040	0.03475	0.03249	0.02808	0.02431	0.02172	0.02003

Continued.

$\overline{w}_{\max}$												
	$\alpha \to 1/2$						$\alpha \to 1$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.02083	0.01932	0.01762	0.01675	0.01633	0.01610	0.02083	0.01932	0.01762	0.01675	0.01633	0.01610
0.1	0.02173	0.02018	0.01822	0.01714	0.01658	0.01628	0.02083	0.01967	0.01805	0.01708	0.01656	0.01626
0.2	0.02317	0.02178	0.01961	0.01813	0.01728	0.01677	0.02083	0.02014	0.01885	0.01781	0.01713	0.01670
0.3	0.02419	0.02312	0.02107	0.01937	0.01823	0.01750	0.02083	0.02042	0.01949	0.01855	0.01780	0.01727
0.4	0.02481	0.02401	0.02227	0.02057	0.01926	0.01834	0.02083	0.02057	0.01991	0.01913	0.01842	0.01784
0.5	0.02518	0.02458	0.02316	0.02159	0.02025	0.01921	0.02083	0.02065	0.02017	0.01955	0.01892	0.01835

Table 2
Maximum deflection of a mixture unified gradient beam with fixed-ends subjected to a transverse point force at the midspan.

$\overline{w}_{max}$												
	$\alpha \to 1/10$						$\alpha \to 1/5$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.00521	0.00369	0.00199	0.00113	0.00070	0.00047	0.00521	0.00369	0.00199	0.00113	0.00070	0.00047
0.1	0.00717	0.00506	0.00276	0.00157	0.00098	0.00066	0.00685	0.00492	0.00271	0.00156	0.00097	0.00066
0.2	0.01193	0.00872	0.00491	0.00284	0.00179	0.00121	0.01037	0.00790	0.00464	0.00275	0.00175	0.00120
0.3	0.01785	0.01368	0.00810	0.00483	0.00309	0.00211	0.01400	0.01133	0.00722	0.00450	0.00295	0.00204
0.4	0.02368	0.01896	0.01191	0.00736	0.00479	0.00331	0.01695	0.01440	0.00994	0.00656	0.00444	0.00314
0.5	0.02877	0.02393	0.01593	0.01023	0.00682	0.00477	0.01912	0.01687	0.01246	0.00868	0.00609	0.00441

Continued.

$\overline{w}_{max}$												
	$\alpha \to 1/2$						$\alpha \to 1$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.00521	0.00369	0.00199	0.00113	0.00070	0.00047	0.00521	0.00369	0.00199	0.00113	0.00070	0.00047
0.1	0.00611	0.00455	0.00260	0.00152	0.00096	0.00065	0.00521	0.00404	0.00243	0.00146	0.00093	0.00064
0.2	0.00754	0.00616	0.00398	0.00251	0.00165	0.00115	0.00521	0.00451	0.00322	0.00218	0.00150	0.00107
0.3	0.00856	0.00749	0.00545	0.00374	0.00260	0.00187	0.00521	0.00479	0.00386	0.00292	0.00218	0.00164
0.4	0.00918	0.00838	0.00665	0.00494	0.00364	0.00271	0.00521	0.00494	0.00428	0.00350	0.00279	0.00221
0.5	0.00955	0.00895	0.00754	0.00597	0.00462	0.00358	0.00521	0.00503	0.00455	0.00392	0.00329	0.00273

Table 3
Midspan deflection of a mixture unified gradient beam with fixed-simple ends subjected to a transverse point force at the midspan.

$\overline{w}_{\mathrm{midspan}}$												
	$\alpha \to 1/10$						$\alpha \to 1/5$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.00911	0.00731	0.00498	0.00345	0.00247	0.00183	0.00911	0.00731	0.00498	0.00345	0.00247	0.00183
0.1	0.01137	0.00894	0.00597	0.00408	0.00290	0.00213	0.01101	0.00877	0.00591	0.00405	0.00289	0.00213
0.2	0.01685	0.01329	0.00873	0.00587	0.00413	0.00303	0.01513	0.01234	0.00838	0.00573	0.00407	0.00299
0.3	0.02369	0.01915	0.01277	0.00860	0.00606	0.00445	0.01951	0.01653	0.01170	0.00815	0.00585	0.00434
0.4	0.03045	0.02538	0.01750	0.01199	0.00853	0.00630	0.02318	0.02035	0.01518	0.01094	0.00802	0.00603
0.5	0.03638	0.03122	0.02242	0.01574	0.01136	0.00848	0.02599	0.02348	0.01841	0.01380	0.01037	0.00793

Continued.

$\overline{w}_{\mathrm{midspan}}$												
	$\alpha \to 1/2$						$\alpha \to 1$					
ζ	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$	$\eta=0$	$\eta=0.1$	$\eta=0.2$	$\eta=0.3$	$\eta=0.4$	$\eta=0.5$
0	0.00911	0.00731	0.00498	0.00345	0.00247	0.00183	0.00911	0.00731	0.00498	0.00345	0.00247	0.00183
0.1	0.01016	0.00832	0.00574	0.00398	0.00285	0.00211	0.00911	0.00769	0.00548	0.00387	0.00280	0.00208
0.2	0.01193	0.01029	0.00751	0.00534	0.00388	0.00289	0.00911	0.00822	0.00643	0.00481	0.00360	0.00274
0.3	0.01333	0.01203	0.00943	0.00706	0.00529	0.00403	0.00911	0.00855	0.00724	0.00580	0.00457	0.00361
0.4	0.01427	0.01328	0.01106	0.00874	0.00681	0.00533	0.00911	0.00874	0.00778	0.00660	0.00546	0.00448
0.5	0.01489	0.01414	0.01232	0.01019	0.00825	0.00666	0.00911	0.00885	0.00814	0.00720	0.00620	0.00527

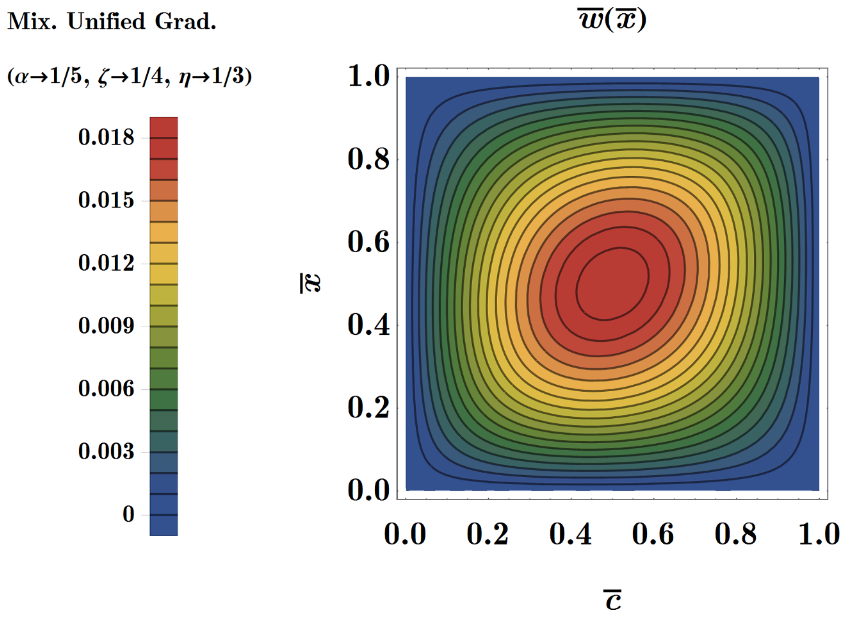

Fig. 11. Contour plot of the flexure of the mixture unified gradient beam with simple supports subject to a transverse force at $\bar{c}$ for prescribed parameter $\alpha = 1/5, \zeta = 1/4, \eta = 1/3$.

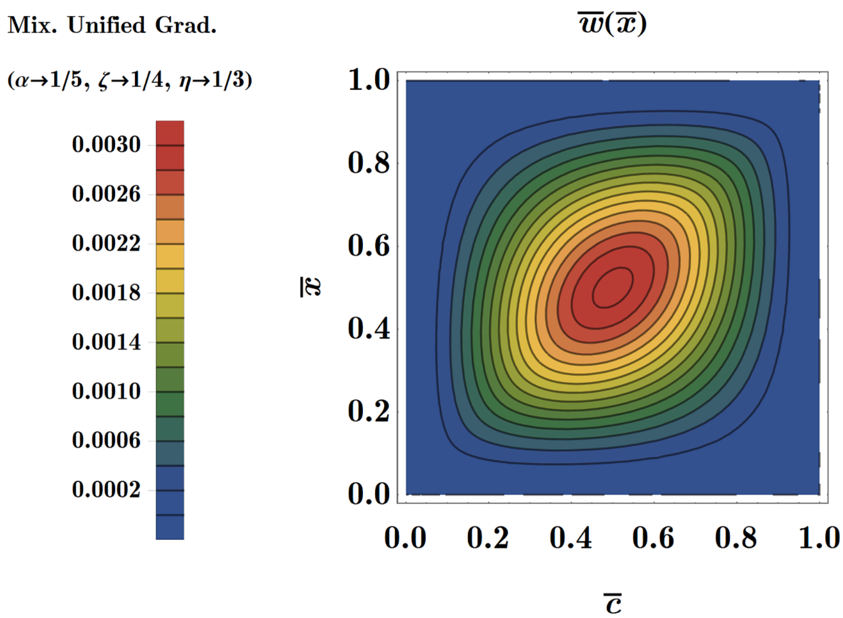

Fig. 12. Contour plot of the flexure of the mixture unified gradient beam with fixed-ends subject to a transverse force at $\bar{c}$ for prescribed parameter $\alpha = 1/5, \zeta = 1/4, \eta = 1/3$.

nanobeam, let us examine the effects of the line of action of the concentrated force $\mathscr{F}$ on the flexure of the mixture unified gradient elastic beam. Implementing the conceived analytical solution approach, contour plots of the spatial variation of the transverse displacement of the nano-sized beam versus the beam centroid axis $\bar{x}$ and the action line of concentrated force $\bar{c}$ are illustrated in Figs. 11 through 13. The elastic nanobeam is assumed to have simple supports, fixed-ends, and fixed-simple boundary conditions, respectively, in Fig. 11, 12, and 13. In numerical illustrations associated with Figs. 11 through 13, the beam centroid axis $\bar{x}$ and the action line of concentrated force $\bar{c}$ are, inevitably, considered to range in the interval $[0,1]$. The mixture parameter $\alpha = 1/5$, the stress gradient characteristic parameter $\zeta = 1/4$, and the strain gradient characteristic parameter $\eta = 1/3$ are, correspondingly, prescribed.

It is notably inferred from the demonstrated numerical results consistent with the flexure of the elastic nanobeam that the transverse displacement of the mixture unified gradient elastic beam is, accordingly, decreased with increasing the number of kine-

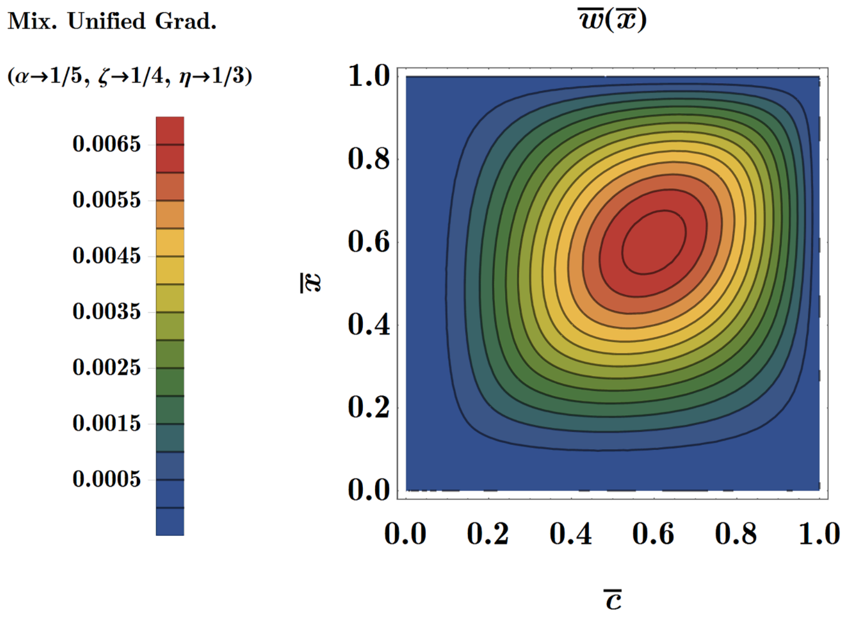

Fig. 13. Contour plot of the flexure of the mixture unified gradient beam with fixed-simple ends subject to a transverse force at $\overline{c}$ for prescribed parameter $\alpha = 1/5, \zeta = 1/4, \eta = 1/3$.

matic boundary constraints of the beam. In consideration of the symmetry of the classical and extra non-standard boundary conditions along with interface conditions for elastic nanobeams with simple supports and fixed-ends, the flexural response of the mixture unified gradient elastic beam is symmetric with respect to the beam centroid axis $\overline{x}$. Consequently, the maximum of the transverse displacement of nano-scale elastic beam occurred when the action line of concentrated force $\overline{c}$ is coincident with the beam midspan. However, the maximum transverse displacement of the mixture unified gradient elastic beam occurs when the action line of concentrated force $\overline{c}$ is getting close to the elastic nanobeam simple support end.

4. Concluding remarks

The flexure mechanics of small-scale elastic beams with discontinuous field variables is investigated by adopting an abstract variational scheme. The mixture unified gradient theory of elasticity is invoked to capture the size-effects. Specifically, the scenario involving discontinuity in field variables, resulting from concentrated loads and couples within the interior parts of the domain, is addressed. To tackle this problem, the variational functional associated with the mixture unified gradient theory is decomposed at the discontinuity cross-section into two contributions corresponding to the domains on the left and right of the cross-section. Founded on the introduced stationary variational principle, mathematically well-posed non-standard interface conditions are derived. The boundary-value problem associated with the flexure of elastic nanobeams with discontinuity is, accordingly, closed. The exact analytical solutions of the transverse displacement of elastic nanobeams are obtained by successive integrations of the differential equations of lower-order while prescribing the variationally consistent and non-standard interface and boundary conditions.

To evince the efficacy of the proposed variational framework, particularly the derived non-standard interface conditions, exact analytical solutions are compared with approximate solutions obtained through a series expansion approach. This comparison is conducted for a simply supported nanobeam subjected to a concentrated load at the midspan. Since each term in the eigenfunction expansion of the flexural moment is a smooth function, the corresponding solution is derived without the need for imposing non-standard interface conditions. The flexural results detected based on the analytical solution demonstrate an excellent agreement with counterpart results obtained via the series expansion method, confirming its effectiveness. The proposed exact analytical solution approach is, afterward, utilized to investigate the size-effect in the flexure of elastic nanobeams with boundary conditions of applicative interest, i.e. elastic nanobeam with clamped-free, simple supports, fixed ends, and fixed-simple ends subject to concentrated loads. Based on the demonstrated numerical results, it is realized that the stress gradient characteristic parameter induces a softening effect on the flexural response of small-scale beams, while the strain gradient characteristic and mixture parameters contribute to a stiffening effect on the beam deflection. The detected nanoscopic findings align with prior results in the literature on the flexural response of mixture unified gradient beams under continuous loadings. Implementing the conceived analytical solution approach, the effects of the line of action of the concentrated force on the flexural behavior of the mixture unified gradient elastic beam are thoroughly discussed. The elucidating numerical results serve as a benchmark for the analysis and design of beam-type blocks of cutting-edge MEMS and NEMS, and thus, the groundwork for future advancements in nano-mechanics. The established variationally consistent framework not only illuminates the path forward but also sparks inspiration for further research endeavors, particularly in the realm of fracture mechanics of ultra-small scale elastic beams.

CRediT authorship contribution statement

S. Ali Faghidian: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Hossein Darban:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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